Week 2: Vector spaces, subspaces and geometry
Tuples or row vectors

Definition.
For \( n \in \mathbb{N} \), and \( x_1, \ldots, x_n \in \mathbb{R} \), we denote the associated \( n \)-tuple or row vector by \((x_1, \ldots, x_n)\). Two row vectors \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) are equal precisely when

\[ x_1 = y_1 \land x_2 = y_2 \land \ldots \land x_n = y_n. \]

Remarks

Note the difference of tuples to sets: \( \{1, 2, 4\} = \{4, 2, 1\} \), but \((1, 2, 4) \neq (4, 2, 1)\).
Interpretation of tuples

Tuples are ordered collections of data. For instance, suppose you want to record, for a group of people,

- shoe size (german units),
- height (in cm), and
- weight (in kg).

This amounts to recording a 3-tuple (or triple) of numbers for each person, e.g., in the order shoe size, height, weight.

Here it is clear that the tuples $(43, 180, 75)$ and $(75, 180, 43)$ are vastly different.
Column vectors. The set $\mathbb{R}^n$

Definition.
For $n \in \mathbb{N}$, and $x_1, \ldots, x_n \in \mathbb{R}$, we denote the associated column vector by

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\text{ or } (x_1, \ldots, x_n)^T.
\]

We define the $n$-dimensional Euclidean space as the set of column vectors

\[
\mathbb{R}^n = \{(x_1, \ldots, x_n)^T : x_1, \ldots, x_n \in \mathbb{R}\}
\]

Elements of $\mathbb{R}^n$ are denoted as $\mathbf{x} = (x_1, \ldots, x_n)^T$. The origin is the vector $\mathbf{0} = (0, \ldots, 0)^T \in \mathbb{R}^n$. 

Vector space operations

Definition.
Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, and $s \in \mathbb{R}$.
Vector addition/subtraction: The sum of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)^T, \quad \mathbf{x} - \mathbf{y} = (x_1 y_1, \ldots, x_n - y_n)^T$$

(1)

Multiplying a vector with a scalar: Scalar multiplication of $s \in \mathbb{R}$ with the vector $\mathbf{x}$ is defined as

$$s \cdot (x_1, \ldots, x_n)^T = (sx_1, \ldots, sx_n)^T.$$  

(2)

The $\cdot$ is often omitted.

Remarks: The definition of the addition generalizes addition in $\mathbb{R} = \mathbb{R}^1$ and in $\mathbb{C} = \mathbb{R}^2$. 
Theorem.
\( \mathbb{R}^n \) fulfills the vector space axioms: Let \( a, b, c \in \mathbb{R}^n \) be arbitrary vectors, and \( s, t \in \mathbb{R} \).

V.1 \( a + b = b + a \)

V.2 \( (a + b) + c = a + (b + c) \)

V.3 \( 0 = (a - a) = 0 \cdot a \)

V.4 \( s(a + b) = sa + sb \).

V.5 \( (s + t)a = sa + ta \)

V.6 \( a = 1a = 0 + a \)
Geometric interpretation of vectors

For $n = 1, 2, 3$, we can think of $\mathbb{R}^n$ as a straight line, a plane and as three-dimensional space, respectively. Elements are visualized both as points or as arrows connecting the points with the origin 0.

Points in $n$-dimensional space, $n = 1, 2, 3$
Geometric interpretation of addition

The sum of two vectors $a, b$ corresponds to the diagonal of the parallelogram with sides $a$ and $b$.

Illustration of sum and difference
Scalar product and length

Definition. Let $a, b \in \mathbb{R}^n$.

1. The scalar product of $a$ and $b$ is defined as

$$a \cdot b = a_1b_1 + a_2b_2 \ldots + a_nb_n.$$ 

2. $|a| = \sqrt{a \cdot a}$ is called length or Euclidian norm of $a$. The distance between two vectors $a$ and $b$ is $|a - b|$.

3. If $a \cdot b = 0$, then $a$ and $b$ are called orthogonal, and we write $a \perp b$.

Remarks:

► Warning: Do not confuse scalar product with scalar multiplication! In scalar products, the $\cdot$ is not omitted.

► The length of a vector generalizes the length of complex numbers.
Properties of scalar product and length

Theorem. For \( a, b, c \in \mathbb{R}^n \) and \( s \in \mathbb{R} \).

(i) \( (sa) \cdot b = s(a \cdot b) \)

(ii) \( a \cdot b = b \cdot a \)

(iii) \( (a + b) \cdot c = a \cdot c + b \cdot c \)

(iv) \( |a| \geq 0 \), with \( |a| = 0 \) only for \( a = 0 \)

(v) \( |sa| = |s| |a| \)

(vi) Cauchy-Schwarz inequality: \( |a \cdot b| \leq |a| |b| \)

(vii) Triangle inequality: \( |a - b| \leq |a + b| \leq |a| + |b| \)
Geometric interpretation of scalar multiplication

As a consequence of part (v) of the theorem: Multiplication by a scalar $s > 0$ amounts to multiplying the length with $s$. Multiplication by $s = -1$ results in a vector pointing in the opposite direction.
Angle and projection

Definition.
Let $a, b \in \mathbb{R}^n \setminus \{0\}$.

(i) The orthogonal projection of $a$ onto $b$ is defined as

$$\text{proj}_b a = \frac{a \cdot b}{|b|^2} b$$

(ii) The angle between $a$ and $b$ is defined as the unique $\alpha \in [0, \pi)$ satisfying

$$\cos(\alpha) = \frac{a \cdot b}{|a| |b|}.$$ 

Note that if $|b| = 1$, the two notions are related via

$$|\text{proj}_b a| = |a| \cos(\alpha).$$
Illustration of angle and projection

Orthogonal projection (left) and angle (right). Projection amounts to dropping a perpendicular from \( a \) onto \( b \).
Theorem. Let $a, b \in \mathbb{R}^n$ with angle $\alpha$, and $c = a - b$.

(i) Cosine Theorem:

$$|c|^2 = |a|^2 + |b|^2 - 2|a||b| \cos(\alpha) .$$  \hspace{1cm} (3)

(ii) Pythagoras’ Theorem: If $a \perp b$, then

$$|c|^2 = |a|^2 + |b|^2 .$$  \hspace{1cm} (4)
**Straight lines**

**Definition.** Let $a, b \in \mathbb{R}^n$, with $b \neq 0$. The straight line through $a$ with direction $b$ is the set

$$L = \{c = a + sb : s \in \mathbb{R}\}. \tag{5}$$

The description (5) is called **parametric form** of $L$, $b$ is called its direction vector.

**Remark:** The line $L$ does not change, if we replace

- $a$ by $a' = a + sb$, and
- $b$ by $b' = rb$.

I.e., the line does not depend on the **length** of the direction vector.
Parallel lines

Definition. Let $L, L' \subset \mathbb{R}^n$ be straight lines, such that

1. $L$ is the straight line through $a$ with direction $b$;
2. $L'$ is the straight line through $a'$ with direction $b'$; and
3. there exists real number $s$ such that $b' = sb$.

Then $L$ and $L'$ are called parallel.
Theorem. Let $L, L' \subset \mathbb{R}^2$ denote straight lines. Then precisely one of the following three cases can occur:

1. $L = L'$;

2. $L$ and $L'$ are parallel, with $L \cap L' = \emptyset$;

3. $L \cap L' = \{x\}$, for a suitable $x \in \mathbb{R}^2$.

Hence, the intersection of two straight lines consists either of zero, one or infinitely many points. How does one decide which case applies? And how does one compute the intersection?
**Theorem.** Let \( \mathbb{L} \subset \mathbb{R}^n \) be a straight line

(i) \( \mathbb{L} \) is uniquely determined by two points \( x, y \in \mathbb{L} \), with \( x \neq y \): Defining \( b = x - y \), one has

\[
\mathbb{L} = \{ c = x + sb : s \in \mathbb{R} \} \quad .
\]

(ii) Assume that \( n = 2 \). There exists a vector \( \mathbf{n} \in \mathbb{R}^2 \) with \( |\mathbf{n}| = 1 \), and \( r \geq 0 \) such that

\[
\mathbb{L} = \{ c \in \mathbb{R}^2 : c \cdot \mathbf{n} = r \} \quad .
\]

Part (i) is very convenient for defining lines, whereas part (ii) will turn out useful for calculations. The equation (10) is called Hesse’s normal form of the line \( \mathbb{L} \), and \( \mathbf{n} \) is the normal vector of \( \mathbb{L} \).
Applications of Hesse’s normal form

Theorem
Consider two straight lines $L, M \subset \mathbb{R}^2$, given by the equations

$$L = \{ c \in \mathbb{R}^2 : c \cdot n = r \} \text{ and } M = \{ c \in \mathbb{R}^2 : c \cdot m = s \},$$

with $n, m$ of length 1, and $r, s > 0$. Then the following statements are true:

(i) $L$ is uniquely defined by $n$ and $r > 0$: $L = M$ if and only if $n = m$ and $r = s$.

(ii) $L \cap M = \emptyset$ if and only if $(n = m$ and $r \neq s$) or $(n = -m)$.
Computation of Hesse’s normal form

Given a line \( L \subset \mathbb{R}^2 \) in parametric form

\[
L = \{ \mathbf{c} = \mathbf{a} + s\mathbf{b} : s \in \mathbb{R} \} ,
\]

how does one compute its Hesse normal form? We are looking for \( \mathbf{n} \) and \( r > 0 \) such that, in particular,

\[
\mathbf{n} \cdot \mathbf{a} = r
\]
\[
\mathbf{n} \cdot (\mathbf{a} + \mathbf{b}) = r .
\]

Subtracting the upper from the lower equation, we obtain

\[
\mathbf{n} \cdot \mathbf{b} = 0 , \text{ which means } \mathbf{n} \perp \mathbf{b} , \text{ or } n_1b_1 + n_2b_2 = 0 .
\]

Together with the requirement \( \mathbf{n} = 1 \), this leaves two possible solutions

\[
\mathbf{n}_{1,2} = \pm \frac{(b_2, -b_1)}{|\mathbf{b}|} .
\]

Among these, we pick \( \mathbf{n} \) in such a way that \( \mathbf{n} \cdot \mathbf{a} \geq 0 \), and let \( r = \mathbf{n} \cdot \mathbf{a} \).
Minimal distance of a point to a line

Given a line \( \mathbb{L} \subset \mathbb{R}^2 \) and a point \( x \in \mathbb{R}^2 \), we define the distance of \( x \) to \( \mathbb{L} \) as

\[
\text{dist}(x, \mathbb{L}) = \min_{c \in \mathbb{L}} |c - x|
\]

**Theorem.** Let \( \mathbb{L} \) be given in Hesse normal form,

\[
\mathbb{L} = \{ c \in \mathbb{R}^2 : c \cdot n = r \}
\]

Then the distance of a point \( x \) to line \( \mathbb{L} \) is computed as

\[
\text{dist}(x, \mathbb{L}) = |x \cdot n - r|
\]

The point on \( \mathbb{L} \) with smallest distance to \( x \) is computed as

\[
c_0 = x + (r - x \cdot n)n
\]
Linear combinations and linear independence

Definition. Let $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$.

1. $b \in \mathbb{R}^n$ is called linear combination of $a_1, \ldots, a_m$ if there exist coefficients $s_1, \ldots, s_m \in \mathbb{R}$ such that

   $$b = s_1a_1 + s_2a_2 + \ldots + s_m a_m.$$

2. The system $(a_j)_{j=1,\ldots,m}$ is called linearly dependent if, for some index $1 \leq i \leq m$, the vector $a_i$ is a linear combination of $(a_j)_{j \neq i}$. Otherwise, it is called linearly independent.

Theorem. (Uniqueness of coefficients) Let $a_1, \ldots, a_m$ be linearly independent, and assume that

$$b = s_1a_1 + s_2a_2 + \ldots + s_m a_m$$

$$= t_1a_1 + t_2a_2 + \ldots + t_m a_m$$

Then $t_1 = s_1$, $t_2 = s_2$, $\ldots$, $t_m = s_m$.
Example for linear independence

Consider the vectors \( \mathbf{a} = (1, 0) \), \( \mathbf{b} = (0, 1)^T \) and \( \mathbf{c} = (1, 1)^T \). Then \( \mathbf{a}, \mathbf{b} \) are linearly independent:

\[
\text{For all } s \in \mathbb{R} : s \mathbf{a} = (s, 0)^T \neq (0, 1)^T = \mathbf{b}.
\]

One shows similarly that \( \mathbf{a}, \mathbf{c} \) are linearly independent, as well as \( \mathbf{b}, \mathbf{c} \).

But: The system \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) is linearly dependent: \( \mathbf{c} = \mathbf{a} + \mathbf{b} \).
The span of a system of vectors

Definition. Let $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$. The span of these vectors is the set

$$\text{span}(a_1, a_2, \ldots, a_m) = \{b \in \mathbb{R}^n : b \text{ is a linear combination of } a_1, a_2, \ldots, a_m\}$$

Theorem. Let $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$, and let $U = \text{span}(a_1, a_2, \ldots, a_m)$. Then $U$ is a subspace, i.e. it fulfills

S.1 $0 \in U$.

S.2 If $b, c \in U$, then $b + c \in U$.

S.3 If $s \in \mathbb{R}$, and $b \in U$, then $sb \in U$.

Remark: The theorem expresses that $U$ is closed under vector addition and scalar multiplication.
Definition of a plane

Definition. Let $a, b, c \in \mathbb{R}^n$, with $a, b$ linearly independent. The plane through $c$ spanned by $a$ and $b$ is the set

$$\mathbb{P} = \{ x \in \mathbb{R}^n : x = c + y, \text{ with } y \in \text{span}(a, b) \}$$

$$= \{ x \in \mathbb{R}^n : x = c + sa + tb, s, t \in \mathbb{R} \}$$

Examples

- The set $\{(r, s, 1)^T : r, s \in \mathbb{R}\}$ is the plane through $(0, 0, 1)^T$ spanned by the vectors $(1, 0, 0)^T$ and $(0, 1, 0)^T$.

- $\mathbb{R}^2$ is the only plane contained in $\mathbb{R}^2$: If $a, b \in \mathbb{R}^2$ are linearly independent, $\text{span}(a, b) = \mathbb{R}^2$. As a consequence, every plane $P \subset \mathbb{R}^2$ fulfills $P = \mathbb{R}^2$. 

Alternative definitions of planes

Theorem. Let $\mathbb{P} \subset \mathbb{R}^n$ be a plane

(i) $\mathbb{P}$ is uniquely determined by three points $x, y, z \in \mathbb{P}$, subject to the condition that

$$a = x - z \text{ and } b = y - z$$

are linearly independent. In this case,

$$\mathbb{P} = \{ c = z + sa + tb : s, t \in \mathbb{R} \} \, .$$

(ii) Assume that $n = 3$. There exists a vector $n \in \mathbb{R}^3$ with $|n| = 1$, and $r \geq 0$ such that

$$\mathbb{P} = \{ c \in \mathbb{R}^3 : c \cdot n = r \} \, .$$

The equation (10) is called Hesse’s normal form of the plane $\mathbb{P}$, and $n$ is called the normal vector of the plane.
Theorem. Let $P, P' \subset \mathbb{R}^3$ denote planes. Then precisely one of the following three cases can occur:

1. $P = P'$;
2. $P \cap P'$ is a straight line;
3. $P \cap P' = \emptyset$.

As before, the different cases are best sorted out using the Hesse normal forms of the two planes.
Applications of Hesse’s normal form for planes

Theorem
Consider planes $\mathbb{P}, \mathbb{M} \subset \mathbb{R}^3$, given by the equations

$$
\mathbb{P} = \{ \mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{n} = r \} \quad \text{and} \quad \mathbb{M} = \{ \mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{m} = s \},
$$

(11)

with $\mathbf{n}, \mathbf{m}$ of length 1, and $r, s > 0$. Then the following statements are true:

(i) $\mathbb{P}$ is uniquely defined by $\mathbf{n}$ and $r > 0$: $\mathbb{P} = \mathbb{M}$ if and only if $\mathbf{n} = \mathbf{m}$ and $r = s$.

(ii) $\mathbb{P} \cap \mathbb{M} = \emptyset$ if and only if $(\mathbf{n} = \mathbf{m}$ and $r \neq s$) or $(\mathbf{n} = -\mathbf{m})$.

(iii) $\mathbb{P} \cap \mathbb{M}$ is a straight line if and only if $\mathbf{n} \neq \pm \mathbf{m}$.

Remaining question: How to compute the HNF of a plane.
Cross product of two vectors

Definition. Given \( a, b \in \mathbb{R}^3 \), the cross product \( a \times b \) is defined as

\[
\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.
\]  

(12)

Theorem: Properties of the cross product.
For all \( a, b, c \in \mathbb{R}^3 \) and \( r \in \mathbb{R} \),

(i) \( r(a \times b) = (ra) \times b = a \times (rb) \), as well as \( (a + b) \times c = a \times c + b \times c \).

(ii) \( a \times b = -b \times a \), in particular \( a \times a = 0 \).

(iii) \( a \times b \) is orthogonal to \( a \) and \( b \).

(iv) If \( a, b \) have angle \( \alpha \), then \( |a \times b| = |a||b| \sin(\alpha) \) (area of the parallelogram with sides \( a, b \)).
Computation of Hesse’s normal form for planes

Given a plane \( P \subset \mathbb{R}^2 \) in parametric form

\[
P = \{ x = z + sa + tb : s, t \in \mathbb{R} \},
\]

we are looking for \( n \) with \( |n| = 1 \) and \( r > 0 \), meeting the requirements

\[
n \cdot a = n \cdot b = 0
\]

\[
n \cdot z = r.
\]

Using the properties of the cross product, we see that

\[
n = \pm \frac{1}{|a| |b| \sin(\alpha)} a \times b
\]

is the suitable candidate. The sign of the right hand side is chosen to guarantee that \( n \cdot z \geq 0 \), and let \( r = n \cdot z \).
Minimal distance of a point to a plane

Given a plane $\mathbb{P} \subset \mathbb{R}^3$ and a point $x \in \mathbb{R}^3$, we define the distance of $x$ to $\mathbb{P}$ as

$$\text{dist}(x, \mathbb{P}) = \min_{c \in \mathbb{P}} |c - x|$$

Theorem. Let $\mathbb{P}$ be given in Hesse normal form,

$$\mathbb{P} = \{ c \in \mathbb{R}^3 : c \cdot n = r \}$$

Then the distance of a point $x$ to $\mathbb{P}$ is computed as

$$\text{dist}(x, \mathbb{P}) = |x \cdot n - r| .$$

The point in $\mathbb{P}$ with smallest distance to $x$ is computed as

$$c_0 = x + (r - x \cdot n)n .$$
Summary

Important notions

- The vector space $\mathbb{R}^n$ and operations defined on it.
- Geometric interpretations of vector addition and scalar multiplication.
- Scalar product and Euclidian length.
- Linear combinations and linear independence.
- Subspaces, lines and planes.
- Hesse’s normal form for lines in $\mathbb{R}^2$ and for planes in $\mathbb{R}^3$.
  - How to compute it, and
  - how to use it (e.g., computing distances)