Calculus and Linear Algebra for Biomedical Engineering

Week 13: Review of selected topics

H. Führ, Lehrstuhl A für Mathematik, RWTH Aachen, WS 07

Topic overview

1. Sequences, series and limits
2. Checking continuity and differentiability
3. Taylor polynomials

Sequences, series and limits

Relevant questions: Given a sequence $(a_n)_{n\in\mathbb{N}}$,

- Does $\lim_{n\to\infty} a_n$ exist?
- Does $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exist?
- If either of the above limit exist, what is its value?

Tools:

- Convergence criteria
- Breaking down a series/sequence into simpler ones
- Keeping a list of basic sequences for which the answers are known

Criteria for sequence convergence

- Necessary criterion: Boundedness.
  If $(a_n)_{n\in\mathbb{N}}$ converges, then there exist $A, B \in \mathbb{R}$ with $A < a_n < B$, for all $n \in \mathbb{N}$.
- Sufficient criterion: Boundedness and monotonicity.
- Necessary and sufficient criterion: Cauchy criterion.
  $(a_n)_{n\in\mathbb{N}}$ converges iff for every $\epsilon > 0$ there exists $N(\epsilon)$ such that $|a_n - a_k| < \epsilon$ for all $n, k > N(\epsilon)$.

Known limits

- If $s > 0$, then

$$\lim_{n\to\infty} n^\alpha s^n = \begin{cases} \infty & s > 1, \text{ or } s = 1, \alpha > 0 \\ 1 & s = 1, \alpha = 0 \\ 0 & s < 1, \text{ or } s = 1, \alpha < 0 \end{cases}$$

- If

$$a_n = \frac{s_k n^k + s_{k-1} n^{k-1} + \ldots + s_0}{t_m n^m + t_{m-1} n^{m-1} + \ldots + t_0},$$

  with $s_k, t_m \neq 0$ and $m, k \in \mathbb{N}_0$, then

$$\lim_{n\to\infty} a_n = \begin{cases} \pm\infty & k > m \\ \frac{s_k}{t_k} & k = m \\ 0 & k < m \end{cases}$$

- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Computing limits from known limits

Suppose we know that

$$\lim_{n\to\infty} a_n = a \ , \quad \lim_{n\to\infty} b_n = b$$

then, for all $r, s \in \mathbb{R}$

$$\lim_{n\to\infty} ra_n + sb_n = ra + sb \ , \lim_{n\to\infty} a_n b_n = ab \ , \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b} \ , \qquad (1)$$

where the last limit is defined if $b \neq 0$.
Furthermore, if f is a function continuous at a, then

$$\lim_{n\to\infty} f(a_n) = f(a) \ .$$

Example

We want to compute the limit

$$\lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^4+1}}$$

We first rewrite the expression as

$$\frac{n^2+1}{\sqrt{n^4+1}} = \sqrt{\frac{(n^2+1)^2}{n^4+1}} = \sqrt{\frac{n^4+2n^2+1}{n^4+1}}$$

The expression in the square root converges to 1 (see slide 4). The square root function is continuous at 1, therefore

$$\lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^4+1}} = \lim_{n\to\infty} \sqrt{\frac{n^4+2n^2+1}{n^4+1}} = \sqrt{1} = 1 \ .$$

Criteria for series convergence

A necessary condition for the convergence of the series $\sum_{n=1}^{\infty} a_n$ is that

$$\lim_{n\to\infty} a_n = 0 \ .$$

Sufficient conditions are

(a) Majorant criterion: Let $\sum_{n=0}^{\infty} z_n$ be an absolutely convergent series such that $|x_n| < |z_n|$. Then $(x_n)_{n\in\mathbb{R}}$ converges absolutely.

(b) Quotient criterion: If there exists a constant c with $0 < c < 1$, such that for all $n \in \mathbb{N}$, with $n > M$, $\left|\frac{x_{n+1}}{x_n}\right| < c$, then $\sum_{n=0}^{\infty} x_n$ converges absolutely.

(c) Leibniz criterion: Suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to zero, and fulfills $|x_{n+1}| < |x_n|$ as well as $x_{n+1} \cdot x_n \leq 0$. Then $\sum_{n=0}^{\infty} x_n$ converges.

Known series

- Harmonic series: $\sum_{n=1}^{\infty} n^{-1} = \infty$. Thus, for $\alpha < -1$, $\sum_{n=1} n^{-\alpha} = \infty$.
- For $\alpha > 1$, $\sum_{n=1}^{\infty} n^{-\alpha} < \infty$.
- Geometric series: If $|q| < 1$, then

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \ .$$

- Exponential series: For fixed $x \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \ .$$

Sums of sequences

Suppose we know that

$$\sum_{n=1}^{\infty} a_n = a \ , \ \ \sum_{n=1}^{\infty} b_n = b$$

with $a, b \in \mathbb{R}$. Then, for all $r, s \in \mathbb{R}$

$$\sum_{n=1}^{\infty} ra_n + sb_n = ra + sb \ . \tag{2}$$

In particular, $\sum_{n=1}^{\infty} ra_n + sb_n$ exists.

Example: WS 5, Ex. 7 (k)

We want to determine whether $\sum_{n=1}^{\infty} a_n$ converges, for

$$a_n = \left(\frac{n}{n+1}\right)^n$$

We observe that

$$\begin{aligned} \left(\frac{n}{n+1}\right)^n &= \left(1 - \frac{1}{n+1}\right)^n \\ &= \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{1-\frac{1}{n+1}} . \end{aligned}$$

Here the denominator converges to 1, and the enumerator converges to $1/e$. Hence $a_n \to 1/e$, in particular $a_n \not\to 0$, and the series diverges.

Example: WS 5, Ex. 7 (j)

We want to determine whether $\sum_{n=1}^{\infty} a_n$ converges, for

$$a_n = \frac{k^{2n}}{n!} \,.$$

Here $k \in \mathbb{N}$ is a constant. We consider the quotient

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{k^{2n+2} n!}{(n+1)! k^{2n}} = \frac{k^2}{n+1} \,.$$

For all $n > 2k^2$, we find that

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{2} \,.$$

Then the quotient criterion implies that $\sum_{n=1}^{\infty} a_n$ converges. In fact,

$$\sum_{n=1} \frac{k^{2n}}{n!} = -1 + \sum_{n=0}^{\infty} \frac{\left(k^2\right)^n}{n!} = e^{k^2} - 1 \,.$$

Example: WS 5, Ex. 7 (m)

We want to determine whether $\sum_{n=1}^{\infty} a_n$ converges, for

$$a_n = \underbrace{\frac{2}{n^2}}_{b_n} + \underbrace{(-1)^n \frac{n^2+1}{n^3}}_{c_n} .$$

We read off from the above list that $\sum_{n=1}^{\infty} n^{-2}$ converges, and thus

$$\sum_{n=1}^{\infty} b_n < \infty .$$

For $\sum_{n=1}^{\infty} c_n$ we observe that the sign of two consecutive series elements changes. Moreover, $c_n \to 0$, hence it remains to check that $(|c_n|)_{n\in\mathbb{N}}$ is eventually strictly decreasing (Leibniz criterion). But

$$|c_n| = \frac{n^2+1}{n^3} = \frac{1}{n} + \frac{1}{n^3} ,$$

which is strictly decreasing. Hence $\sum_{n=1}^{\infty} c_n$ converges.

Checking continuity and/or differentiability

Recall from Lecture 7:

- Every differentiable function is continuous.
- “Most” functions possessing an explicit expression are differentiable
- Exceptions: Piecewise defined functions, points outside the domain of definition.

Problem: For a function f, given as an expression using known functions, determine the domain of continuity (differentiability)

- Determine the set S of $x \in \mathbb{R}$ for which $f(x)$ is not well-defined (for differentiability, consider the value $|0|$ as not well-defined).
- On $\mathbb{R} \setminus S$, f is continuous (differentiable)
- For $x \in S$, check continuity (differentiability) by considering $\lim_{y \to x} f(y)$ ($\lim_{y \to x} f'(y)$).

Known function classes

- For $\alpha \in \mathbb{R}$, the function $f(x) = x^\alpha$, is differentiable on $(0, \infty)$ with derivative $f'(x) = \alpha x^{\alpha-1}$. This includes the constant function $f(x) = 1 = x^0$, with derivative $f'(x) = 0$.
  For $n \in \mathbb{N}$, the function $f(x) = x^n$ is differentiable on $\mathbb{R}$, with derivative $f'(x) = nx^{n-1}$. The quotient rule entails for $g(x) = x^{-n}$, that $f'(x) = -nx^{-n+1}$.
- As a consequence, polynomials $f : \mathbb{R} \to \mathbb{R}$ are differentiable.
- $f(x) = |x|$ is continuous on $\mathbb{R}$, differentiable on $\mathbb{R} \setminus \{0\}$.
- Trigonometric functions: E.g., $\sin, \cos$ are differentiable, with $\sin' = \cos$, $\cos' = -\sin$.
- The exponential function is differentiable, with $\exp' = \exp$.
- Any function that is obtained from the above list by composition and/or algebraic operations, is again differentiable wherever it is well-defined.

Example: WS 8, Ex. 2 (e)

Problem: Given

$$f(x) = \frac{x^2|x+1|}{|x-2|},$$

decide where f is continuous, where it is differentiable, compute f'.

Note: f is is made up of known functions.
$f(x)$ is not defined for $x = 2$.
For $x = -1$, the enumerator may not be differentiable.

$\Rightarrow f$ is continuous on $\mathbb{R} \setminus \{2\}$, and differentiable on $\mathbb{R} \setminus \{-1, 2\}$.

For $x \to 2$, we find $f(x) \to \infty$. In particular, f is not continuous at 2.

Computing derivatives

For the computation of f', we write

$$f(x) = \begin{cases} \frac{x^2(x+1)}{x-2} & x < -1 \\ \frac{-x^2(x+1)}{x-2} & -1 < x < 2 \\ \frac{x^2(x+1)}{x-2} & 2 < x \end{cases}$$

Each of the pieces is now differentiated separately, which yields

$$f'(x) = \begin{cases} \dfrac{(3x^2+2x)(x-2) - x^2(x+1)}{(x-2)^2} & x < -1 \text{ or } x > 2 \\ -\dfrac{(3x^2+2x)(x-2) - x^2(x+1)}{(x-2)^2} & -1 < x < 2 \end{cases}$$

As $x \to -1$, $x < -1$, we obtain $f'(x) \to -1$, and $f'(x) \to 1$, for $x \to -1, x > 1$. Thus f is not differentiable at -1.

Taylor polynomial and residue

Problem: Given $n+1$ times differentiable $f : (a,b) \to \mathbb{R}$ and x_0, compute the Taylor polynomial

$$T_{n,x_0}(y) = f(x_0) + f'(x_0)(y-x_0) + \frac{f''(x_0)}{2}(y-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(y-x_0)^n$$

and estimate the residue.

$$R_{n,x_0}(y) = f(y) - T_{n,x_0}(y)$$

Necessary steps:

- Compute $f'(x_0), f''(x_0), \ldots, f^{(n+1)}(x_0)$, write down T_{n,x_0}.
- For an estimate of R_{n,x_0}, find a constant $B > 0$ such that

  $$\forall y \in (a,b) \ : \ |f^{(n+1)}(y)| \leq M$$

  Then

  $$\forall y \in (a,b) \ : \ |R_{n,x_0}(y)| \leq \frac{M(b-a)^{n+1}}{(n+1)!} \ .$$

Example: WS 8, Ex. 6 (b)

Problem: Compute the Taylor polynomial T_{2,x_0}, for $x_0 = \pi/2$,

$$f(x) = \cos(\frac{\pi}{4}\sin(x)) \; .$$

and estimate the remainder $R_{2,x_0}(y)$ for $|y - x_0| < 0.1$.
We compute the derivatives

$$\begin{aligned} f'(x) &= -\sin(\frac{\pi}{4}\sin(x))\frac{\pi}{4}\cos(x) \\ f''(x) &= -\cos(\frac{\pi}{4}\sin(x))\left(\frac{\pi}{4}\right)^2\cos^2(x) + \sin(\frac{\pi}{4}\sin(x))\frac{\pi}{4}\sin(x) \end{aligned}$$

Using $\sin(\pi/2) = 1$, $\cos(\pi/2) = 1$, $\sin(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$, we find

$$\begin{aligned} f(x_0) &= \sqrt{2}/2 \; , \; f'(x_0) = 0 \; , \; f''(x_0) = \frac{\sqrt{2}\pi}{8} \\ T_{2,x_0} &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{16}(x - \frac{\pi}{2})^2 \end{aligned}$$

Example: WS 8, Ex. 6 (b)

An estimate of the error term requires an upper bound for $|f'''(x)|$:

$$\begin{aligned} f'''(x) &= \sin(\frac{\pi}{4}\sin(x))\left(\left(\frac{\pi}{4}\right)^3\cos^3(x)+\frac{\pi}{4}\cos(x)\right) \\ &+ 3\left(\frac{\pi}{4}\right)^2\cos(\frac{\pi}{4}\sin(x))\sin(x)\cos(x) \end{aligned}$$

for all x with $|x-\frac{\pi}{2}|<0.1$. We use a rather crude estimate, namely $|\sin(x)|<1$, $|\cos(x)|<1$, and obtain

$$|f'''(x)| \le \left(\frac{\pi}{4}\right)^3+\left(\frac{\pi}{4}\right)+3\left(\frac{\pi}{4}\right)^2 \approx 3.1204221$$

Accordingly, if $|x-\frac{\pi}{2}|<0.1$, then

$$|R_{2,x_0}(x)| \le \frac{3.1204221}{3!}\left|x-\frac{\pi}{2}\right|^3 < 0.00052007$$