Calculus and Linear Algebra for Biomedical Engineering

Week 10: Riemann integral and its properties

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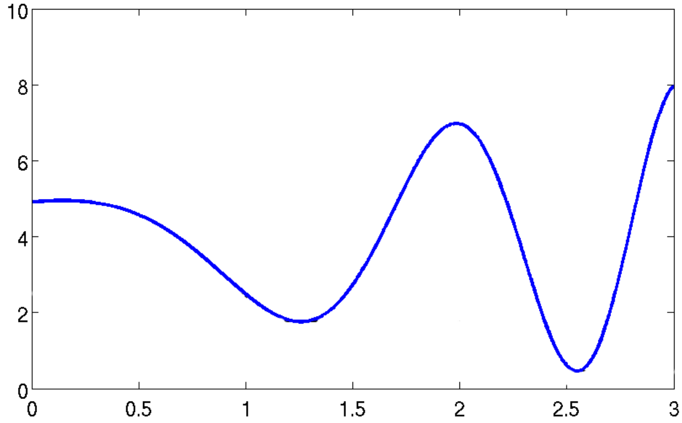
Motivation: Computing flow from flow rates

We observe the flow of water through a drain, which varies with time.



The result is a flow rate, in litres/second, continuously recorded over a time interval $[a, b]$. From these data, we want to determine the total amount A of water that has passed through the valve during the interval.

This value corresponds to the area under the graph of f

Answer for constant rate

If the flow is constant, say equal to c, the answer is easily obtained:

$$A = (b - a) \cdot c$$

This corresponds to the formula

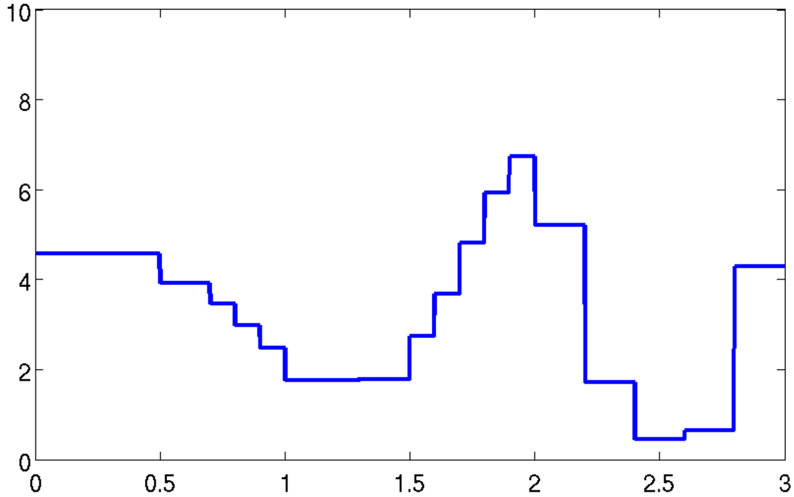
$$\text{area} = \text{width} \cdot \text{height} \ .$$

for rectangular areas.

The idea to calculate the area under arbitrary graphs is to approximate the graph by piecewise constant functions.

Area under the graph: Piecewise constant functions

A piecewise constant function or step function is a function $f : [a, b] \to \mathbb{R}$ that consists of finitely many constant pieces



Here, the region under the graph is made up out of rectangles and its area is computed by summing the areas of the rectangles.

Partition

Definition. Let $I = [a, b] \subset \mathbb{R}$ be some interval. A partition of I is given by a finite subset $\mathcal{P} = \{x_0, \ldots, x_n\}$ satisfying $\{a, b\} \in \mathcal{P}$. Without loss of generality,

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b \ .$$

Example: The set $\mathcal{P} = \{0, 0.3, 0.5, 0.8, 1.0\}$ defines a partition of the interval $[0, 1]$.

Approximation by step functions

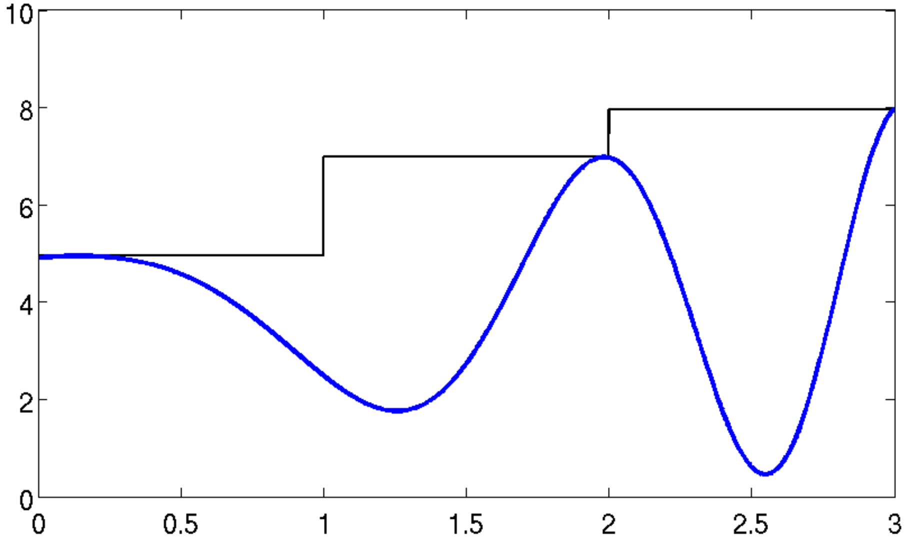
Definition.
Let $f : [a,b] \to \mathbb{R}$ be a function, and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ a partition.
We define

$$\begin{aligned} \overline{M_k}(f) &= \sup\{f(x) : x_k < x < x_{k+1}\} \\ \underline{M_k}(f) &= \inf\{f(x) : x_k < x < x_{k+1}\} \end{aligned}$$

Interpretation: $\overline{M_k}$ and $\underline{M_k}$ provide optimal approximation of the graph of f by step functions with jumps in $\mathcal{P}$, one from above, one from below.

Example: Approximation from above

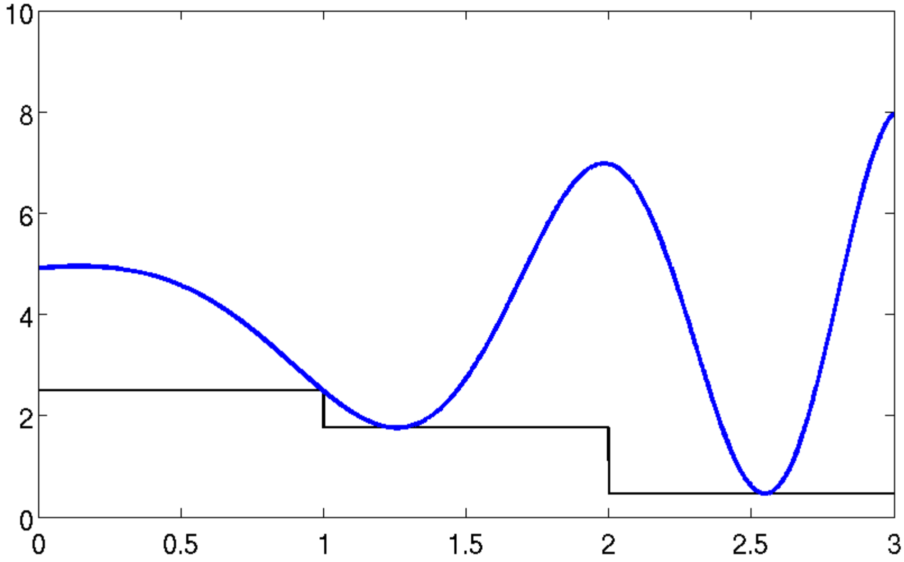
A function defined on $[0,3]$, partition $\mathcal{P}=\{0,1,2,3\}$.
Blue: Function graph, Black: Step function associated to $\overline{M_k}$



Example: Approximation from below

A function defined on $[0,3]$, partition $\mathcal{P} = \{0,1,2,3\}$.

Blue: Function graph, Black: Step function associated to $\underline{M_k}$



Upper and lower sum

Definition. Let $f : [a, b] \to \mathbb{R}$, and let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, with $a = x_0 < x_1 < \dots < x_n = b$.. We write

$$\overline{S}(\mathcal{P}) = \sum_{k=1}^{n} \overline{M_{k-1}}(x_k - x_{k-1})$$

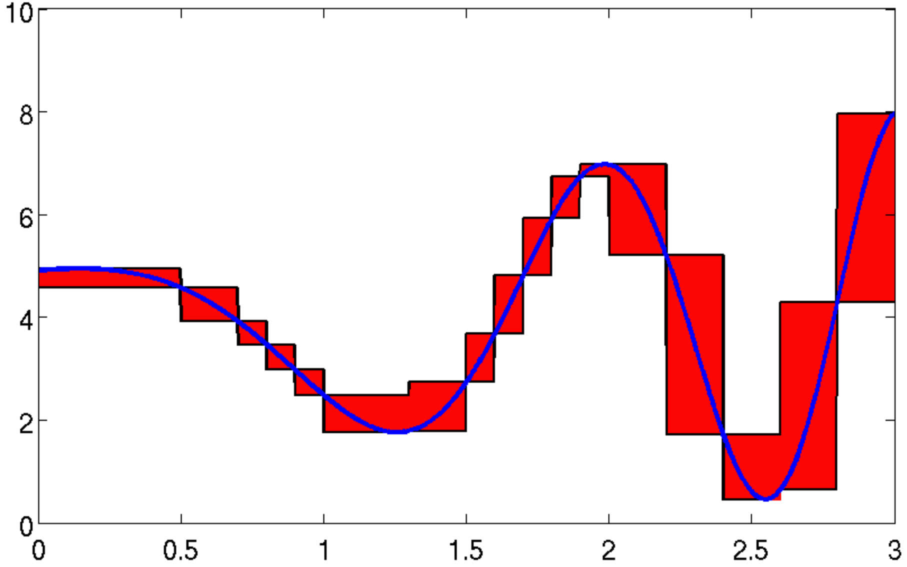
$$\underline{S}(\mathcal{P}) = \sum_{k=1}^{n} \underline{M_{k-1}}(x_k - x_{k-1})$$

Interpretation:

- The area below the step function with values $\overline{M_{k-1}}$ contains the area below the graph of f. Hence $\overline{S}(\mathcal{P})$ is greater or equal to the area below the graph of f.
- Likewise: $\underline{S}(\mathcal{P})$ is smaller or equal to the area below the graph of f.

Graphical interpretation of upper and lower sum

The difference $\overline{S}(\mathcal{P}) - \underline{S}(\mathcal{P})$ is the area between upper and lower step function approximation



Refinement of a partition

Definition. Let $\mathcal{P}_1, \mathcal{P}_2$ be two partitions of $[a, b]$. Then $\mathcal{P}_1$ is called refinement of $\mathcal{P}_2$ if $\mathcal{P}_1 \supset \mathcal{P}_2$.

Interpretation:

- If $\mathcal{P}_1 \supset \mathcal{P}_2$, then

$$\underline{S}(\mathcal{P}_2) \leq \underline{S}(\mathcal{P}_1) \leq \overline{S}(\mathcal{P}_1) \leq \overline{S}(\mathcal{P}_2)$$

  Hence the area between upper and lower approximation decreases.
- The two should approximate the same value, as the partition gets finer and finer.

Illustration for refinement

A function $f : [0, 3] \to \mathbb{R}$, partition $\{0, 1, 2, 3\}$, lower and upper approximation

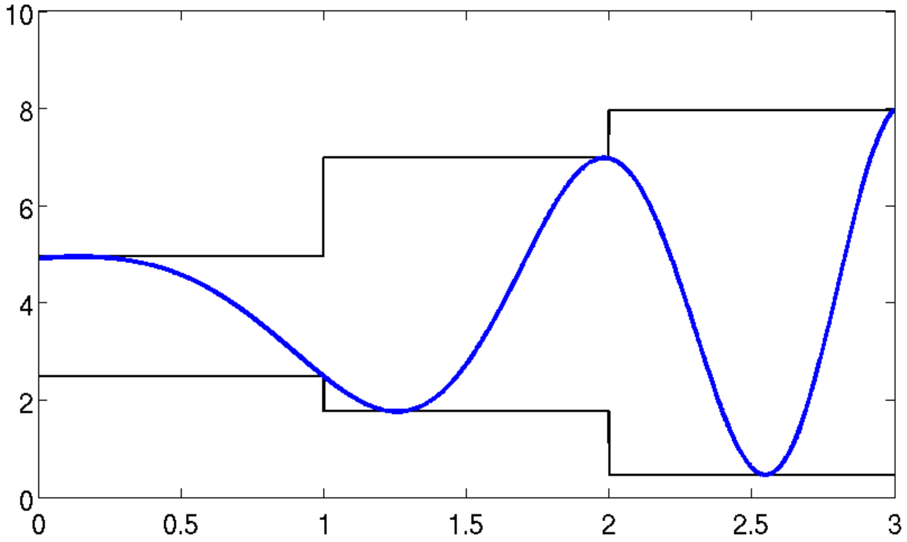
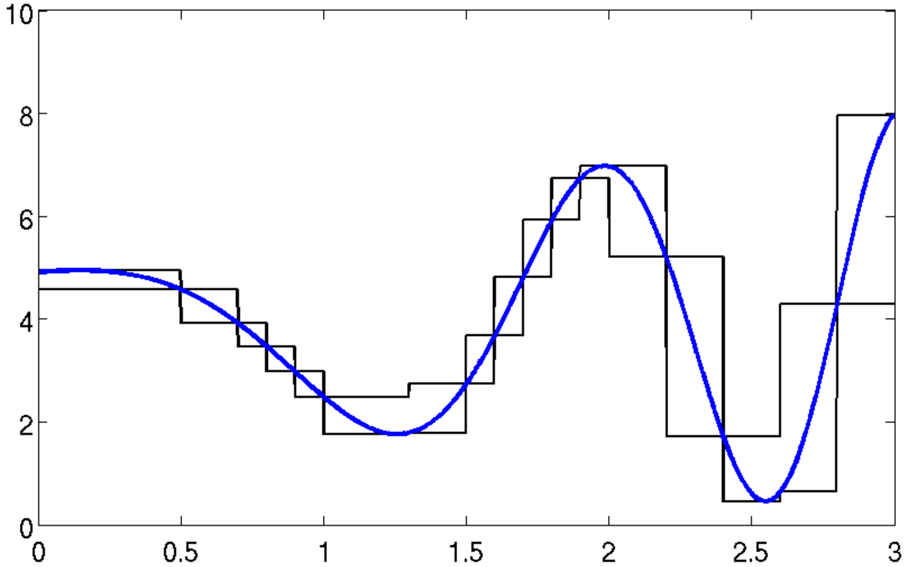


Illustration for refinement

The same function, lower and upper approximation for the refinement $\{0, 0.5, 0.7, 0.8, 0.9, 1, 1.3, 1.5, 1.6, 1.7, 1.8, 1.9, 2, 2.2, 2.4, 2.6, 2.8, 3\}$.



Riemann integrable function

Definition. The function $f : [a, b] \longrightarrow \mathbb{R}$ is called (Riemann) integrable if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$\overline{S}(\mathcal{P}) - \underline{S}(\mathcal{P}) < \epsilon$$

Note: This implies

$$\overline{S}(\mathcal{P}') - \underline{S}(\mathcal{P}') < \epsilon$$

for every refinement $\mathcal{P}'$ of $\mathcal{P}$.

Convergence of upper and lower sums

Theorem 1.
Let f be a Riemann integrable function. Let $\mathcal{P}_n$ be a sequence of partitions satisfying $\delta_n \to 0$, where δ_n is the maximal distance of two neighboring elements of $\mathcal{P}_n$.
Then

$$I(f) = \lim_{n\to\infty} \overline{S}(\mathcal{P}_n)$$

exists, with

$$I(f) = \lim_{n\to\infty} \underline{S}(\mathcal{P}_n).$$

Moreover, $I(f)$ is the same for all sequences of partitions with $\delta_n \to 0$.

Definition of the Riemann integral

Definition. If f is integrable, $I(f)$ as in Theorem 1. $I(f)$ is called the (Riemann) integral of f over $[a, b]$, and denoted as

$$\int_a^b f(x)dx \; .$$

a is called lower bound of the integral, b is called upper bound of the integral, and f is called the integrand.
Furthermore, we define, for $a < b$,

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

as well as

$$\int_a^a f(x)dx = 0$$

Criteria for Riemann integrability

Sufficient conditions:

- If f is continuous on $[a, b]$, then f is integrable.
- If f is monotonic and bounded on $[a, b]$, then f is integrable.

Example: A bounded function that is not integrable:

$$f : [0, 1] \longrightarrow \mathbb{R} \ , \ \ f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

For every partition $\mathcal{P}$, one finds

$$\overline{S}(\mathcal{P}) = 1 \neq -1 = \underline{S}(\mathcal{P}) \ .$$

Properties of the Riemann integral

Theorem 2.

Let f, g be integrable over the interval with bounds a, b, let $s \in \mathbb{R}$

- sf is integrable, with $\int_a^b sf(x)dx = s\int_a^b f(x)dx$.
- $f + g$ is integrable, with $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
- Let c in $\mathbb{R}$ be such that f is integrable over $[b, c]$. Then f is integrable over $[a, c]$, with

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \ .$$

- If f is integrable, then $|f|$ is integrable as well, with

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

Monotonicity of integrals

Theorem 3.
Let $a \leq b$, let $f : [a,b] \to \mathbb{R}$ be integrable and bounded, with

$$m \leq f(x) \leq M \ , \quad \text{for all } x \in [a,b]$$

Then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \ .$$

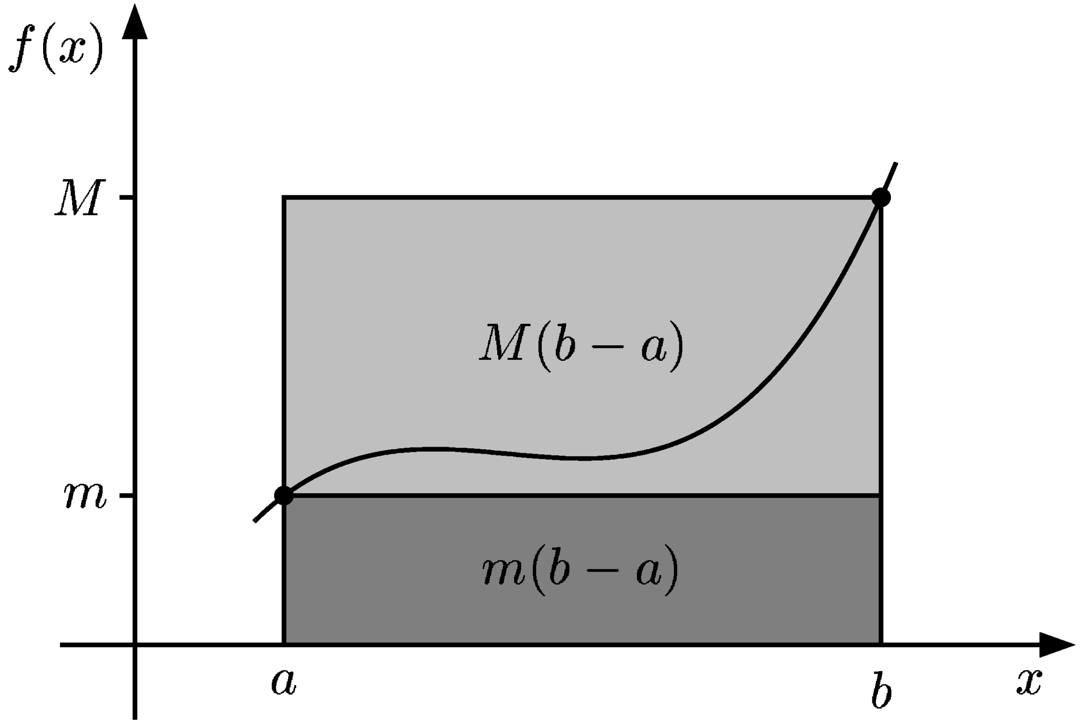
This applies in particular, when f is continuous on $[a,b]$, and

$$m = \min_{x\in[a,b]} f(x) \ , \ M = \max_{x\in[a,b]} f(x) \ .$$

More generally, if $f, g : [a,b] \to \mathbb{R}$ are integrable, with $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx \ .$$

Illustration for the estimate



Fundamental Theorem of Calculus

Theorem 4.
Let $f : [a,b] \to \mathbb{R}$ be continuous. We define

$$F : [a,b] \to \mathbb{R} \ , \ F(y) = \int_a^y f(x)dx$$

Then F is continuous on $[a,b]$, differentiable on (a,b), with

$$F'(x) = f(x) \ , \ \forall x \in (a,b) \ .$$

Conversely, suppose that $G : [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b) with $G' = f$. Then the integral is computed as

$$\int_a^b f(x)dx = G|_a^b := G(b) - G(a)$$

Integration and antiderivatives

Remarks: Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

- A differentiable function F with $F' = f$ is called antiderivative or primitive of f. Hence f has a primitive given by

$$F(y) = \int_a^y f(x)dx \ .$$

- Two primitives F, G of f only differ by a constant: $F(x) = G(x) - c$, with $c \in \mathbb{R}$ fixed. By letting

$$F(y) = \int_a^y f(x)dx$$

  one obtains the unique primitive of f satisfying $F(a) = 0$.

- It is customary to denote primitives as $F = \int f(x)dx$ (without bounds), and refer to them as indefinite integrals of f.

Application: The length of a curve

Definition.
Let $f : [a, b] \longrightarrow \mathbb{R}^n$ be given, i.e.,

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T.$$

The set

$$\mathcal{C} = \{f(x) : x \in [a, b]\}$$

is called a curve in $\mathbb{R}^n$, and f is called parameterization of $\mathcal{C}$.
We assume that all f_i are continuously differentiable on (a, b) and continuous on $[a, b]$. We define the length of $\mathcal{C}$ as

$$l(\mathcal{C}) = \int_a^b \sqrt{f_1'(x)^2 + f_2'(x)^2 + \ldots + f_n'(x)^2} dx$$

Example: Circumference of the circle

We consider the map $f : [0, 2\pi] \to \mathbb{R}^2$, with $f(x) = (\sin(x), \cos(x))$. The resulting curve is the unit circle.

We compute

$$f_1'(x) = \cos(x)\ ,\ \ f_2'(x) = -\sin(x)$$

and thus, using $\sin^2 + \cos^2 = 1$,

$$\int_0^{2\pi} \sqrt{f_1'(x)^2 + f_2'(x)^2} dx = \int_0^{2\pi} 1 dx = 2\pi\ .$$

Example: Length of a graph

We want to determine the length of the graph G_f of $f(t) = t^2$, for $t \in [0,1]$. G_f is parameterized by

$$g : [0,1] \to \mathbb{R}^2\ ,\ \ g(t) = (t, t^2)^T\ .$$

Using $g_1'(t) = 1$, $g_2'(t) = 2t$, we obtain

$$l(G_f) = \int_0^1 \sqrt{1+4t^2}dt\ .$$

One can check that

$$F(t) = \frac{1}{4}\left(2t\sqrt{1+4t^2} + \ln(2t + \sqrt{1+4t^2})\right)$$

is a primitive of $g(t) = \sqrt{1+4t^2}$. Hence,

$$l(G_f) = F|_0^1 = \frac{1}{4}\left(2\sqrt{5} + \ln(2+\sqrt{5})\right) - 0$$

Summary

- Definition and interpretation of integrals; area under the graph
- Properties of the integral: Linearity, monotonicity
- Evaluation of integrals via antiderivatives
  ($\rightsquigarrow$ New problem: How to obtain antiderivatives)
- Application of integrals: Curve length