Some constructions of modular forms for the Weil representation and applications

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Introduction

Let V be a simple vertex operator algebra satisfying certain regularity conditions with irreducible modules V_γ. Define

$$F_\gamma(\tau) = \operatorname{tr}|_{V_\gamma} q^{L_0 - c/24} \,.$$

Then $F = \sum F_\gamma e^\gamma$ is a modular form for the Weil representation of $SL_2(\mathbb{Z})$.

Borcherds' singular theta correspondence maps modular forms for the Weil representation of $SL_2(\mathbb{Z})$ to automorphic forms on orthogonal groups.

The Weil representation

Let D be a discriminant form of even signature with quadratic form $\gamma \mapsto \gamma^2/2$. The Weil representation of $\Gamma = SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$ is defined by

$$\rho_D(T)\, e^{\gamma} = e(-\gamma^2/2)\, e^{\gamma}$$

$$\rho_D(S)\, e^{\gamma} = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e(\gamma\beta)\, e^{\beta}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of Γ.

The Weil representation

Theorem

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(-a\beta_c^2/2)e(-b\beta\gamma)e(-bd\gamma^2/2)e^{d\gamma+\beta}$$

with $\xi = e(\operatorname{sign}(D)/4) \prod \xi_p$.

The Weil representation

Let $F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau) e^\gamma$ be a holomorphic function on the upper halfplane with values in $\mathbb{C}[D]$ and k an integer. Then F is a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau + d)^k \rho_D(M) F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and F is meromorphic at $i\infty$.

Example

Let L be a positive definite even lattice of even rank $2k$. For $\gamma \in D_L$ define

$$\theta_\gamma(\tau) = \sum_{\alpha \in \gamma + L} q^{\alpha^2/2}.$$

Then

$$\theta(\tau) = \sum_{\gamma \in D_L} \theta_\gamma(\tau) e^\gamma$$

is a modular form for the dual Weil representation $\overline{\rho}_{D_L}$ of weight k which is holomorphic at $i\infty$.

Induction from congruence subgroups

Let D be a discriminant form of even signature. The level of D is the smallest positive integer k such that $k\gamma^2/2 = 0 \mod 1$ for all $\gamma \in D$. Suppose the level of D divides N. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then the formula for ρ_D gives

$$\rho_D(M)e^\gamma = \left(\frac{a}{|D|}\right) e\big((a-1)\,\mathrm{oddity}(D)/8\,\big)\, e(-bd\gamma^2/2)\, e^{d\gamma}\,.$$

Induction from congruence subgroups

Theorem

Let D be a discriminant form of even signature and level dividing N.

Let f be a scalar valued modular form for $\Gamma_0(N)$ of weight k and character χ_D and let H be an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ as a set. Then

$$F_{\Gamma_0(N),f,H} = \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in H} f|_M \, \rho_D(M^{-1}) e^\gamma$$

is modular form for ρ_D of weight k.

Induction from congruence subgroups

Let f be a scalar valued modular form on $\Gamma_1(N)$ of weight k and character χ_γ. Then

$$F_{\Gamma_1(N),f,\gamma} = \sum_{M \in \Gamma_1(N)\backslash\Gamma} f|_M \, \rho_D(M^{-1}) e^\gamma$$

is modular form for ρ_D of weight k.

Every modular form for Weil representation can be obtained as a linear combination of liftings from $\Gamma_1(N)$.

Induction from congruence subgroups

The Fourier expansions of the inductions can be calculated explicitly using the above formula for Weil representation.
For $\Gamma_1(N)$ we find

Theorem

The function $F_{\Gamma_1(N),f,\gamma}$ can be written as a sum $\sum F_s$ over the cusps of $\Gamma_1(N)$ where

$$F_s = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in a\gamma + D^{c*}} e(d(\mu - a\gamma)_c^2/2) e(b\mu\gamma)$$
$$e(-ab\gamma^2/2)\, t\, g_{mt,j_\mu} \left\{ e^{\mu} + (-1)^k e(\mathrm{sign}(D)/4) e^{-\mu} \right\}$$

if $N > 2$ and s is regular and similarly in the other cases.

Induction from isotropic subgroups

Let D be a discriminant form of even signature. Let H be an isotropic subgroup of D and $H^{\perp}$ the orthogonal complement of H in D. Then $D_H = H^{\perp}/H$ is a discriminant form. Let $F_{D_H} = \sum_{\gamma \in D_H} F_{D_H,\gamma}\, e^{\gamma}$ be a modular form for ρ_{D_H}. Define

$$F = \sum_{\gamma \in H^{\perp}} F_{D_H,\gamma+H}\, e^{\gamma} \, .$$

Then

Theorem

F is a modular form for ρ_D.

Discriminant forms of squarefree level

Theorem

Let D be a discriminant form of squarefree level N and $F = \sum_{\gamma \in D} F_\gamma e^\gamma$ a modular form for ρ_D which is invariant under $\mathrm{Aut}(D)$. Then the complex vector space W spanned by the components F_γ, $\gamma \in D$ is generated by the functions $F_0|_M$, $M \in \Gamma$.

Discriminant forms of squarefree level

Theorem

Let D be a discriminant form of squarefree level N and I_k the set of isotropic elements of order k. Let $F = \sum_{\gamma \in D} F_\gamma e^\gamma$ be a modular form for ρ_D which is invariant under $\mathrm{Aut}(D)$. Let N_R be the product over the primes with nonvanishing I_p. For $k|N_R$ define $F_k = F_\gamma$ where γ is any element in I_k. Then the functions F_k span the subspace W_0 of W with T-eigenvalue 0. Define

$$\begin{array}{rccl} \Phi : & W_0 & \longrightarrow & W_0 \\ & f & \longmapsto & \text{0-component of } F_{\Gamma_0(N),f,0} \,. \end{array}$$

Discriminant forms of squarefree level

Then

$$\Phi(F_k) = \sum_{j|N_R} a_{jk} F_j$$

with

$$a_{jk} = \frac{N}{|D|} |I_j| \sum_{c|(N/j,N/k)} \frac{|D_c|}{c}.$$

The matrix $A = (a_{jk})$ has determinant

$$\det(A) = \left(\frac{N}{|D|}\right)^{\sigma(N_R)} \left(\sum_{d|N/N_R} \frac{|D_d|}{d} \right)^{\sigma(N_R)} \prod_{d|N_R} |I_d| \prod_{d|N_R} \frac{|D_d|}{d}.$$

In particular Φ is invertible.

Discriminant forms of squarefree level

Corollary

Let D be a discriminant form of squarefree level N and F a modular form for ρ_D which is invariant under $\mathrm{Aut}(D)$. Then $F = F_{\Gamma_0(N),f,0}$ for a suitable modular form on $\Gamma_0(N)$ with character χ_D.

Applications

1. Let N be the Niemeier lattice with root system E_8^3 and g be a permutation of the three E_8-components of order 3. Then $N^g \cong \sqrt{3}E_8$ and $N^{g\perp} \cong A_2 \otimes E_8$. The theta function $\theta_{N^{g\perp}}$ defines a modular form for the discriminant form of N^g. This function is invariant under $\mathrm{Aut}(N^g)$ because the centralizer of g in $\mathrm{Aut}(N)$ induces the full automorphism group of N^g. Let $L = N^g \oplus \left(\begin{smallmatrix} -2 & 3 \\ 3 & 0 \end{smallmatrix}\right)$. Then $\theta_{N^{g\perp}}$ induces a modular form on L. Denote the quotient of this form by the invariant 3Δ by $F_{\theta_{N^{g\perp}}/3\Delta}$. Define $\eta_g(\tau) = \eta(3\tau)^8$. Then

$$F = F_{\theta_{N^{g\perp}}/3\Delta} + \frac{1}{3}\, F_{\Gamma_0(9),1/\eta_g,0}$$

is a modular form for the Weil representation of $L \oplus II_{1,1}$ with nonnegative integral coefficients, reflective poles and $[F_0](0) = 8$.

Applications

The theta lift of F has singular weight and is given by

$$e((\rho, Z)) \prod_{\alpha \in L'^{+}} \big(1 - e((\alpha, Z))\big)^{[F_{\alpha+L}](-\alpha^2/2)}$$
$$= \sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} \big(1 - e((3nw\rho, Z))\big)^{8} .$$

This is the denominator identity of a generalized Kac-Moody algebra whose real simple roots are the simple roots of W and imaginary simple roots are the positive multiples of 3ρ with multiplicity 8.

2. Let N be the Niemeier lattice with root system D_{12}^2. Define $g \in \mathrm{Aut}(N)$ by $g(x,y) = (y,x)$. Then $N^g \cong N^{g\perp} \cong \sqrt{2}D_{12}^+$. The theta function $\theta_{N^{g\perp}}$ defines a modular form for the discriminant form of N^g. This function is again invariant under $\mathrm{Aut}(N^g)$ because $C(g)$ induces $\mathrm{Aut}(N^g)$. The lattice N^g contains a sublattice $K \cong \sqrt{2}D_{12}$ of genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$. Then $H = N^g/K$ is an isotropic subgroup of D_K. The function $\theta_{N^{g\perp}}/2\Delta$ induces a modular form $F_{\theta_{N^{g\perp}}/2\Delta}$ for the Weil representation of K. Define $\eta_g(\tau) = \eta(2\tau)^{12}$ and

$$F = F_{\theta_{N^{g\perp}}/2\Delta} + \frac{1}{2}\,F_{\Gamma_0(4),1/\eta_g,0} - \frac{1}{4}\,F_{\Gamma_0(4),1/\eta_g,H}\,.$$

Applications

The elements of norm $1/2 \mod 1$ in D_K decompose into 3 orbits under $\mathrm{Aut}(K)$ of length $132, 1848, 132$. The components of F are given by

$$F_0 = q^{-1} + 12 + 300q + 5792q^2 + 84186q^3 + 949920q^4 + \dots$$

and

$$F_\gamma = 12 + 288q + 5792q^2 + 84096q^3 + 949920q^4 + \dots$$

if $\gamma \in D_K^2 \backslash \{0\}$,

$$F_\gamma = 4 + 224q + 5344q^2 + 81792q^3 + 939232q^4 + \dots$$

if $\gamma^2/2 = 0 \bmod 1$ and $\gamma \notin D_K^2$,

Applications

$$F_\gamma = q^{-1/2} + 44q^{1/2} + 1242q^{3/2} + 22216q^{5/2} + \dots$$

if $\gamma^2/2 = 1/2 \bmod 1$ and γ is in one of the orbits of length 132,

$$F_\gamma = 32q^{1/2} + 1152q^{3/2} + 21696q^{5/2} + 284928q^{7/2} + \dots$$

if $\gamma^2/2 = 1/2 \bmod 1$ and γ is in the orbit of length 1848,

$$F_\gamma = q^{-1/4} + 90q^{3/4} + 2535q^{7/4} + 42614q^{11/4} + \dots$$

if $\gamma^2/2 = 1/4 \bmod 1$ and

$$F_\gamma = 12q^{1/4} + 520q^{5/4} + 10908q^{9/4} + 153960q^{13/4} + \dots$$

if $\gamma^2/2 = 3/4 \bmod 1$.

Applications

Let $L = K \oplus II_{1,1}$ and $M = L \oplus II_{1,1}$. Then F induces a modular form for D_M. The theta lift Ψ of F is a holomorphic automorpic form of singular weight. The level one expansion of Ψ is given by

$$e((\rho, Z)) \prod_{\alpha \in L'^{+}} \left(1 - e((\alpha, Z))\right)^{[F_{\alpha+L}](-\alpha^2/2)}$$
$$= \sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} \left(1 - e((2nw\rho, Z))\right)^{12}.$$

This is the denominator identity of a generalized Kac-Moody algebra whose simple roots and multiplicities are explicitly known.

3. Let Λ be the Leech lattice and $g \in \mathrm{Aut}(\Lambda)$ of cycle shape 3.21. Define

$$L = \Lambda^g \oplus \sqrt{7} \left(\begin{smallmatrix} -2 & 3 \\ 3 & 0 \end{smallmatrix} \right)$$

and $M = L \oplus II_{1,1}$. Then M has genus $II_{4,2}(3^{+2}9^{-1}7^{+3})$. Let

$$\Lambda^{g,3} = \Lambda^{g^3} \cap \Lambda^{g\perp} \cong \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix} \right) \otimes A_2 \,.$$

Let $\eta_g(\tau) = \eta(3\tau)\eta(21\tau)$ and $h(\tau) = \eta_g(\tau/3)$.

Then

$$F = \frac{1}{3}\, F_{\Gamma_0(63),1/\eta_g,0} + \frac{1}{12}\, F_{\Gamma_0(63),\theta_{\Lambda^g,3}/\eta_{g^3},D^{21}} + \frac{1}{216} \sum_{\gamma\in\gamma_1+D^{21}} F_{\Gamma_1(63),1/h,\gamma}$$

is a modular form for the Weil representation of M'/M with nonnegative integral coefficients, reflective poles and $[F_0](0) = 2$.

Applications

The theta lift of F has singular weight and is given by

$$e((\rho, Z)) \prod_{\alpha \in L'^{+}} \big(1 - e((\alpha, Z))\big)^{[F_{\alpha+L}](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, \eta_g((w\rho, Z))\,.$$

This is the denominator identity of a generalized Kac-Moody algebra whose simple roots and multiplicities are explicitly known.

Applications

4. The fake monster algebra G is a generalized Kac-Moody algebra acted on by Co_0.

Borcherds' conjecture (1995)

The twisted denominator identities of G under Co_0 are automorphic forms of singular weight on orthogonal groups.

The above methods can be used to give a complete proof of this conjecture.