Rademacher's series for the partition function

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Within this talk we want to consider the partition function and its representation in terms of Rademacher's functions. This provides a possibility to calculate it efficiently since it takes integral values and hence every approximation which is close enough yields the knowledge of the exact result.

§1 Sketching the proof

We need some preparation to carry out the proof and thus it is a good idea to see what we are aiming at. The generating function for the partition function p is well known and equals

$$F(z) = \prod_{m=1}^{\infty} \frac{1}{1 - z^m} = \sum_{n=0}^{\infty} p(n) z^n \tag{1}$$

on the unit disc. By Cauchy's residue theorem as shown in [Krieg, Analysis IV, p.504, 3.1] we have

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)}{\zeta^{n+1}} d\zeta,$$

where C is any contour homotopic to the positively orientated circle of radius $e^{-2\pi}$ within the punctured unit disc.

To deduce a converging series from this integral, we have to chose appropriate C's and in addition corresponding finite fragmentations of these C's. Most of the work, that we have to do, will be dedicated to this problem. A further problem will be the need for a functional equation for F. We want to treat this first.

§2 A functional equation for F

We recall the functional equation of Dedekind's η-function for a matrix

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

satisfying $c > 0$. For such a M we have

$$\eta(M\langle z\rangle) = \epsilon(a,b,c,d) \cdot (-i(cz+d))^{\frac{1}{2}} \cdot \eta(z)$$

with

$$\epsilon(a,b,c,d) = \exp\left(\pi i \left(\frac{a+d}{12c} + s(-d,c)\right)\right)$$

and s Dedekind's sum

$$s(h,k) = \sum_{n=1}^{k-1} \frac{n}{k}\left(\frac{hn}{k} - \left\lfloor \frac{hn}{k} \right\rfloor - \frac{1}{2}\right).$$

The product expansion of η

$$\eta(z) = e^{\pi i \frac{z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i m z}\right)$$

yields

$$F(e^{2\pi i z}) = e^{\frac{\pi i z}{12}} \cdot \eta(z)^{-1}.$$

We can transport this to a functional equation of F.

(2.1) Theorem
Let $z \in \mathbb{E}$ with $\mathrm{Re}(z) > 0$ and let $k, h, H \in \mathbb{N}$ such that $(h,k) = 1$ and $hH \equiv -1 \pmod{k}$. Define

$$x = \exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right), \qquad x' = \exp\left(\frac{2\pi i H}{k} - \frac{2\pi}{z}\right).$$

Then F satisfies

$$F(x) = e^{\pi i s(h,k)} \left(\frac{z}{k}\right)^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(x'). \qquad \diamond$$

Proof

Using the functional equation of η this is a straight forward calculation. Set $\tau = (iz/k + h)/k \in \mathbb{H}$ and $\tau' = M\langle\tau\rangle$ with M as above. By the last equations we have

$$\begin{aligned}
F(e^{2\pi i\tau}) &= e^{\frac{\pi i\tau}{12}} \cdot \eta(\tau)^{-1} \\
&= e^{\frac{\pi i\tau}{12}} \cdot \epsilon(a,b,c,d) \cdot (-i(c\tau+d))^{\frac{1}{2}} \cdot \eta(\tau')^{-1} \\
&= F(e^{2\pi i\tau'}) \cdot e^{\frac{\pi i(\tau-\tau')}{12}} \cdot \epsilon(a,b,c,d) \cdot (-i(c\tau+d))^{\frac{1}{2}} \\
&= F(e^{2\pi i\tau'}) \cdot e^{\pi i s(-d,c)} \cdot \exp\left(\frac{\pi i}{12}\left(\tau - \tau' + \frac{a+d}{c}\right)\right) \cdot (-i(c\tau+d))^{\frac{1}{2}}.
\end{aligned}$$

We now choose $a = H$, $b = -(hH+1)/k$, $c = k$ and $d = -h$. Considering $\det(M) = H(-h) - (k(-(hH+1)/k)) = 1$ we find $M \in \mathrm{SL}_2(\mathbb{Z})$ as assumed. Then

$$\begin{aligned}
\tau' &= \left(H\frac{\frac{iz}{k}+h}{k} - \frac{hH+1}{k}\right) \cdot \left(k\frac{\frac{iz}{k}+h}{k} - h\right)^{-1} \\
&= \left(\frac{\frac{iHz}{k}-1}{k}\right)\left(\frac{iz}{k}\right)^{-1} \\
&= \frac{H}{k} + \frac{i}{z},
\end{aligned}$$

and we have

$$\begin{aligned}
(-i(c\tau+d))^{\frac{1}{2}} &= (-i(k\tau-h))^{\frac{1}{2}} \\
&= \left(-i\left(i\frac{z}{k}+h-h\right)\right)^{\frac{1}{2}} \\
&= \left(\frac{z}{k}\right)^{\frac{1}{2}} \qquad \text{and} \\
\tau - \tau' + \frac{a+d}{c} &= \frac{iz/k + h - ikz^{-1} - H + H - h}{k} \\
&= i\left(\frac{z}{k^2} - \frac{1}{z}\right).
\end{aligned}$$

Inserting this into the first equation we have

$$F(x) = F(e^{2\pi i\tau}) = F(e^{2\pi i\tau'}) \cdot e^{\pi i s(-d,c)} \cdot \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) \cdot \left(\frac{z}{k}\right)^{\frac{1}{2}},$$

which yields the claim. □

§3 The path of integration

Next we turn to the path of integration. Later we want to apply the biholomorphic map $z \mapsto e^{2\pi i z}$ from $S := \{z \in \mathbb{H} : \mathrm{Re}(z) \in [0,1)\}$ to the unit disc. Therefore we will consider a path of integration from i to $i+1$.

— *Farey fractions* —

We introduce some subsets of $[0,1]$.

(3.1) Definition
For $N \in \mathbb{N}$ we call the set

$$F_N := \left\{ \frac{k}{l} : k,\, l \in \underline{n},\, (k,l) = 1 \right\}$$

the set of Farey fractions of order N. Here we denote $\underline{n} = \{1, \dots n\}$. We call two fractions consecutive in F_N, iff they are consecutive in $F_N \subseteq [0,1]$ with respect to their magnitude. ◇

(3.2) Remarks

(i) Obviously $F_N \subseteq F_{N+k}$ for all $k \in \mathbb{N}$.

(ii) If $0 < a/b < c/d$ their mediant $(a+c)/(b+d)$ lies between them. This follows from $(a+c)/(b+d) - a/b = (bc-ad)/(b(b+d)) > 0$ and $c/d - (a+c)/(b+d) = (bc-ad)/(d(b+d)) > 0$. Pay attention not to confuse it with the median of a list of numbers. ◇

We want to investigate the set $F_{N+1} \setminus F_N$. Consecutive fractions turn out to be the key.

(3.3) Lemma
Given $0 \leq a/b < c/d \leq 1$ with $bc - ad = 1$. Then a/b and c/d are consecutive fractions in F_N if

$$\max(b,d) \leq N \leq b + d - 1.$$

◇

Proof
The condition $bc - ad = 1$ yields $(a,b) = (c,d) = 1$. For $\max(b,d) \leq N$ we see $b, d \in \underline{N}$ and thus $a/b,\, c/d \in F_N$ for those N.

Now suppose in addition $N \leq b + d - 1$ and that there is a fraction $h/k \in F_N$ such that $a/b < h/k < c/d$. It follows $bh - ak \geq 1$ and $ck - dh \geq 1$. We use $bc - ad = 1$ and find

$$b + d > N \geq k = k(bc - ad) = b(ck - dh) + d(bh - ak) \geq b + d.$$

This is a contradiction and hence a/b and c/d are consecutive in F_N. $\square$

We use the last equation to deduce a further proposition.

(3.4) Proposition
Given $0 \leq a/b < c/d \leq 1$ with $bc - ad = 1$. Set $h := a + b$ and $k := c + d$. Then the mediant h/k of a/b and c/d satisfies

$$bh - ak = 1, \qquad ck - dh = 1.$$

In particular $(h, k) = 1$. $\diamond$

Proof
Since $a/b < (a + c)/(b + d) < c/d$ we see $bh - ak, ck - dh \geq 1$. Then

$$k = b(ck - dh) + d(bh - ak) \geq b + d \tag{2}$$

as shown in the proof of (3.3) enforces $bh - ak = ck - dh = 1$ to obtain $k = b + d$. $\square$

Now we are able to prove the desired result. It is

(3.5) Theorem
We have $F_N \subseteq F_{N+1}$. Each fraction in $F_{N+1} \setminus F_N$ is the mediant of its neighbours which are consecutive in F_N. Moreover given consecutive fractions a/b and c/d in F_N we have $bc - ad = 1$. $\diamond$

Proof
We use induction on n, where the claim is clear for $\{0/1, 1/1\} = F_1 \subseteq F_2 = \{0/1, 1/2, 1/1\}$.

For the induction step we put together (3.3) and (3.4). Consider consecutive fractions a/b and c/d in F_N for a fixed N. For $N + 1 < b + d$ they will be consecutive in F_{N+1} by (3.3). Otherwise the mediant h/k is in between a/b and c/d by (3.4). But then again by (3.3) no further fraction can exist in F_{N+1} between a/b and h/k or h/k and c/d since $N + 1 = k = b + d < 2b + d = b + k$ and $N + 1 = k = b + d < b + 2d = k + d$ respectively. Moreover by (3.4) the fraction h/k satisfies $bh - ak = 1$ and $ck - dh = 1$ and this proofs the last claim. $\square$

— *Ford circles* —

To construct our path of integration we will use sections of certain circles, the Ford circles. This is the reason for which we now study them.

(3.6) Definition
Given a fraction h/k with $(h,k)=1$. The Ford circle defined by this fraction is the circle in the complex plane with radius $1/(2k^2)$ and centre at the point $h/k+i/(2k^2)$. It is denoted by $C(h,k)$. $\diamond$

(3.7) Remark
The Ford circle $C(h,k)$ is exactly the circle of radius $1/(2k^2)$ in the closed upper half plan, which has the real line as tangent and touches it in h/k. $\diamond$

(3.8) Proposition
Two Ford circles $C(a,b)$ and $C(c,d)$ are either tangent to each other or they do not intersect. They are tangent iff $bc-ad=\pm 1$. Thus exactly those Ford circles are tangent, which correspond to fractions which are consecutive in F_N for one $N\in\mathbb{N}$.$\diamond$

Proof
The square of the distance of the centre points is

$$D^2=\left(\frac{a}{b}-\frac{c}{d}\right)^2+\left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2.$$

The square of the sum of their radii r and R is

$$(r+R)^2=\left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2.$$

Hence the difference yields

$$\begin{aligned}D^2-(r+R)^2&=\left(\frac{ad-bc}{bd}\right)^2+\left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2-\left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2\\&=\frac{(ad-bc)^2}{b^2d^2}-\frac{4}{4b^2d^2}\\&=\frac{(ad-bc)^2-1}{b^2d^2}\geq 0.\end{aligned}$$

The last inequality holds since $ad-bc\neq 0$ is integral. We now see that the circles never intersect and they are tangent if, and only if, $(ab-bc)^2=1$. $\square$

Our next question is, where these points of tangency are located. The answer is

(3.9) Lemma

Let $a/b < c/d$ be two consecutive Farey fractions. The point of contact of $C(a,b)$ with $C(c,d)$ is the point

$$s = \frac{c}{d} - \frac{b}{d(b^2+d^2)} + \frac{i}{b^2+d^2}.$$

Moreover, the point of contact s is on the semicircle in the closed upper half plane whose diameter is the interval $[a/b, c/d]$. ◇

Proof

We use the theorem on intersecting lines, Thales' theorem and the altitude theorem, already proved in school. The doubtful reader may be referred to [Krieg, Ebene Geometrie, 2007] for proofs of these theorems in analytic coordinates.

We define x and y in terms of

$$s = \left(\frac{c}{d} - x\right) + i\left(\frac{1}{2d^2} - y\right).$$

By the theorem on intersecting lines we get

$$\begin{aligned} \frac{x}{\frac{c}{d}-\frac{a}{b}} &\overset{\text{intersecting lines}}{=} \frac{\frac{1}{2d^2}}{\frac{1}{2b^2}+\frac{1}{2d^2}} = \frac{b^2}{b^2+d^2} \quad \text{and} \\ \frac{y}{\frac{1}{2d^2}} &\overset{\text{intersecting lines}}{=} \frac{\frac{1}{2d^2}-\frac{1}{2b^2}}{\frac{1}{2d^2}+\frac{1}{2b^2}} = \frac{b^2-d^2}{b^2+d^2}. \end{aligned}$$

Solving for x and y we have

$$\begin{aligned} x &= (bc-ad)\frac{b}{d(b^2+d^2)} = \frac{b}{d(b^2+d^2)} \quad \text{and} \\ y &= \frac{1}{2d^2}\frac{b^2-d^2}{b^2+d^2} \end{aligned}$$

and finally inserting this we establish

$$\begin{aligned} s &= \left(\frac{c}{d} - \frac{b}{d(b^2+d^2)}\right) + i\left(\frac{1}{2d^2}\left(1 - \frac{b^2-d^2}{b^2+d^2}\right)\right) \\ &= \left(\frac{c}{d} - \frac{b}{d(b^2+d^2)}\right) + i\left(\frac{1}{2d^2}\frac{b^2+d^2-b^2+d^2}{b^2+d^2}\right) \\ &= \left(\frac{c}{d} - \frac{b}{d(b^2+d^2)}\right) + i\left(\frac{1}{b^2+d^2}\right). \end{aligned}$$

By Thales' theorem, it now suffices to prove that the triangle $\Delta(a/b, c/d, s)$ is right-angled. To proof this we establish the equation given by Euklid's altitude theorem.

$$\left(\mathrm{Re}(s) - \frac{a}{b}\right)\left(\frac{c}{d} - \mathrm{Re}(s)\right) \overset{!}{=} \mathrm{Im}(s)^2.$$

This can be verified directly.

$$\begin{aligned}\left(\mathrm{Re}(s) - \frac{a}{b}\right)\left(\frac{c}{d} - \mathrm{Re}(s)\right) &= \left(\frac{c}{d} - \frac{b}{d(b^2+d^2)} - \frac{a}{b}\right)\left(\frac{c}{d} - \frac{c}{d} + \frac{b}{d(b^2+d^2)}\right) \\ &= \left(\frac{\overbrace{bc-ad}^{=1}}{bd} - \frac{b}{d(b^2+d^2)}\right)\frac{b}{d(b^2+d^2)} \\ &= \frac{1}{d}\frac{b}{d(b^2+d^2)}\frac{b^2+d^2-b^2}{b(b^2+d^2)} = \frac{1}{(b^2+d^2)^2} = \mathrm{Im}(s)^2. \qquad \square\end{aligned}$$

(3.10) Definition
Fix $N \in \mathbb{N}$ and let $a/b < h/k$ be consecutive fractions in F_N where a/b uniquely determined by h/k. Then $s_l(h,k)$ is the point of tangency of $C(a,b)$ with $C(h,k)$, the lower point of contact with respect to h/k. Given consecutive fractions $h/k < c/d$ we define $s_h(h,k)$ to be the point of tangency of $C(h,k)$ with $C(c,d)$, the higher point of contact with respect to h/k. Moreover, if $h/k = 0/1$ we define $s_l(h,k) := i$ and if $h/k = 1/1$ we define $s_h(h,k) = 1+i$. If the referred circle is obvious we write $s_l = s_l(h,k)$ and $s_h = s_h(h,k)$ for better readability. ◇

— *A transformation of Ford circles* —

Later we wish to integrate over Ford circles and their sections. Thus, now, we need to investigate the effect of some transformations.

(3.11) Lemma
The transformation

$$t : z \mapsto -ik^2\left(z - \frac{h}{k}\right)$$

maps the Ford circle $C(h,k)$ onto a circle K of radius $1/2$ with centre $z_0 = 1/2$. For consecutive fractions $a/b < h/k$ and $h/k < c/d$ respectively in F_N the points of contact of $C(a,b)$ with $C(h,k)$ and of $C(h,k)$ with $C(c,d)$ respectively are mapped to

$$t(s_l) = \frac{k^2}{k^2+b^2} + i\frac{kb}{k^2+b^2} \quad \text{and}$$
$$t(s_h) = \frac{k^2}{k^2+d^2} - i\frac{kd}{k^2+d^2}.$$

Regarding $C(0,1)$ and $C(1,1)$ we have

$$t(s_l(0,1)) = t(s_h(1,1)) = 1.$$

Moreover the upper arc joining s_l and s_h maps onto the arc of K which does not touch the imaginary axis. ◇

Proof

The translation tra : $z \mapsto z - h/k$ maps $C(h,k)$ onto the circle with centre $i/(2k^2)$. Then ro : $z \mapsto -ik^2 z$ rotates the centre to the real axis and changes the radius to $1/2$. Given $a/b < h/k < c/d$, we verify the equations by (3.9).

$$\begin{aligned}
\text{ro}\circ\text{tra}(s_l) &= \text{ro}\circ\text{tra}\left(\frac{k}{h} - \frac{b}{k(b^2+k^2)} + \frac{i}{b^2+k^2}\right)\\
&= -ik^2\left(\left(\frac{k}{h} - \frac{b}{k(b^2+k^2)} + \frac{i}{b^2+k^2}\right) - \frac{h}{k}\right)\\
&= -i\frac{bk}{b^2+k^2} + \frac{k^2}{b^2+k^2}
\end{aligned}$$

and analogously

$$\begin{aligned}
\text{ro}\circ\text{tra}(s_h) &= \text{ro}\circ\text{tra}\left(\frac{c}{d} - \frac{k}{d(k^2+d^2)} + \frac{i}{k^2+d^2}\right)\\
&= -ik^2\left(\left(\frac{c}{d} - \frac{k}{d(k^2+d^2)} + \frac{i}{k^2+d^2}\right) - \frac{h}{k}\right)\\
&= -ik^2\frac{ck(k^2+d^2) - k^2 - hd(k^2+d^2)}{kd(k^2+d^2)} + \frac{k^2}{k^2+d^2}\\
&= -ik^2\frac{\overbrace{(ck-hd)}^{=1}(k^2+d^2) - k^2}{kd(k^2+d^2)} + \frac{k^2}{k^2+d^2}\\
&= -i\frac{k^2d^2}{kd(k^2+d^2)} + \frac{k^2}{k^2+d^2}\\
&= \frac{k^2}{k^2+d^2} - i\frac{kd}{k^2+d^2}.
\end{aligned}$$

Two further calculations yield

$$\begin{aligned}\mathrm{ro}\circ\mathrm{tra}(s_l(0,1)) &= \mathrm{ro}\circ\mathrm{tra}(i) = -i\left(i-\frac{0}{1}\right) = 1,\\ \mathrm{ro}\circ\mathrm{tra}(s_h(1,1)) &= \mathrm{ro}\circ\mathrm{tra}(i+1) = -i\left(i+1-\frac{1}{1}\right) = 1\end{aligned}$$

Finally the last assertion follows from $t(h/k) = 0$. □

We will need some estimates, too. They treat s_l and s_h of the last theorem.

(3.12) Lemma
Let s_l, s_h and the transformation t as in (3.11). Suppose $a/b < h/k$ and $h/k < c/d$ respectively are consecutive in F_N. Then we have

$$|t(s_l)| = \frac{k}{\sqrt{k^2+b^2}}, \qquad |t(s_h)| = \frac{k}{\sqrt{k^2+d^2}}.$$

Furthermore, if z is on the chord joining $t(s_l(h,k))$ and $t(s_h(h,k))$ for consecutive fractions $a/b < h/k < c/d$ or the chord joining $t(s_h(0,1))$ and $t(s_l(1,1))$, we have

$$|z| < \frac{\sqrt{2}k}{N}.$$

The length of this chord does not exceed $2\sqrt{2}k/N$. ◇

Proof
Using the assertions of (3.11), given $a/b < h/k$ and $h/k < c/d$ respectively, we have

$$\begin{aligned}|t(s_l)|^2 &= \frac{k^4+k^2b^2}{(k^2+b^2)^2} = \frac{k^2}{k^2+b^2} \quad \text{and}\\ |t(s_h)|^2 &= \frac{k^4+k^2d^2}{(k^2+d^2)^2} = \frac{k^2}{k^2+d^2}.\end{aligned}$$

For the second claim, assume z is on the chord joining two points p and q. Then $|z| \leq \max(|p|,|q|)$ and hence it suffices to proof the claim for $z = t(s_l(h,k))$ and $z = t(s_h(h,k))$ with $a/b < h/k < c/d$ consecutively in F_N and for $z = t(s_h(0,1))$ and $z = t(s_l(1,1))$ respectively. This follows from the equations, we already deduced, and the following estimates.

We utilize the inequality relating the arithmetic mean to the root mean square and exploiting the fact that the fractions are consecutive in F_N and thus by (3.3) the inequalities $k+d \geq N+1$ and $k+b \geq N+1$ respectively hold. We have

$$\sqrt{k^2+b^2} \geq \frac{k+b}{\sqrt{2}} \geq \frac{N+1}{\sqrt{2}} > \frac{N}{\sqrt{2}} \quad \text{and similarly}$$
$$\sqrt{k^2+d^2} \geq \frac{k+d}{\sqrt{2}} \geq \frac{N+1}{\sqrt{2}} > \frac{N}{\sqrt{2}}.$$

Then the claim follows by the first equations. Finally the chord's length is less than $|t(s_l)| + |t(s_h)| = 2\sqrt{2}k/N$. □

— The path of integration —

We are now ready to introduce the path of integration.

(3.13) Definition
Fix $N \in \mathbb{N}$ and let $U_N(C(h,k))$ be the path starting in s_l and ending in s_h on the upper arc of $C(h,k)$, with s_l and s_h as in (3.10). We set the N-th path of integration to be

$$I_N := \bigoplus_{\frac{h}{k} \in F_N} U_N(C(h,k)),$$

where the direct sum is taken in order of the magnitude of h/k. ◇

(3.14) Remark
By (3.9) the path of integration is connected and it is disjoint to the real axis. ◇

To apply the Cauchy's residue theorem we will transform the punctured unit disc to the strip $S = \{z \in \mathbb{H} : \mathrm{Re}(z) \in [0,1)\}$. We will proof

(3.15) Lemma
The chord between i and $i+1$ is homotopic to I_N. ◇

Proof
We use induction on N. For $N = 1$ we have

$$I_1 = (t \in [0,1] \mapsto \frac{t}{2} + \frac{i}{2}(1 + \sqrt{1-t^2}))$$
$$\oplus (t \in [0,1] \mapsto \frac{1+t}{2} + \frac{i}{2}(1 + \sqrt{1-(1-t)^2})).$$

Thus the convex combination

$$H : [0,1] \times [0,1], (\lambda, t) \mapsto \lambda + (1-\lambda) I_1(t)$$

is a homotopy of I_1 and the chord between i and $i+1$.

Next assume I_N to be homotopic to the cord between i and $1+i$ for a fixed N. Then it suffices to prove I_{N+1} to be homotopoic to I_N. To do this we investigate the difference. By (3.5) we see that only finitely many curves are inserted. Namely suppose a fraction $h/k \in F_{N+1}$ was inserted between $a/b, c/d \in F_N$. Then we can write

$$\begin{aligned} &U_{N+1}(C(a,b)) \oplus U_{N+1}(C(h,k)) \oplus U_{N+1}(C(c,d)) \\ = &R_l \oplus U_N(C(a,b)) \oplus M_l \oplus U_{N+1}(C(h,k)) \oplus M_r \oplus U_N(C(c,d)) \oplus R_r, \end{aligned}$$

where R_l and R_r either vanish or are rests occurring from further insertions in F_{N+1} left respectively right to a/b respectively c/d. Moreover M_l is the arc between the point of contact of $C(a,b)$ with $C(c,d)$ and the point of contact of $C(a,b)$ and $C(h,k)$. The M_r is defined analogously. Thus $M_l \oplus U_{N+1}(C(h,k)) \oplus M_r$ is homotopic to the point curve and applying this construction to every inserted fraction we obtain the desired homotopy. □

§4 Rademacher's series for $p(n)$

(4.1) Theorem
If $n \in \mathbb{N}$ the partition function $p(n)$ is represented by the convergent series

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{N} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) + \mathcal{O}(N^{-\frac{1}{2}}), \qquad \text{where}$$

$$A_k(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k}$$

and $s(h,k)$ is Dedekind's sum. ◇

Proof
We have already seen

$$p(n) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta^{n+1}} d\zeta,$$

where C is any contour homotopic in the punctured unit disc to the positively orientated circle of radius $e^{-2\pi}$.

By applying the biholomorphic map $z \mapsto e^{2\pi i z}$ we transform $S = \{z \in \mathbb{H} : \mathrm{Re}(z) \in [0,1)\}$ to the punctured unit disc and the cord between i and $1+i$ to the circle of radius $e^{-2\pi}$ with positive orientation . Then by (3.15) every I_N is an appropriate path of integration. Summing this up we have

$$\begin{aligned}
p(n) &= \int_{I_N} F(e^{2\pi i z})(e^{2\pi i z})^{-(n+1)} e^{2\pi i z} dz \\
&= \int_{I_N} F(e^{2\pi i z})(e^{-2\pi i n z}) dz.
\end{aligned}$$

We first split the path of integration according to the definition of I_N.

$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \le h \le k \\ (h,k)=1}} \int_{U_N(C(h,k))} F(e^{2\pi i z})(e^{-2\pi i n z}) dz.$$

We will calculate each integral separately and introduce error terms. Therefor for fixed h, k like in the sum we apply the biholomorphic map $t^{-1} : z \mapsto h/k + iz/k^2$ with $t : z \mapsto -ik^2(z - h/k)$ as in (3.11). We see by this lemma that we have to calculate

$$\begin{aligned}
&\int_{U_N(C(h,k))} F(e^{2\pi i z})(e^{-2\pi i n z}) dz \\
&= \int_{t(U_N(C(h,k)))} F\left(\exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right)\right) e^{-2\pi i n h/k} e^{2\pi n z/k^2} \frac{i}{k^2} dz \\
&= i k^{-2} e^{-2\pi i n h/k} \int_{t(U_N(C(h,k)))} e^{2\pi n z/k^2} F\left(\exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right)\right) dz.
\end{aligned}$$

We are now ready to apply the functional equation (2.1).

$$\begin{aligned}
&F\left(\exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right)\right) \\
&= e^{\pi i s(h,k)} k^{-\frac{1}{2}} \Psi_k(z) F\left(\exp\left(\frac{2\pi i H}{k} - \frac{2\pi}{z}\right)\right), \quad \text{where} \\
&\Psi_k(z) = z^{\frac{1}{2}} \exp\left(\frac{\pi}{12 z} - \frac{\pi z}{12 k^2}\right) \quad \text{and} \quad hH \equiv -1 \pmod{k}.
\end{aligned}$$

We define

$$I_1(h,k) := \int\limits_{t(U_N(C(h,k)))} \Psi_k(z) e^{2\pi nz/k^2} dz \quad \text{for } k \geq 2,$$

$$I_2(h,k) := \int\limits_{t(U_N(C(h,k)))} \Psi_k(z) e^{2\pi nz/k^2} \left(F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z} \right) \right) - 1 \right) dz \quad \text{for } k \geq 2.$$

Now observe that for $k = 1$ we may chose $H = 0$ and then the integrand is independent of h. Furthermore by (3.11) we see that $t(U_N(C(1,1))) \oplus t(U_N(C(0,1)))$ is the arc from $s_l(1,1)$ to $s_h(0,1)$ not touching the imaginary axis. Thus for ease of notation we may define the special case $k = 1$, too.

$$I_1(0,1) := \int\limits_{t(U_N(C(1,1)))\oplus t(U_N(C(0,1)))} \Psi_k(z) e^{2\pi nz/k^2}\Big|_{k=1} dz \quad \text{and}$$

$$I_2(0,1) := \int\limits_{t(U_N(C(1,1)))\oplus t(U_N(C(0,1)))} \Psi_k(z) e^{2\pi nz/k^2}$$

$$\cdot \left(F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z} \right) \right) - 1 \right)\Big|_{k=1,H=0} dz.$$

If we consider that for $k = 1$ the additional terms are independent of h, too, we may combine the cases $h = 0$ and $h = 1$ in one expression involving $I(0,1)$.

$$\begin{aligned}
p(n) &= \sum_{k=1}^{N} \sum_{\substack{0\leq h\leq k\\(h,k)=1}} \int\limits_{U_N(C(h,k))} F(e^{2\pi iz})(e^{-2\pi inz}) dz \\
&= \sum_{k=1}^{N} \sum_{\substack{0\leq h\leq k\\(h,k)=1}} ik^{-2} e^{-2\pi inh/k} \int\limits_{t(U_N(C(h,k)))} e^{2\pi nz/k^2} F\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2} \right) \right) dz \\
&= \sum_{k=1}^{N} \sum_{\substack{0\leq h\leq k\\(h,k)=1}} ik^{-2} e^{-2\pi inh/k} \\
&\qquad \int\limits_{t(U_N(C(h,k)))} e^{2\pi nz/k^2} e^{\pi is(h,k)} k^{-\frac{1}{2}} \Psi_k(z) F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z} \right) \right) dz \\
&= \sum_{k=1}^{N} \sum_{\substack{0\leq h<k\\(h,k)=1}} ik^{-\frac{5}{2}} e^{\pi is(h,k)} e^{-2\pi inh/k} (I_1(h,k) + I_2(h,k)).
\end{aligned}$$

Note that we have to write $0 \leq h < k$ to exclude the case $k = 1 = h$.

We first deal with $I_2(h,k)$ and therefor temporarily fix N. Since the integrand is holomorpic on the right half plane, we may change the path of integration $t(U_N(C(h,k)))$ and $t(U_N(C(1,1))) \oplus t(U_N(C(0,1)))$ respectively to the chord between $s_l(h,k)$ and $s_h(h,k)$ and the chord joining $t(s_h(1,1)$ and $t(s_l(0,1))$ respectively as in (3.10). Its length does not exceed $2\sqrt{2}k/N$ as we proofed in (3.12). On the chord we have $|z| < \sqrt{2}k/N$. Furthermore, note that $\mathrm{Re}(1/z) \geq 1$ for any z on the chord, since this chord is contained in the circle with radius $1/2$ and centre $1/2$.

We estimate the integrand on the chord.

$$\begin{aligned}
&\left|\Psi_k(z)e^{2\pi nz/k^2}\left(F\left(\exp\left(\frac{2\pi iH}{k}-\frac{2\pi}{z}\right)\right)-1\right)\right| \\
&= |z|^{\frac{1}{2}}\exp\left(\frac{\pi}{12}\mathrm{Re}\left(\frac{1}{z}\right)-\frac{\pi}{12k^2}\mathrm{Re}(z)\right)\cdot e^{2\pi n\mathrm{Re}(z)/k^2}\left|\sum_{m=1}^{\infty}p(m)e^{2\pi iHm/k-2\pi m/z}\right| \\
&\leq |z|^{\frac{1}{2}}\exp\left(\frac{\pi}{12}\mathrm{Re}\left(\frac{1}{z}\right)\right)e^{2\pi n/k^2}\sum_{m=1}^{\infty}p(m)e^{-2\pi m\mathrm{Re}(1/z)} \\
&\leq |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(m)e^{-2\pi(m-1/24)\mathrm{Re}(1/z)} \\
&\leq |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(m)e^{-2\pi(m-1/24)} \\
&= |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(m)e^{-2\pi(24m-1)/24} \\
&< |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(24m-1)e^{-2\pi(24m-1)/24} \\
&= |z|^{\frac{1}{2}}e^{2\pi n}\sum_{m=1}^{\infty}p(24m-1)y^{(24m-1)} \\
&= c|z|^{\frac{1}{2}},
\end{aligned}$$

where $y = e^{-2\pi/24}$ and c is a constant, which is independent of k and which is lower than infinity since the partition function is a dominating series and $|y| < 1$.

Since z is on the chord we have $|z| < \sqrt{2}k/N$. Hence the integrand is bounded by

$2^{1/4}c(k/N)^{1/2}$. Since the chord's length is less than $2\sqrt{2}k/N$, we find

$$|I_2(h,k)| < 2\sqrt{2}\frac{k}{N}c|z|^{\frac{1}{2}} < 2\sqrt{2}\frac{k}{N}c\left(\sqrt{2}\frac{k}{N}\right)^{\frac{1}{2}} < Ck^{\frac{3}{2}}N^{-\frac{3}{2}}$$

for an appropriate constant C.

Inserting this into the inner sum we have

$$\begin{aligned}\left|\sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-\frac{5}{2}}e^{\pi is(h,k)}e^{-2\pi inh/k}I_2(h,k)\right| &< \sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} Ck^{-1}N^{-\frac{3}{2}}\\ &\le CN^{-\frac{3}{2}}\sum_{k=1}^{N}1 = CN^{-\frac{1}{2}},\end{aligned}$$

and thus

$$p(n) = \sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-\frac{5}{2}}e^{\pi is(h,k)}e^{-2\pi inh/k}I_1(h,k) + \mathcal{O}(N^{-\frac{1}{2}}).$$

Our next step will be to simplify the integral $I_1(h,k)$. In (3.11) we have already seen that s_l is mapped to the upper half plane by the transformation t and thus the path of integration has negative orientation. We want to complete it to the full circle K_- of radius $1/2$ and centre point $1/2$ with negative orientation. To estimate the difference we see that $|t(s_l)|, |t(s_h)| < \sqrt{2}k/N$ if $k \ge 2$ and that $|t(s_h(0,1))|, |t(s_l(1,1))| < \sqrt{2}/N$ by lemma (3.12). For any point z on $K_- \setminus \{0\}$ we have $\mathrm{Re}(1/z) = 1$ and $0 < \mathrm{Re}(z) \le 1$ and thus the integrand has absolute value

$$\begin{aligned}|\Psi_k(z)e^{2\pi nz/k^2}| &= \underbrace{e^{2\pi n\mathrm{Re}(z)/k^2}}_{\le e^{2\pi n}}\underbrace{|z|^{\frac{1}{2}}}_{\le 2^{1/4}k^{1/2}N^{-1/2}}\underbrace{\exp\left(\frac{\pi}{12}\mathrm{Re}\left(\frac{1}{z}\right)-\frac{\pi}{12k^2}\mathrm{Re}(z)\right)}_{\le e^{\pi/12}}\\ &\le e^{2\pi n}e^{\pi/12}2^{1/4}k^{1/2}N^{-1/2}.\end{aligned}$$

Thus the integrand is bounded on the circle almost everywhere. We estimating the length of the negatively orientated arc from $t(s_l)$ to $t(s_h)$ by $\pi(|t(s_l) - t(s_h)|) \le \pi(|t(s_l)| + |t(s_h)|) < \pi 2\sqrt{2}k/N$ for $k \ge 2$ and by

$\pi(|t(s_h(0,1))| + |t(s_l(1,1))|) < \pi 2\sqrt{2}/N$ for $k = 1$. Then we get

$$\begin{aligned} I_1(h,k) &= \int\limits_{t(U_N(C(h,k)))} \Psi_k(z) e^{2\pi n z/k^2} dz \quad \text{for } k \geq 2, \\ &= \int\limits_{K_-} \Psi_k(z) e^{2\pi n z/k^2} dz - \int\limits_{K_- \ominus t(U_N(C(h,k)))} \Psi_k(z) e^{2\pi n z/k^2} dz \\ &= \int\limits_{K_-} \Psi_k(z) e^{2\pi n z/k^2} dz + \mathcal{O}\left(\pi 2\sqrt{2}k/N e^{2\pi n} e^{\pi/12} 2^{1/4} k^{1/2} N^{-1/2}\right) \\ &= \int\limits_{K_-} \Psi_k(z) e^{2\pi n z/k^2} dz + \mathcal{O}(k^{\frac{3}{2}} N^{-\frac{3}{2}}). \end{aligned}$$

Using exactly the same estimates as for $I_2(h,k)$, we deduce

$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} i k^{-\frac{5}{2}} e^{\pi i s(h,k)} e^{-2\pi i n h/k} \int\limits_{K_-} \Psi_k(z) e^{2\pi n z/k^2} dz + \mathcal{O}(N^{-\frac{1}{2}}).$$

Setting

$$A_k(n) = \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k}$$

and regarding the integral's independence of h we have

$$p(n) = i \sum_{k=1}^{N} A_k(n) k^{-\frac{5}{2}} \int\limits_{K_-} \Psi_k(z) e^{2\pi n z/k^2} dz + \mathcal{O}(N^{-\frac{1}{2}}).$$

Now it suffices to proof the last two lemmas

(4.2) Lemma

We have

$$\begin{aligned} &\int\limits_{\pi/12 - i\infty}^{\pi/12 + i\infty} z^{-\frac{5}{2}} \exp\left(z + \frac{\pi^2}{z 6k^2}\left(n - \frac{1}{24}\right)\right) dz \\ &\quad = i 2^{\frac{5}{2}} 3^{\frac{3}{2}} \pi^{-\frac{5}{2}} k^3 d(n,k) \quad \text{,where} \\ &d(n,k) := \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}}\right). \end{aligned}$$

◇

Proof

We will use real analysis to do this and cite Watson p.181. For any $c > 0$ and $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 0$ we have

$$I_\nu(\tau) = \frac{(\frac{1}{2}\tau)^\nu}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} t^{-\nu-1} e^{t+\tau^2/(4t)} dt,$$

where I_ν is a Bessel function. Another representation is know if $\nu = 3/2$ and $c = \pi/12$. Then we have

$$I_{\frac{3}{2}}(\tau) = \sqrt{\frac{2\tau}{\pi}} \frac{d}{d\tau} \left(\frac{\sinh \tau}{\tau} \right).$$

We set

$$\tau(n) = \frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}.$$

The chain rule yields

$$\frac{d}{dn}\left(\frac{\sinh \tau(n)}{\tau(n)}\right) = \frac{d}{d\tau}\left(\frac{\sinh \tau}{\tau}\right)\bigg|_{\tau=\tau(n)} \cdot \frac{d\tau(n)}{dn}.$$

Combining this we have

$$\begin{aligned} d(n,k) := \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) &= \sqrt{\frac{2}{3}}\frac{\pi}{k} \cdot \frac{d}{d\tau}\left(\frac{\sinh \tau}{\tau}\right)\bigg|_{\tau=\tau(n)} \\ &\quad \cdot \sqrt{\frac{2}{3}}\frac{\pi}{k}\frac{1}{2}\left(n-\frac{1}{24}\right)^{-\frac{1}{2}} \\ &= \frac{\pi^2}{3k^2}\frac{d}{d\tau}\left(\frac{\sinh \tau}{\tau}\right)\bigg|_{\tau=\tau(n)} \cdot \left(n-\frac{1}{24}\right)^{-\frac{1}{2}}, \end{aligned}$$

and furthermore

$$\begin{aligned} I_{\frac{3}{2}}(\tau(n)) = \sqrt{\frac{2\tau(n)}{\pi}}\frac{d}{d\tau}\left(\frac{\sinh \tau}{\tau}\right)\bigg|_{\tau=\tau(n)} &= \sqrt{\frac{2\tau(n)}{\pi}}\frac{3k^2}{\pi^2}\left(n-\frac{1}{24}\right)^{\frac{1}{2}} d(n,k) \\ &= \frac{3k^2}{\pi^2}\sqrt{\frac{2\sqrt{2}}{\sqrt{3}k}\left(n-\frac{1}{24}\right)^{\frac{1}{2}}}\left(n-\frac{1}{24}\right)^{\frac{1}{2}} d(n,k) \\ &= 6^{\frac{3}{4}}\pi^{-2}k^{\frac{3}{2}}\left(n-\frac{1}{24}\right)^{\frac{3}{4}} d(n,k). \end{aligned}$$

The final conclusion is

$$\begin{aligned}
&\int\limits_{\pi/12-i\infty}^{\pi/12+i\infty} z^{-\frac{5}{2}} \exp\left(z+\frac{\pi^2}{z6k^2}\left(n-\frac{1}{24}\right)\right) dz \\
&= I_{\frac{3}{2}}(\tau(n))\frac{2\pi i}{(\frac{1}{2}\tau(n))^{\frac{3}{2}}} \\
&= 6^{\frac{3}{4}}\pi^{-2}k^{\frac{3}{2}}\left(n-\frac{1}{24}\right)^{\frac{3}{4}} d(n,k)\frac{2\pi i}{(\frac{1}{2})^{\frac{3}{2}}}\left(\frac{\pi}{k}\right)^{-\frac{3}{2}}\left(\frac{2}{3}\left(n-\frac{1}{24}\right)\right)^{-\frac{3}{4}} \\
&= i2^{\frac{5}{2}}3^{\frac{3}{2}}\pi^{-\frac{5}{2}}k^3 d(n,k).
\end{aligned}$$

□

(4.3) Lemma
With all notations as in (4.1) and (4.2) we have

$$ik^{-\frac{5}{2}}\int\limits_{K_-}\Psi_k(z)e^{2\pi nz/k^2}dz = \frac{\sqrt{k}}{\sqrt{2}\pi}d(n,k).$$

◇

Proof
We start considering

$$\begin{aligned}
ik^{-\frac{5}{2}}\int\limits_{K_-}\Psi_k(z)e^{2\pi nz/k^2}dz &= ik^{-\frac{5}{2}}\int\limits_{K_-} z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)e^{2\pi nz/k^2}dz \\
&= ik^{-\frac{5}{2}}\int\limits_{K_-} z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}+\frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right)\right)dz.
\end{aligned}$$

The transformation of the punctured plane $z \mapsto 1/z$ is a diffeomorphism mapping the line $\{z \in \mathbb{C} : \Re(z) = 1\}$ onto $K_- \setminus \{0\}$ and by applying it we have :

$$\begin{aligned}
ik^{-\frac{5}{2}}\int\limits_{K_-}\Psi_k(z)e^{2\pi nz/k^2}dz &= ik^{-\frac{5}{2}}\int\limits_{1-i\infty}^{1+i\infty} z^{-\frac{1}{2}}\exp\left(\frac{\pi z}{12}+\frac{2\pi}{zk^2}\left(n-\frac{1}{24}\right)\right)(-1)z^{-2}dz \\
&= -ik^{-\frac{5}{2}}\int\limits_{1-i\infty}^{1+i\infty} z^{-\frac{5}{2}}\exp\left(\frac{\pi z}{12}+\frac{2\pi}{zk^2}\left(n-\frac{1}{24}\right)\right)dz.
\end{aligned}$$

Another transformation $z \mapsto 12z/\pi$ and lemma (4.2) yields

$$
\begin{aligned}
& ik^{-\frac{5}{2}} \int\limits_{K_-} \Psi_k(z) e^{2\pi n z/k^2} dz \\
&= -ik^{-\frac{5}{2}} \left(\frac{\pi}{12}\right)^{\frac{5}{2}} \int\limits_{\pi/12 - i\infty}^{\pi/12 + i\infty} z^{-\frac{5}{2}} \exp\left(z + \frac{\pi^2}{z 6 k^2}\left(n - \frac{1}{24}\right)\right) \frac{12}{\pi} dz \\
&= -ik^{-\frac{5}{2}} \left(\frac{\pi}{12}\right)^{\frac{3}{2}} 2^{\frac{5}{2}} 3^{\frac{3}{2}} \pi^{-\frac{5}{2}} d(n,k) i k^3 \\
&= \frac{\sqrt{k}}{\sqrt{2}\pi} d(n,k).
\end{aligned}
$$

□