Calculus and Linear Algebra for Biomedical Engineering

Week 12: Extensions of the Riemann integral: Improper integrals, two-dimensional integrals and volumes

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Improper integrals: Motivation

We again consider the flow rate $f(t)$ of water running into a container. We want to ensure that the container does not overflow at any point in time.

We assume that we can reliably predict $f(t)$ for any point t in the future. Then we have to ensure that the volume V of the container fulfills the inequality

$$\int_0^t f(x)dx \leq V \ ,$$

for all $t > 0$. Assuming that $f \geq 0$, this boils down to the question whether the container can hold all water that flows into it during the unbounded time interval $(0, \infty)$, which could be reformulated as

$$\int_0^\infty f(x)dx \leq V \ .$$

Improper integrals: Definition

Definition.
Let $f:(a,b)\to\mathbb{R}$, with $a,b\in\mathbb{R}\cup\{\pm\infty\}$. Assume that f is Riemann integrable over all intervals $[c,d]\subset(a,b)$, where $-\infty<c<d<\infty$. Pick any $c\in(a,b)$. If both

$$\lim_{t\to a, t\geq a}\int_t^c f(x)dx \text{ and } \lim_{t\to b, t\leq b}\int_c^t f(x)dx$$

exist in $\mathbb{R}$, then $\int_a^b f(x)dx$ is called convergent improper integral, and we let

$$\int_a^b f(x)dx = \lim_{t\to a, t\geq a}\int_t^c f(x)dx + \lim_{t\to b, t\leq b}\int_c^t f(x)dx\ .$$

The choice of c affects neither convergence nor the value of the improper integral.

Special case: One-sided improper integrals

Remarks. If $f : (-\infty, b) \to \mathbb{R}$ is Riemann integrable over all intervals $[a, b]$ for $a < b$, the improper integral simplifies to

$$\int_a^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx \ ,$$

provided the limit exists.
Likewise, if $f : (a, \infty) \to \mathbb{R}$ is Riemann integrable over all intervals $[a, b]$ for $a < b$, then

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx \ ,$$

if the limit exists.

Improper integration extends Riemann integration

The improper integral is an extension of Riemann integration in the following sense:

- If f is Riemann integrable over $[a, b]$, then the improper integral exists and coincides with the Riemann integral.
- However, there are cases where the Riemann integral is not applicable, e.g.,
  - The integration domain is unbounded; or
  - the function is not Riemann-integrable over $[a, b]$,

  but the improper integral converges.
- Computation of improper integrals $\int_a^b f(x)dx$ proceeds in two steps:
  - Compute primitive F of f on (a, b)
  - Compute $\lim_{t\to a} F(t)$, $\lim_{t\to b} F(t)$

First example: Unbounded integration domain

We want to compute the improper integral

$$\int_{-\infty}^{\infty} e^{-|x|} dx \ .$$

For the computation of the improper integral, the point $c = 0$ is the natural choice. Using $e^t \to 0$ for $t \to -\infty$, we find

$$\lim_{t \to -\infty} \int_t^0 e^{-|x|} dx \;=\; \lim_{t \to -\infty} \int_t^0 e^x dx = \lim_{t \to -\infty} e^x \Big|_t^0 = 1$$
$$\lim_{t \to \infty} \int_0^t e^{-|x|} dx \;=\; \lim_{t \to \infty} \int_0^t e^{-|x|} dx = \lim_{t \to \infty} -e^{-x} \Big|_0^t = 1 \ .$$

Hence

$$\int_{-\infty}^{\infty} e^{-|x|} dx = 2 \ .$$

Second example: Integrating an unbounded function

For $\alpha \in \mathbb{R}$, we want to compute $\int_0^1 x^\alpha dx$. For $\alpha \neq -1$, we obtain

$$\begin{aligned} \int_0^1 x^{-\alpha} dx &= \lim_{t \to 0} \int_t^1 x^\alpha dx = \lim_{t \to 0} \frac{1}{\alpha + 1} x^{\alpha+1}|_t^1 = \lim_{t \to 0} \frac{1 - t^{\alpha+1}}{\alpha + 1} \\ &= \begin{cases} \frac{1}{\alpha+1} & \alpha + 1 > 0 \\ \infty & \alpha + 1 < 0 \end{cases} \end{aligned}$$

For $\alpha = -1$, we obtain

$$\int_0^1 x^{-1} dx = \lim_{t \to 0} \ln(1) - \ln(t) = -\infty \ .$$

Hence the improper integral converges precisely if $\alpha > -1$.
Note: For $\alpha < 0$, the integrand is unbounded on any interval $(0, \epsilon)$, hence the Riemann integral does not converge. Hence the case $-1 < \alpha < 0$ is not covered by the Riemann integral.

Properties of improper integrals

Theorem. Let $f, g : (a,b) \to \mathbb{R}$ be given, and $s \in \mathbb{R}$.

- If f, g are improperly integrable, then so is $f + sg$, with $\int_a^b f(x) + sg(x)dx = \int_a^b f(x)dx + s\int_a^b g(x)dx$.
- Assume that $0 \leq f(x) \leq g(x)$, for all $x \in (a,b)$. Then

$$\int_a^b g(x)dx \text{ converges} \quad \Rightarrow \quad \int_a^b f(x)dx \text{ converges}$$
$$\int_a^b f(x)dx \text{ diverges} \quad \Rightarrow \quad \int_a^b g(x)dx \text{ diverges}$$

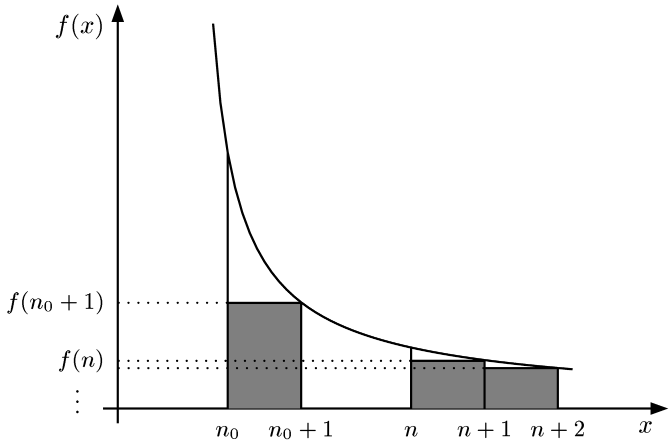
- If $\int_a^b |f(x)|dx$ converges, so does $\int_a^b f(x)dx$, with

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \ .$$

Convergence of integrals vs. Convergence of series

Theorem. Let $n_0 \in \mathbb{N}$, and suppose that $f : [n_0, \infty) \to \mathbb{R}$ is positive and monotonically decreasing. Then

$$\int_{n_0}^{\infty} f(x)dx < \infty \Rightarrow \sum_{n=n_0}^{\infty} f(n) < \infty \, .$$



Example: Estimating sums by integrals

We want to determine for which $\alpha \in \mathbb{R}$ the sum $\sum_{n=1}^{\infty} n^{\alpha}$ is finite. Now $\sum_{n=1} n^{-1} = \infty$ (harmonic series) entails for all $\alpha > -1$ that

$$\sum_{n=1}^{\infty} n^{\alpha} \geq \sum_{n=1}^{\infty} n^{-1} = \infty \ .$$

Hence it remains to consider the case $\alpha < -1$. Since $f(x) = x^{\alpha}$ is decreasing on $[1, \infty)$, we can decide this by computing

$$\int_1^{\infty} f(x)dx \ = \ \lim_{t \to \infty} \int_1^t x^{\alpha} dx = \lim_{t \to \infty} \frac{t^{\alpha+1} - 1}{\alpha + 1} = \frac{-1}{\alpha + 1} < \infty \ .$$

This implies $\sum_{n=1}^{\infty} n^{\alpha} < \infty$ for all $\alpha < -1$.

Higher-dimensional integrals and Volume: Motivation

We consider a rectangular pool with variable depth. I.e., the pool is located in a rectangle $[a, b] \times [c, d]$, and its depth is a function $f : [a, b] \times [c, d] \to \mathbb{R}$. We call $b - a$ the length and $d - c$ the breadth of the pool.

Our task is to compute the volume of the pool. Again, if the depth of the pool is a constant K, this is easy:

$$\text{Volume} = \text{length} \times \text{breadth} \times \text{depth} = (b - a)(d - c)K \ .$$

For variable depths, we can hope to approximate the volume by a procedure analogous to the one-dimensional case:

- Cut the domain $[a, b] \times [c, d]$ into small rectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$
- Approximate f by constants on each rectangle
- Sum up the approximate volumes above the rectangles

Two-dimensional partitions

Definition. Let $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be a function. A partition $\mathcal{P}$ of $[a,b] \times [c,d]$ is given by a pair $(\mathcal{P}_1, \mathcal{P}_2)$ of partitions of $[a,b]$ and $[c,d]$, respectively.

Observe: A partition $\mathcal{P}$ cuts the rectangle $[a,b) \times [c,d)$ into the rectangles

$$R_{k,l} = [x_{k-1}, x_k) \times [y_{l-1}, y_l) \ , \ k = 1, \ldots, n, l = 1, \ldots, m$$

We define

$$\overline{M_{k,l}} = \sup\{f(x) : x \in R_{k,l}\} \ , \ \underline{M_{k,l}} = \inf\{f(x) : x \in R_{k,l}\}$$

Riemann-integrable functions in two dimensions

Definition. Let $f : [a, b] \times [c, d] \to \mathbb{R}$, and $\mathcal{P}$ a partition of $[a, b] \times [c, d]$. We define

$$\overline{S(\mathcal{P})} = \sum_{k,l} \overline{M_{k,l}}(x_k - x_{k-1})(y_l - y_{l-1})$$

$$\underline{S(\mathcal{P})} = \sum_{k,l} \underline{M_{k,l}}(x_k - x_{k-1})(y_l - y_{l-1})$$

Definition. The function $f : [a, b] \times [c, d] \to \mathbb{R}$ is called **Riemann integrable** if for all $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that

$$\overline{S(\mathcal{P})} - \underline{S(\mathcal{P})} < \epsilon \ .$$

Definition of the Riemann integral in two dimensions

Definition. Let Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be Riemann-integrable. Let partitions $\mathcal{P}^n = (\mathcal{P}_1^n, \mathcal{P}_2^n)$ (for $n \in \mathbb{N}$)) be given such that

$$\delta_n = \text{ maximal diameter of the rectangles in } \mathcal{P}^n$$

goes to zero. Then there exists a unique number $I(f)$ such that

$$I(f) = \lim_{n\to\infty} \underline{S(\mathcal{P}^n)} = \overline{S(\mathcal{P}^n)} \ .$$

$I(f)$ is called the Riemann integral of f, and denoted as

$$I(f) = \iint_{[a,b]\times[c,d]} f(x,y)d(x,y) \ .$$

Properties of the Riemann integral

Let $f, g : [a, b] \times [c, d] \longrightarrow \mathbb{R}$ be Riemann integrable, and $s \in \mathbb{R}$

- $f+sg$ is Riemann integrable, with $\iint\limits_{[a,b]\times[c,d]} f(x,y)+sg(x,y)d(x,y) = \iint\limits_{[a,b]\times[c,d]} f(x,y)d(x,y) + s \iint\limits_{[a,b]\times[c,d]} g(x,y)d(x,y)$.
- If $f(x,y) \leq g(x,y)$, for all $(x,y) \in (a,b) \times (c,d)$. Then
$$\iint\limits_{[a,b]\times[c,d]} f(x,y)d(x,y) \leq \iint\limits_{[a,b]\times[c,d]} g(x,y)d(x,y)$$
- $|f|$ is also Riemann integrable, with
$$\left| \iint\limits_{[a,b]\times[c,d]} f(x,y)d(x,y) \right| \leq \iint\limits_{[a,b]\times[c,d]} |f(x,y)|d(x,y)$$

Computing two-dimensional integrals

Theorem. Every continuous $f : [a,b] \times [c,d] \to \mathbb{R}$ is integrable, and the integral can be computed in either of the following ways:

- The function $g_1 : [a,b] \mapsto \mathbb{R}$, $g_1(x) = \int_c^d f(x,y)dy$ is continuous, and

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)d(x,y) = \int_a^b g_1(x)dx$$

- The function $g_2 : [c,d] \mapsto \mathbb{R}$, $g_2(y) = \int_a^b f(x,y)dx$ is continuous, and

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)d(x,y) = \int_a^b g_2(y)dx$$

Hence

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)d(x,y) = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx\ .$$

First example: A pool with linearly increasing depth

We have a pool of length 50 and breadth 25. We assume that the pool is 80 centimeters deep at the shallow end, and over the length of 50 meters the depth increases linearly up to 3 meters. In other words, the depth function $f : [0, 50] \times [0, 25] \to \mathbb{R}$ is given by

$$f(x, y) = 0.8 + \frac{2.2}{50}x \ .$$

Then the volume of the pool is computed as

$$\int_0^{50} \int_0^{25} 0.8 + \frac{2.2}{50} x dy dx = \int_0^{50} 25 \cdot (0.8 + \frac{2.2}{50} x) dx = \left(20x + \frac{1.1}{2} x^2 \right) \Big|_0^{50} = 2375 \ ,$$

i.e., 2375 cubic metres of water are needed to fill the pool.
Alternatively, we could have computed

$$\int_0^{25} \int_0^{50} 0.8 + \frac{2.2}{50} x dx dy = \int_0^{25} \left(0.8x + \frac{1.1}{50} x^2 \right) \Big|_0^{50} dy = \int_0^{25} 95 dy = 2375$$

Second example

We want to compute the integral

$$\iint\limits_{[0,1]\times[0,1]} e^{xy} d(x,y)$$

$$\int_0^1 \int_0^1 e^{xy} dx dy = \int_0^1 \left(\frac{e^{xy}}{y}\right)\Bigg|_0^1 dy = \int_0^1 \frac{e^y - 1}{y} dy \ .$$

Using the substitution $s = e^y$, $ds = \frac{dy}{y}$, we continue

$$\int_0^1 \frac{e^y - 1}{y} dy = \int_{e^0}^{e^1} s - 1 ds = \left(\frac{s^2}{2} - s\right)\Bigg|_1^e = \frac{e^2}{2} - e + \frac{1}{2} \approx 1.4726$$

More general integration domains

More general setting: Let $G \subset \mathbb{R}^2$ denote a bounded set, $f : G \to \mathbb{R}$. How should we define $\iint\limits_G f(x,y)d(x,y)$?

Answer: Since G is bounded, $G \subset [a,b] \times [c,d]$, for suitable $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. We extend f to a function $g : [a,b] \times [c,d] \to \mathbb{R}$ by letting

$$g(x,y) = \begin{cases} f(x,y) & (x,y) \in G \\ 0 & (x,y) \in [a,b] \times [c,d] \setminus G \end{cases}$$

We then declare f integrable iff g is integrable, and define

$$\iint\limits_G f(x,y)d(x,y) = \iint\limits_{[a,b]\times[c,d]} g(x,y)d(x,y)$$

This works at least for reasonable G and f.

Projectable sets

Definition. Let $G \subset \mathbb{R}^2$ be a bounded set. Then G is called

- y-projectable if there exist continuous functions $\overline{y}, \underline{y} : [a, b] \to \mathbb{R}$ such that

$$G = \{(x, y) : \underline{y}(x) \leq y \leq \overline{y}(x) \ , \ x \in [a, b]\} \ .$$

- x-projectable if there exist continuous functions $\overline{x}, \underline{x} : [c, d] \to \mathbb{R}$ such that

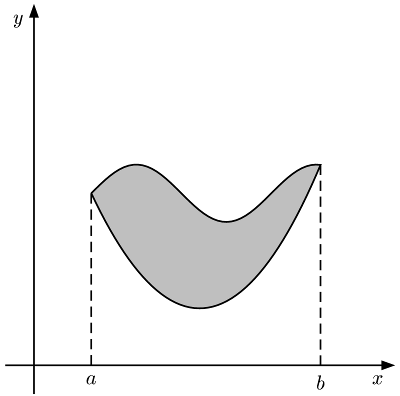
$$G = \{(x, y) : \underline{x}(y) \leq x \leq \overline{x}(y) \ , \ y \in [c, d]\}$$

- projectable if it is either x- or y-projectable.

Illustration: Projectable sets

A projectable set is described by a pair of curves as (lower and upper, or left and right) boundaries.

Left to right: y-projectable, x-projectable, both x- and y-projectable



Integration over projectable sets

Theorem. Let $f : G \to \mathbb{R}$ be continuous, where $G \subset \mathbb{R}^2$ is a bounded projectable set. Then f is integrable, and

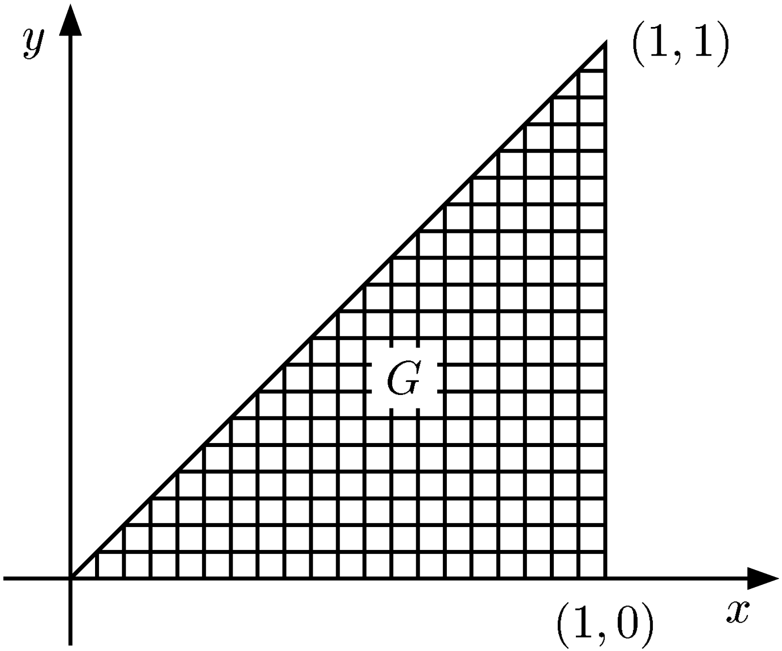
- If G is y-projectable, i.e., $G = \{(x,y) : \underline{y}(x) \le y \le \overline{y}(x)\ ,\ \ x \in [a,b]\}$ then
$$\iint\limits_G f(x,y)d(x,y) = \int_a^b \int_{\underline{y}(x)}^{\overline{y}(x)} f(x,y)dydx$$
- If G is x-projectable, i.e., $G = \{(x,y) : \underline{x}(y) \le x \le \overline{x}(y)\ ,\ \ y \in [c,d]\}$, then
$$\iint\limits_G f(x,y)d(x,y) = \int_c^d \int_{\underline{x}(y)}^{\overline{x}(y)} f(x,y)dxdy\ .$$

If G is both x and y-projectable, we have in particular that

$$\int_a^b \int_{\underline{y}(x)}^{\overline{y}(x)} f(x,y)dydx = \int_c^d \int_{\underline{x}(y)}^{\overline{x}(y)} f(x,y)dxdy$$

Example: Integration over a triangle

Suppose that the integration domain is given by the triangle with corners $(0,0), (1,0), (1,1)$.



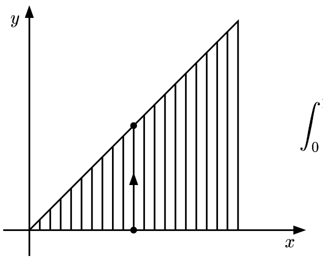
Two alternatives for integration

Observation: The domain is both x- and y-projectable, with

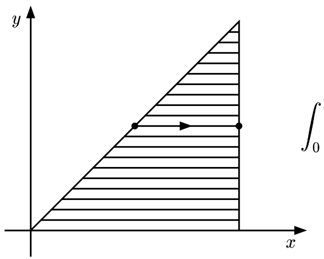
$$\underline{y}(x) = 0\ ,\ \overline{y}(x) = x\ ,\ \underline{x}(y) = y\ ,\ \overline{x}(y) = 1$$

Hence

$$\iint\limits_G f(x,y)d(x,y) = \int_0^1 \left(\int_0^x f(x,y)dy \right) dx = \int_0^1 \left(\int_y^1 f(x,y)dx \right) dy$$



$\int_0^1 \left(\int_0^x f(x,y)\, dy \right) dx$



$\int_0^1 \left(\int_y^1 f(x,y)\, dx \right) dy$

Summary: Integration over projectable domain

Suppose that $f : G \to \mathbb{R}$ is continuous, and G is y-projectable. Then

$$\int\int_G f(x,y)d(x,y)$$

is determined by the following steps:

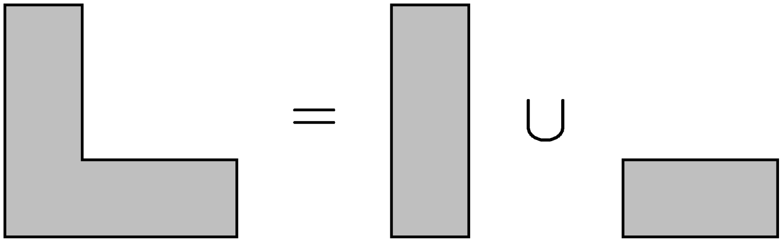
- Determine lower and upper bound for outer integration variable x, i.e., $a \leq x \leq b$
- Determine lower and upper bound for inner integration variable y, as function of x. I.e., $\underline{y}(x) \leq y \leq \overline{y}(x)$.
- For each $x \in [a,b]$, determine a primitive F_x of the function $f_x : y \mapsto f(x,y)$. In this step, x is just a constant.
- Compute $\int_a^b F_x(\overline{y}(x)) - F_x(\underline{y}(x))dx$

For x-projectable domains, exchange the roles of x- and y-coordinates.

Further extensions of the Riemann integral

Generalizations are possible and useful for

- Domains that can be pieced together by projectable domains



- Higher dimensions: Given a suitable $f : [a, b] \times [c, d] \times [r, t] \to \mathbb{R}$, we approximate f by constants on subcubes $[x_i, x_{i+1}) \times [y_l, y_{l+1}) \times [z_j, z_{j+1}]$, etc.
  All properties we observed for the two-dimensional integrals generalize to higher-dimensional integrals in a straightforward way.

Summary

- Improper integration requires
  - Computing the primitive
  - Taking limits towards integration boundaries
- Integration over rectangles requires
  - Computing primitives with respect to one of the integration variables
  - Evaluating at the boundaries, and integrating the result over the remaining variable
- Integration over projectable domains requires
  - Computing the integration boundaries for one variable as function of the other
  - Evaluating the inner integral at the boundaries
  - Integrating the result over the remaining variable