Calculus and Linear Algebra for Biomedical Engineering

Week 1: Complex Numbers, Trigonometric Functions

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Motivation

Recall from last week:

- Extensions of number domains (e.g., from $\mathbb{N}$ to $\mathbb{Z}$, from $\mathbb{Z}$ to $\mathbb{Q}$, from $\mathbb{Q}$ to $\mathbb{R}$) are motivated partly by the desire to extend operations (e.g., subtraction, division, powers).
- A negative real number does not have a square root in $\mathbb{R}$.

Definition. The set $\mathbb{C}$ of complex numbers is defined as

$$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\} \quad .$$

where (x, y) denotes an ordered pair of real numbers.
Note: Two ordered pairs (a, b) and (x, y) are equal if and only if $a = x$ and $b = y$. In particular, $(a, b) = (b, a)$ only if $a = b$.

Operations on complex numbers

Definition. Let $(a, b), (x, y) \in \mathbb{C}$.

- The **sum** resp. **difference** is defined as

$$(a, b) + (x, y) = (a + x, b + y) \quad , \quad (a, b) - (x, y) = (a - x, b - y) \quad . \tag{1}$$

- The **product** is defined as

$$(a, b) \cdot (x, y) = (ax - by, ay + bx) \quad . \tag{2}$$

Interpretation of complex numbers

Pairs are useful for the rigourous definition of complex numbers. For carrying out computations with complex numbers, other notations are preferred.

- We identify $(x, 0) \in \mathbb{C}$ with $x \in \mathbb{R}$.
  Note that now, $x + y$ could mean the usual sum of real numbers, or the result of the addition $(x, 0) + (y, 0)$ in $\mathbb{C}$. However, the latter is $(x + y, 0)$, which we identify with $x + y$.
- We define the imaginary unit as $i := (0, 1) \in \mathbb{C}$. Note that $i^2 = -1$.
- We can now write arbitrary complex numbers as

$$z = (x, y) = (x, 0) + (0, y) = x + (0, 1) \cdot y = x + iy \quad .$$

- In the new notation, sum and product become

$$(a+ib)+(x+iy) = (a+x)+i(b+y) \;, \; (a+ib)\cdot(x+iy) = (ax-by)+i(ay+bx) \;. \tag{3}$$

Axioms for sums and products

Theorem. All axioms regarding sums (differences) and products in $\mathbb{R}$ carry over to $\mathbb{C}$. In particular, the following properties can be verified directly.

- Addition is commutative and associative.
- Multiplication is commutative and associative.
- The distributive law relating addition and multiplication holds.
- Using the identifications from above, specifically $1 = (1, 0)$ and $0 = (0, 0)$, we find for arbitrary $z \in \mathbb{C}$

$$z = z \cdot 1 = z + 0 \; , \; z \cdot 0 = 0 \quad .$$

Complex conjugate, Real and Imaginary part, Modulus

Definition. For $z = x + iy \in \mathbb{C}$, we introduce the following notions:

$$\mathrm{Re}(z) = x \quad , \quad \text{the real part of } z \tag{4}$$

$$\mathrm{Im}(z) = y \quad , \quad \text{the imaginary part of } z \tag{5}$$

$$\overline{z} = x - iy \quad , \quad \text{the complex conjugate of } z \tag{6}$$

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2} \quad , \quad \text{the modulus or length of } z \tag{7}$$

Note that $x^2 + y^2 \geq 0$, hence $|z|$ is well-defined and positive. $|z|$ is also called absolute value of z.

Useful formulas:

$$z = \mathrm{Re}(z) + i\mathrm{Im}(z) \quad , \quad \mathrm{Re}(z) = \frac{z + \overline{z}}{2} \quad , \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2}$$

Complex division

Theorem. For $z = x + iy \in \mathbb{C} \setminus \{0\}$, write

$$z^{-1} = |z|^{-2} \cdot \overline{z} \quad .$$

Then

$$z \cdot z^{-1} = z \cdot \frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1 \quad . \tag{8}$$

Remarks: This allows to define division by $z \in \mathbb{C} \setminus \{0\}$, via

$$\frac{w}{z} = w \cdot z^{-1}$$

All properties known for division in $\mathbb{R}$ remain true in $\mathbb{C}$.

Properties of the modulus

Note that for a real number $x = x + i0$, one computes $|x + i0| = \sqrt{x^2 + 0^2} = |x|$. Hence the modulus of a real number is the same, whether we regard x as a real or complex number.

Theorem. (Rules for the absolute value:)
Let $w, z \in \mathbb{C}$

- $|z| \geq 0$, and $|z| = 0 \Leftrightarrow z = 0$.
- $|wz| = |w|\,|z|$.
- $|z| = |-z| = |\overline{z}|$
- $|w + z| \leq |w| + |z|$.

The last property is known as the triangle inequality. Useful reformulations are

$$\big|\,|w| - |z|\,\big| \leq |w + z| \leq |w| + |z| \quad .$$

Computing real and imaginary parts of a quotient

General procedure: Given a quotient $z = \dfrac{x+iy}{a+ib}$, multiply denominator and enumerator with the complex conjugate of the enumerator,

$$z = \frac{x+iy}{a+ib} = \frac{(x+iy)(a-ib)}{(a+ib)(a-ib)} = \frac{(x+iy)(a-ib)}{a^2+b^2} \quad .$$

Now the denominator is a real number, and we only need to compute the enumerator using formula (3).

Example: Computing the real and imaginary part of $z = \frac{5+3i}{1-2i}$:

$$z = \frac{5+3i}{1-2i} = \frac{(5+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-1+13i}{5} = -\frac{1}{5} + \frac{13}{5}i.$$

Hence $\mathrm{Re}(z) = \frac{1}{5}$ and $\mathrm{Im}(z) = \frac{13}{5}$.

Summary: Algebraic Properties of $\mathbb{C}$

So far, we have seen that the set $\mathbb{C}$ further expands our number domain:

$$\emptyset \subset \mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

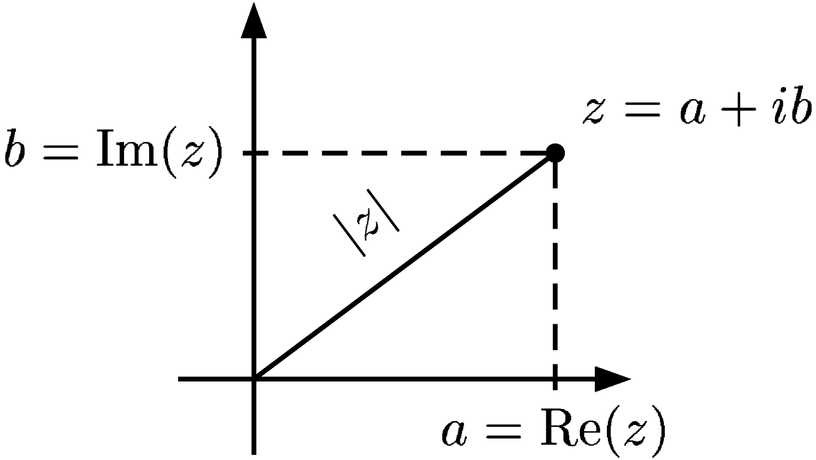
Algebraic operations, eg. taking sums, differences, products, quotients, are extended to $\mathbb{C}$, and the same computational rules as for $\mathbb{R}$ apply to $\mathbb{C}$ also. (Exception: monotonicity).

Note: We can now take the square root of -1:

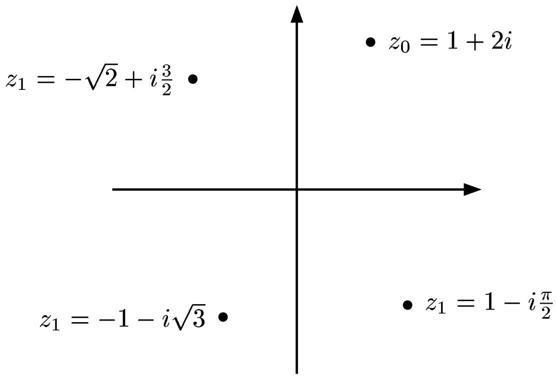
$$i^2 = (0 + i \cdot 1) \cdot (0 + i \cdot 1) = 0^2 - 1^2 + 1 \cdot 0 + 0 \cdot 1 = -1$$

Geometric interpretation of complex numbers

A complex number $a + ib$ is a pair of coordinates describing a point (a, b) in the plane. The modulus is the distance to the origin. The a-axis is called real axis, the b-axis is the imaginary axis.

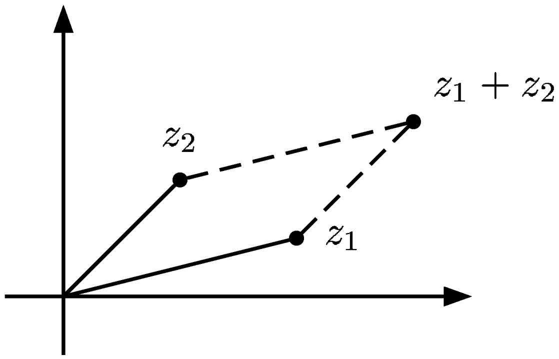


Some more examples



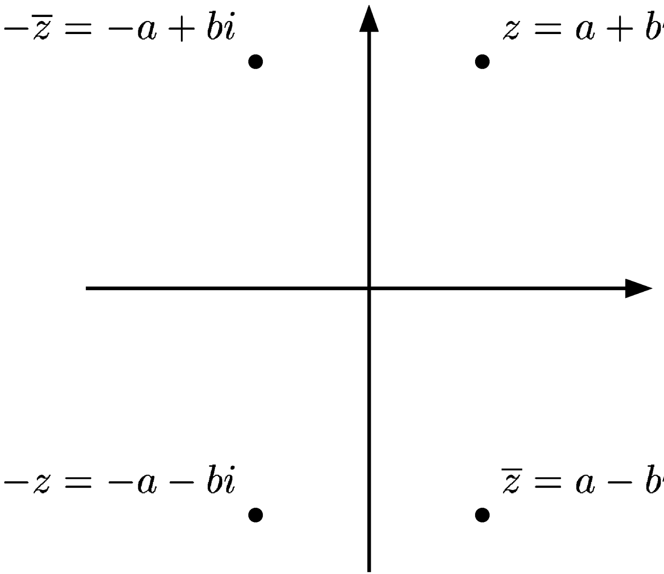
Geometric interpretation of addition

The sum of two complex numbers z_1, z_2 corresponds to the diagonal of the parallelogram with sides z_1 and z_2.
Triangle inequality: The sum of the sidelengths is greater than or equal to the length of the diagonal.



Geometric interpretation of complex conjugate

Taking complex conjugates or negatives of complex numbers amounts to reflection about a coordinate axis or about the origin



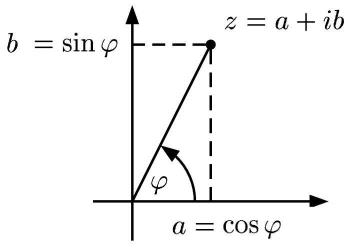
Trigonometric functions

The geometric interpretation of the product requires polar coordinates and trigonometric functions.
Consider a complex number $z = a + ib$ with

$$1 = |z| = \sqrt{a^2 + b^2}$$

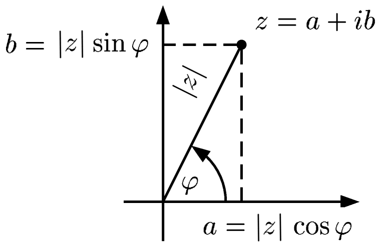
z describes an angle φ with the real axis, and $|z| = 1$ leads to the equations

$$a = \cos(\varphi) \quad , \quad b = \sin(\varphi)$$



Polar coordinates

For a general nonzero complex number $z = a + ib$, we write $z = |z|w$, where w has length 1. We then get the same picture, except that all lengths are multiplied by $|z|$:



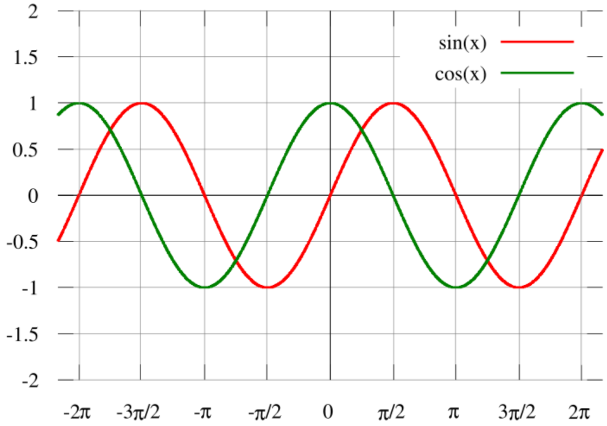
Thus, every $z \in \mathbb{C} \setminus \{0\}$, can be written (uniquely) as

$$z = r\,(\cos(\varphi) + i \sin(\varphi)) \text{ , where } r > 0 \text{ and } -\pi < \varphi \leq \pi \; .$$

(r, φ) are the polar coordinates of z, with $r = |z|$, the length of z.
φ is called argument of z, denoted $\arg(z)$.

Plots of sine and cosine

Plots of sine and cosine:



Useful values of sine and cosine:

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1

Important properties of sine and cosine

Theorem.

- **Symmetry:** For all angles α
$$\sin(-\alpha) = -\sin(\alpha) \quad , \quad \cos(-\alpha) = \cos(\alpha) \quad , \quad \sin(\alpha + \frac{\pi}{2}) = \cos(\alpha)$$
- **Periodicity:** For all angles α
$$\sin(\alpha) = \sin(\alpha + 2\pi) \; , \; \cos(\alpha) = \cos(\alpha + 2\pi) \quad .$$
- For all angles α
$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$
- **Addition Theorem.** For all $\alpha, \beta \in \mathbb{R}$,
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \tag{9}$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha) \tag{10}$$

Tangent and Arctangent

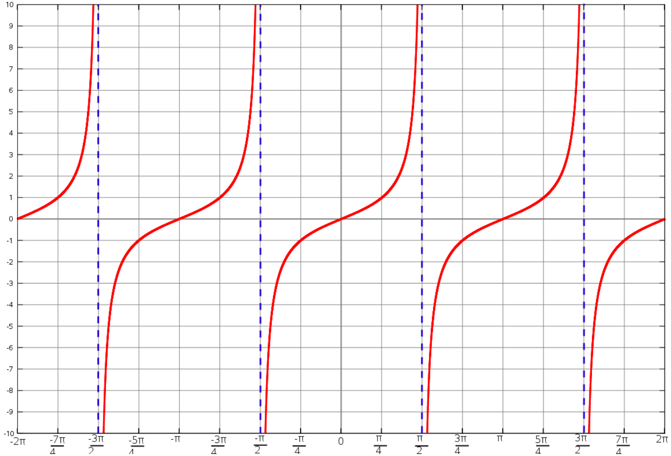
Definition. For $x \in \mathbb{R}$, we define the tangent function as

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Properties of the tangent function:

1. $\tan$ is π-periodic. It is undefined for all $x = k\pi + \pi/2$, with $k \in \mathbb{Z}$.
2. For all $x \in \mathbb{R}$, $\tan(-x) = -\tan(x)$.
3. For every $y \in \mathbb{R}$ there is a unique x with $-\pi/2 < x < \pi/2$ such that $y = \tan(x)$.
4. Let $y \in \mathbb{R}$. Suppose that $-\pi/2 < x < \pi/2$ is the unique number with $\tan(x) = y$. We define $\arctan(y) = x$, the arctangent of y.

Plot of tangent



Useful values of $\tan$:

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\tan(\alpha)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	n.d.	0

Geometric interpretation of the product

Let $z_1 = r_1(\cos(\varphi_1) + i\sin(\varphi_1))$ and $w = r_2(\cos(\varphi_2) + i\sin(\varphi_2))$. Then

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2 (\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2)) \\
&+ i\,(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)))
\end{aligned}
$$

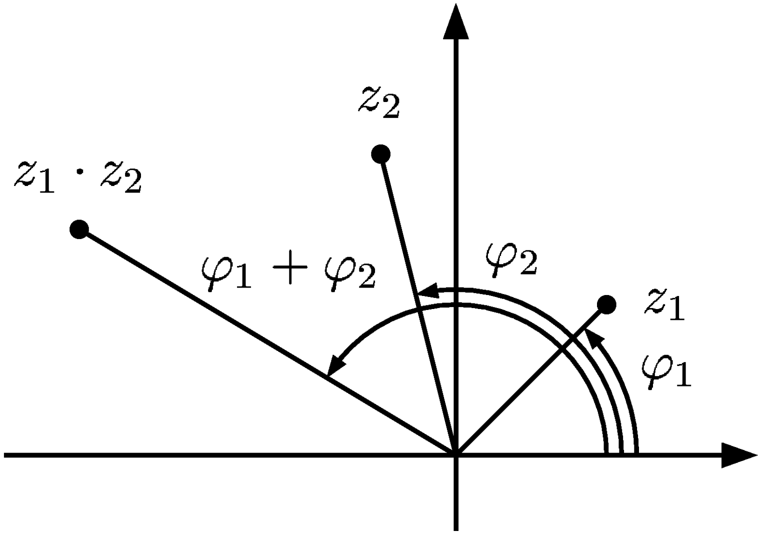
Now the addition theorem for trigonometric functions yields

$$
z_1 z_2 = \underbrace{r_1 r_2}_{=|z_1 z_2|} (\cos(\underbrace{\varphi_1 + \varphi_2}_{\arg(z_1 z_2)}) + i\sin(\varphi_1 + \varphi_2)) \quad ,
$$

i.e., $z_1 z_2$ has polar coordinates $(r_1 r_2, \varphi_1 + \varphi_2)$. Hence, one computes the polar coordinates of $z_1 \cdot z_2$ by

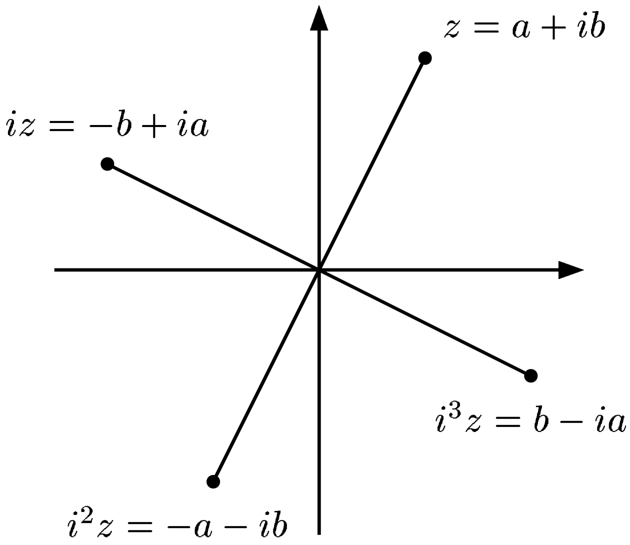
- multiplying the lengths of z_1 and z_2, and
- adding the arguments of z_1 and z_2 (possibly adding or subtracting 2π to remain in $(-\pi, \pi]$)

Illustration of multiplication



Example: Multiplication by powers of i

i has polar coordinates $(0, \pi/2)$. Hence multiplying by i is the same as rotating by 90 degrees counterclockwise:



Computing polar coordinates

Given a complex number $z = a + ib$, its length is easily computed as

$$|z| = \sqrt{a^2 + b^2} \ .$$

The argument is defined as $-\pi < \alpha \leq \pi$ such that

$$|z| \cos(\alpha) = a \ , \ |z| \sin(\alpha) = b \quad .$$

Hence $\tan(\alpha) = \frac{b}{a}$ is a necessary requirement.

General formula: The argument of $z = a + ib$ is given by

$$\alpha = \begin{cases} \arctan(b/a) & a > 0 \\ \pi - \arctan(b/a) & a < 0 < b \\ -\pi + \arctan(b/a) & a, b < 0 \\ 0 & a = 0 \wedge b > 0 \\ \pi & a = 0 \wedge b < 0 \end{cases}$$

Application: Integer powers and inverse

Theorem. (DeMoivre)
Let $n \in \mathbb{N}$, and $z = r(\cos(\varphi) + i \sin(\varphi))$. Then

$$z^n = r^n(\cos(n\varphi) + i \sin(n\varphi))$$

i.e., z^n has polar coordinates $(r^n, n\varphi)$.

Inverse: $1 = (1, 0) \in \mathbb{C}$ has length 1 and argument $\alpha = 0$. Hence, for arbitrary $z \in \mathbb{C} \setminus \{0\}$

- $1 = |z \cdot z^{-1}| = |z|\,|z^{-1}|$, hence $|z^{-1}| = \frac{1}{|z|}$
- the arguments add up to zero, hence z^{-1} has argument $-\varphi$

therefore the polar coordinates of z^{-1} are $(r^{-1}, -\varphi)$.

$\Rightarrow$ DeMoivre's theorem holds for all $n \in \mathbb{Z}$

Application: Taking roots

Let $n \in \mathbb{N}$ and $w \in \mathbb{C} \setminus \{0\}$, with polar coordinates (r, φ). We want to find all solutions of the equation

$$z^n = w \quad .$$

By periodicity of $\sin, \cos$, there are precisely n such complex numbers, having polar coordinates

$$\left(r^{1/n}, \frac{\varphi}{n}\right) \;,\; \left(r^{1/n}, \frac{\varphi + 2\pi}{n}\right) \;,\; \ldots \;,\; \left(r^{1/n}, \frac{\varphi + 2\pi(n-1)}{n}\right) \quad .$$

Note: Some of the angles

$$\frac{\varphi}{n}, \frac{\varphi + 2\pi}{n}, \ldots, \frac{\varphi + 2\pi(n-1)}{n}$$

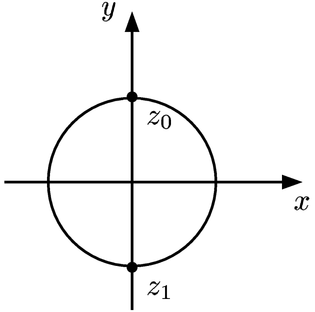
are greater than π. We subtract 2π from these angles to obtain angles in the prescribed interval $(-\pi, \pi]$.

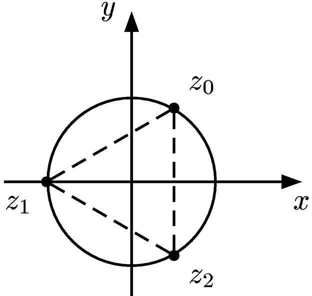
Examples: Roots of -1

Example $w = -1$ has polar coordinates $(1, \pi)$. Hence its square roots have polar coordinates $(1, \frac{\pi}{2})$ and $(1, -\frac{\pi}{2})$, corresponding to $z_{0,1} = \pm i$. The cubic roots of -1 have length 1 and arguments $\frac{\pi}{3}, \pi, \frac{-\pi}{3}$, yielding

$$z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2}\ ,\ \ z_1 = -1\ ,\ \ z_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Left: Square roots, right: cubic roots of -1





Further examples: Quadratic equation

Note: An arbitrary quadratic equation

$$z^2 + az + b = 0 \ ,$$

with $a, b \in \mathbb{C}$, has at least one complex solution. Just as for real coefficients, we can derive the formula

$$z_{1,2} = \frac{a \pm \sqrt{a^2 - 4b}}{2} \ ,$$

describing all possible solutions, and the root can now be evaluated for every choice of a and b.

Remark: In fact, much more is true. Given any polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

with $a_{n-1}, \ldots, a_0 \in \mathbb{C}$, there exists $z \in \mathbb{C}$ with $f(z) = 0$.

Summary

- Complex numbers and operations on them: Sums, products, inverses of complex numbers
- Imaginary and real parts, complex conjugates
- Polar coordinates: Modulus, argument and their uses
- Geometric interpretation of complex numbers and operations on them
- Computation of powers and roots of complex numbers