Vector invariants in arbitrary characteristic

Frank Grosshans

Aachen, RWTH

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Outline

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  - §1. The general problem of vector invariants

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  - §5. Main Theorem

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  - §6. Finite groups; Classical groups

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  - §7. Connections to representation theory

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  - (Multiply an $n \times d$ matrix with entries x_{ij} by g on right.)

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  - $GL_d * {}^H k[M_{n,n}] \subset {}^H k[M_{n,d}]$
  - General problem restated: does $GL_d * {}^H k[M_{n,n}] = {}^H k[M_{n,d}]$?

§2. Weyl's Theorem

- Weyl's Theorem and example
  - **Theorem** [Weyl, The Classical Groups, 2nd ed., p.44]. Suppose that char $k = 0$. Then, $GL_d * {}^H k[M_{n,n}] = {}^H k[M_{n,d}]$

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  - The orthogonal group, an integral

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  - **Theorem**. Let V, W be finite-dimensional G-modules with $V \subset W$. If $V^U = W^U$, then $V = W$.

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  - Have ${}^{H}k[M_{n,d}]^U \subset {}^{H}k[M_{n,n}]^U \subset (GL_d *^{H} k[M_{n,n}])^U$.
  - Conclude that $GL_d * {}^{H}k[M_{n,n}] = {}^{H}k[M_{n,n}]$.

- Finding counter-examples
  - If $GL_d * {}^H k[M_{n,n}] = {}^H k[M_{n,d}]$ for all d, then there is a positive integer N so that ${}^H k[M_{n,d}]$ is generated by polynomials of degree $\leq N$ for all d.

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  - Thus, if the maximal degree of the generators for ${}^H k[M_{n,d}]$ increases with d, then $GL_d * {}^H k[M_{n,n}] \not\subseteq {}^H k[M_{n,d}]$ when d is sufficiently large.

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  - **Theorem** (Richman, 1996). H finite, char $k = p$, p divides $|H|$, then every set of k-algebra generators for ${}^{H}k[M_{n,d}]$ contains a generator of degree $d(p-1)/(p^{|H|-1}-1)$

- $p-$root closure
  - Definition. Let char $k = p > 0$ and let R and S be commutative k - algebras with $R \subset S$. We say that S is contained in the p - root closure of R if for every $s \in S$, there is a non- negative integer m so that $s^{p^m} \in R$.

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  - **Main Theorem**. H closed subgroup of GL_n. Then ${}^H k[M_{n,d}]$ is contained in the p - root closure of $GL_d * {}^H k[M_{n,n}]$. (If $p = 0$, have equality.)

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  - do *not* have compete reducibility; char $k = 2$, $V =< v, w >$, $G = GL_2$, look at $S^2(V)$

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- Integral extensions
  - **Definition**. A commutative k-algebra, G linear algebraic group with identity e. A *rational action* of G on A is given by a mapping $G \times A \longrightarrow A$, denoted by $(g, a) \longrightarrow ga$ so that: (i) $g(g\acute{}a) = (gg\acute{})a$ and $ea = a$ for all $g, g\acute{} \in G, a \in A$; (ii) the mapping $a \longrightarrow ga$ is a k-algebra automorphism for all $g \in G$; (iii) every element in A is contained in a finite-dimensional subspace of A which is invariant under G and on which G acts by a rational representation.

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  - **Theorem**. G reductive, A commutative k-algebra on which G acts rationally. Then A is integral over $G{\cdot}A^U$, smallest G -invariant algebra containing A^U.

- Proof of Main Theorem
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  - Separating orbits (Draisma, Kemper, Wehlau): let $x, y \in M_{n,d}$. If there is an $F \in^H k[M_{n,d}]$ with $F(x) \neq F(y)$, then there is an $F_o \in GL_d *^H k[M_{n,n}]$ with $F_o(x) \neq F_o(y)$.

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  - (van der Kallen). Suppose that char $k = p > 0$. Let X and Y be affine varieties and let $f : X \to Y$ be a proper bijective morphism. Then $k[X]$ is contained in the p-root closure of $k[Y]$.

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  - Put $X = Spec^H k[M_{n,d}]$, $Y = Spec GL_d * {}^H k[M_{n,n}]^U$

- Finite groups
  - char $k = 0$ or char $k = p$ where $p \nmid |H|$ (non-modular case)

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  - **Problem 2**: Explain Richman's theorem

§6. More examples

- Classical groups
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  - invariants for char $k = p > 0$ same as for char $k = 0$ (Igusa, Rota, De Concini, Procesi)

§7. Connections to representation theory

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  - When is $GL_d * {}^H k[M_{n,n}] = {}^H k[M_{n,d}]$?
  - Why are the invariants of the classical groups the same in all characteristics?
  - Why is Richman's theorem true?
  - Answers (?): lie in the study of the representation of GL_d on ${}^H k[M_{n,d}]$.

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- Graded algebra
  - Can construct a graded algebra, $gr({}^{H}k[M_{n,d}])$. There is an GL_d - equivariant algebra monomorphism $\Phi : gr({}^{H}k[M_{n,d}]) \to \bigoplus V_i$ where the V_i are Schur modules.

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  - Any Schur module has unique (up to scalar) highest weight vector.
  - In the case of ${}^H k[M_{n,d}]$, these highest weight vectors are all in ${}^H k[M_{n,n}]$.
  - But, in general, V_i is not the linear span of the $g * v_o$ where v_o is a highest weight vector.

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  - By restriction, get algebra monomorphism $\Phi': gr(GL_d * {}^H k[M_{n,d}]) \to \oplus V_i$
  - (C1) Φ' is surjective
  - (C2) $gr(GL_d * {}^H k[M_{n,d}])$ has a good GL_d - filtration.
  - (C3) $GL_d * {}^H k[M_{n,n}] = {}^H k[M_{n,d}]$
  - (C1) if and only if (C2).

- Three conditions
  - By restriction, get algebra monomorphism $\acute{\Phi}: gr(GL_d * {}^H k[M_{n,d}]) \to \oplus V_i$
  - (C1) $\acute{\Phi}$ is surjective
  - (C2) $gr(GL_d * {}^H k[M_{n,d}])$ has a good GL_d - filtration.
  - (C3) $GL_d * {}^H k[M_{n,n}] = {}^H k[M_{n,d}]$
  - (C1) if and only if (C2).
  - (C2) implies (C3)

- Three conditions
  - **Theorem**. $U =$ maximal unipotent subgroup of GL_d consisting of upper triangular matrices with 1's on diagonal, $T =$ diagonal matrices. Suppose that ${}^{H}k[M_{n,d}]^{U} = k[a_1, \ldots, a_r]$ with a_i having T-weight ϖ_i. If Schur module with highest weight ϖ_i is irreducible for $i = 1, \ldots, r$, then Φ is surjective and $GL_d * {}^{H}k[M_{n,n}] = {}^{H}k[M_{n,d}]$.

§7. Connections to representation theory

- Three conditions
  - **Theorem.** $U =$ maximal unipotent subgroup of GL_d consisting of upper triangular matrices with 1's on diagonal, $T =$ diagonal matrices. Suppose that ${}^{H}k[M_{n,d}]^{U} = k[a_1, \ldots, a_r]$ with a_i having T-weight ϖ_i. If Schur module with highest weight ϖ_i is irreducible for $i = 1, \ldots, r$, then Φ is surjective and $GL_d * {}^{H}k[M_{n,n}] = {}^{H}k[M_{n,d}]$.
  - Examples: classical groups

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Slides - Beamer

- This document illustrates the appearance of a presentation created with the shell **Slides - Beamer**.
- The LaTeX Beamer document class produces presentations, handouts, and transparency slides as typeset PDF files.
- DVI output is not available.
- The class provides
  - Control of layout, color, and fonts
  - A variety of list and list display mechanisms
  - Dynamic transitions between slides
  - Presentations containing text, mathematics, graphics, and animations
- A single document file contains an entire Beamer presentation.
- Each slide in the presentation is created inside a frame environment.
- To produce a sample presentation in *SWP* or *SW*, typeset this shell document with PDFLaTeX .

Beamer Files

- The document class base file for this shell is `beamer.cls`.
- To see the available class options, choose Typeset, choose Options and Packages, select the Class Options tab, and then click the Modify button.
- This shell specifies showing all notes but otherwise uses the default class options.
- The typesetting specification for this shell document uses these options and packages with the defaults indicated:

Options and Packages	**Defaults**
Document class options	Show notes
Packages:	
hyperref	Standard
mathpazo	None
multimedia	None

Using This Shell

- The front matter of this shell has a number of sample entries that you should replace with your own.
- Replace the body of this document with your own text. To start with a blank document, delete all of the text in this document.
- Changes to the typeset format of this shell and its associated LaTeX formatting file (`beamer.cls`) are not supported by MacKichan Software, Inc. If you want to make such changes, please consult the LaTeX manuals or a local LaTeX expert.
- If you modify this document and export it as "Slides - Beamer.shl" in the `Shells\Other\SW` directory, it will become your new Slides - Beamer shell.

What is Beamer?

- Beamer is a LaTeX document class that produces beautiful PDFLaTeX presentations and transparency slides.
- Beamer presentations feature
  - PDFLaTeX output
- To produce a sample presentation in *SWP* or *SW*, typeset this shell document with PDFLaTeX .

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Creating frames

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  2. Place the text for the frame between the BeginFrame and EndFrame fields.
  3. Enter the frame title and subtitle.
     If you used the Frame fragment, apply the Frame title and Frame subtitle text tags as necessary.

Learn more about Beamer

- This shell and the associated fragments provide basic support for Beamer in *SWP* and *SW*.
- To learn more about Beamer, see SWSamples/PackageSample-beamer.tex in your program installation.
- For complete information, read the BeamerUserGuide.pdf manual found via a link at the end of SWSamples/PackageSample-beamer.tex.
- For support, contact **support@mackichan.com**.