Calculus and Linear Algebra for Biomedical Engineering

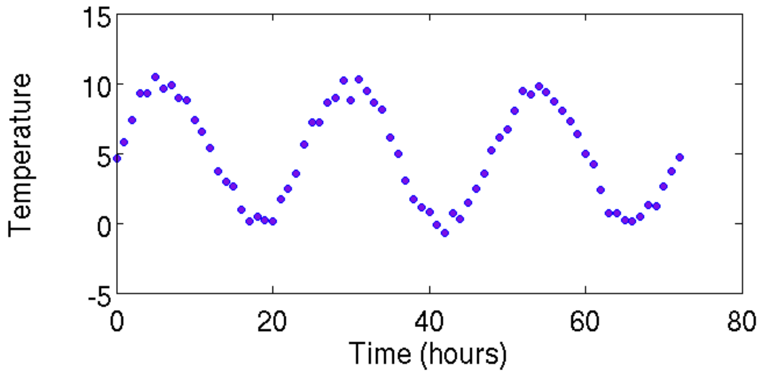
Week 5: Functions and graphs

H. Führ, Lehrstuhl A für Mathematik, RWTH Aachen, WS 07

Motivation: Measurements at fixed intervals

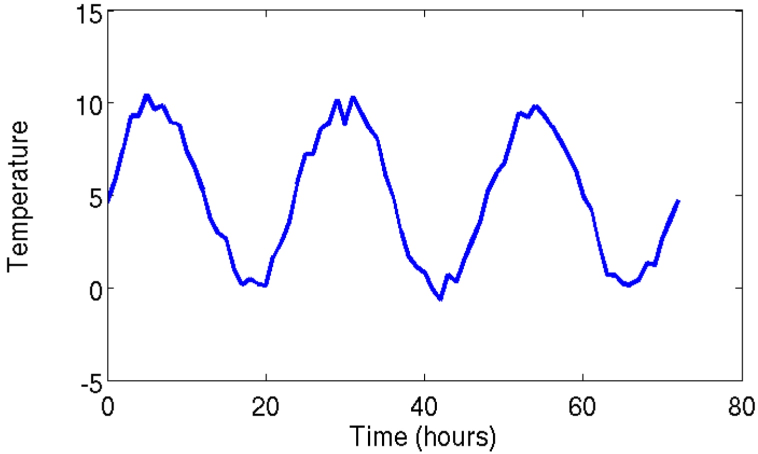
Consider a sequence $y_0, y_1, y_2, \ldots$ of real numbers, obtained e.g. by measuring the temperature at a given spatial point, at times $t = 0, 1, 2, \ldots$. (in hours)

Standard visualization of data as scatter plot:



Measurements at arbitrary points in time

The one hour time interval is clearly arbitrary, we could have taken measurements at times $t = 0.0, 0.1, 0.2, \ldots$, or even at $t = \sqrt{2}, \pi, \ldots,$

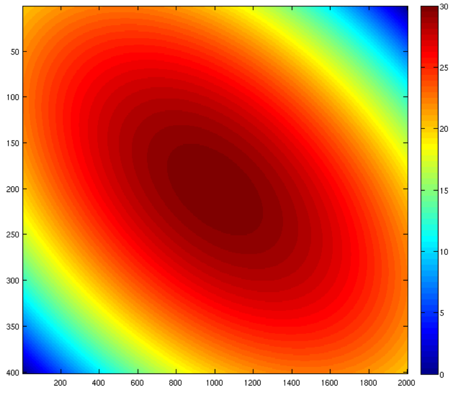


In other words, we think of temperature as a function of the real variable t.

Measurements at arbitrary points in time and space

We could also imagine more elaborate measurements, say measuring temperature in more than one point. Again, the points could be arbitrary, and it makes sense to think of temperature as a function of several (spatial and temporal) variables.

Example: Heat distribution in a two-dimensional object (color-coded)



Definition of functions

Definition. Let $n, m \in \mathbb{N}$, and $D \subset \mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^m$ is a rule that assigns each $\mathbf{x} \in D$ a unique element $\mathbf{y} \in \mathbb{R}^m$. This element is denoted as $f(x)$. We also write $f : D \ni \mathbf{x} \mapsto f(\mathbf{x})$. A mapping $f : D \to \mathbb{R}$ is called a function.

Examples:

- $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \frac{x}{4} - \frac{1}{2}$ is an affine function
- $f : \mathbb{R}_0^+ \to \mathbb{R}$, $f(x) = \sqrt{x}$
- An example of a piecewise defined function is

$$f(x) = \begin{cases} 1 & x > 0 \\ x/2 & x \leq 0 \end{cases}$$

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, with $f(x) = |x|$

Further examples

- A matrix $A \in \mathbb{R}^{m \times n}$ defines $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ via $f(\mathbf{x}) = A \cdot \mathbf{x}$.
- By assigning each element z its polar coordinates, we define $f : \mathbb{C} \setminus \{0\} \ni z \mapsto (|z|, \arg(z)) \in (0, \infty) \times (-\pi, \pi]$
- Vector addition is a mapping, if we identify pairs $(\mathbf{x}, \mathbf{y})$ of vectors in $\mathbf{R}^n$ with vectors $(x_1, \ldots, x_n, y_1, \ldots, y_n)^T \in \mathbb{R}^{2n}$:

$$+ : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n \ , \ (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y}$$

  Similarly, scalar multiplication is a mapping

$$\cdot \ : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \ , \ (r, x_1, \ldots, x_n)^T \mapsto (rx_1, \ldots, rx_n)^T \ .$$

- Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, with $\mathbf{a}$, $\mathbf{b}$ linearly independent. Let $\mathbb{P}$ denote the plane through $\mathbf{c}$ spanned by $\mathbf{a}$ and $\mathbf{b}$. Then $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$, with $f(r, s) = r\mathbf{a} + s\mathbf{b} + \mathbf{c}$, is a mapping with $f(\mathbb{R}^2) = \mathbb{P}$.

Domain and range of a mapping

Definition. If $f : D \to \mathbb{R}^m$ is a mapping, we call

- D the domain of f
- $f(D) = \{f(\mathbf{x}) : \mathbf{x} \in D\}$ the range of f

Examples

- $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \frac{x}{4} - \frac{1}{2}$, has range $\mathbb{R}$
- $f : \mathbb{R}_0^+ \to \mathbb{R}$, $f(x) = \sqrt{x}$ has range $\mathbb{R}_0^+$
- The piecewise defined function

$$f(x) = \begin{cases} 1 & x > 0 \\ x/2 & x \leq 0 \end{cases}$$

  has range $(-\infty, 0] \cup \{1\}$.
- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, with $f(x) = |x|$, has domain $\mathbb{R}_0^+$.

The graph of a mapping

Definition. Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$. The graph of f is the set

$$G_f = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in D\} \subset \mathbb{R}^{n+m}\ .$$

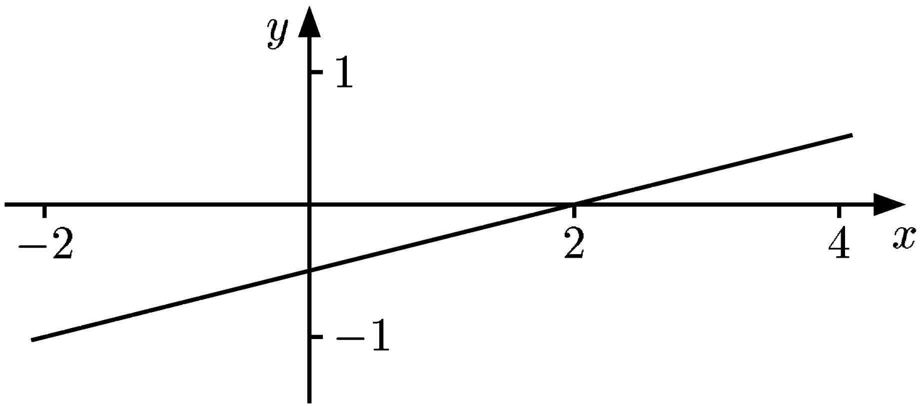
Observations:

1. The graph G_f has the property that for every $\mathbf{x} \in \mathbb{R}^n$ there is at most one $\mathbf{y}$ such that $(\mathbf{x}, \mathbf{y}) \in G_f$
2. Conversely, if $G \subset \mathbb{R}^{n+m}$ has the property from 1., then there is a mapping f such that $G = G_f$.

Visualization of the graph

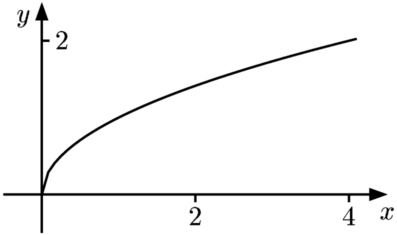
For mappings $f : D \longrightarrow \mathbb{R}$, with $D \subset \mathbb{R}^n$, and $n = 1, 2$, the graph can be visualized. For $n = 1$, the graph is a curve in $\mathbb{R}^2$.

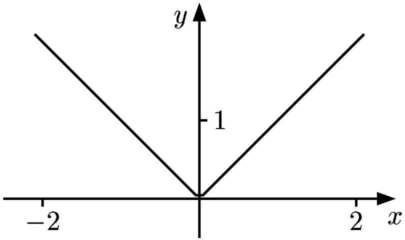
Example: The affine function $f(x) = \frac{x}{4} - \frac{1}{2}$

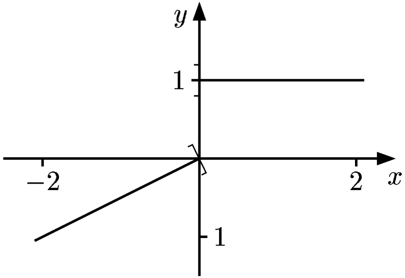


More examples of graphs

Left to right: Square root, absolute value, piecewise defined function

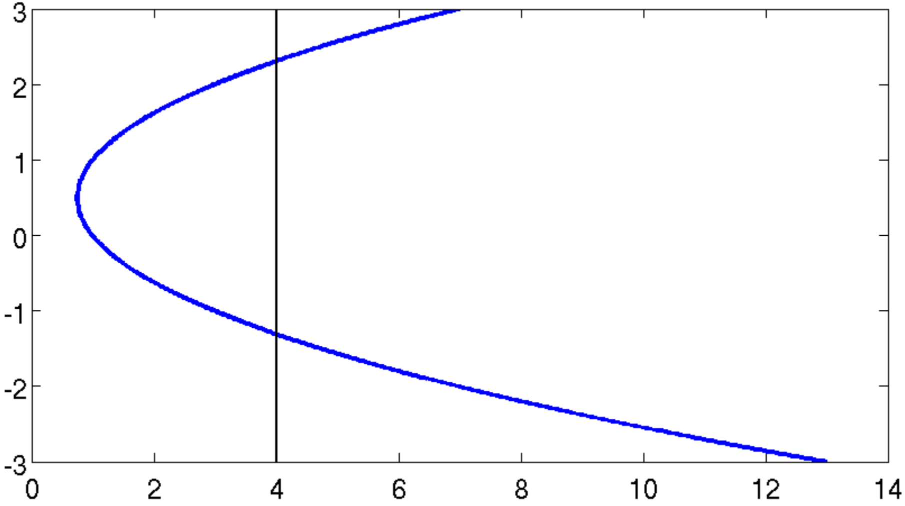






A curve that is not a graph

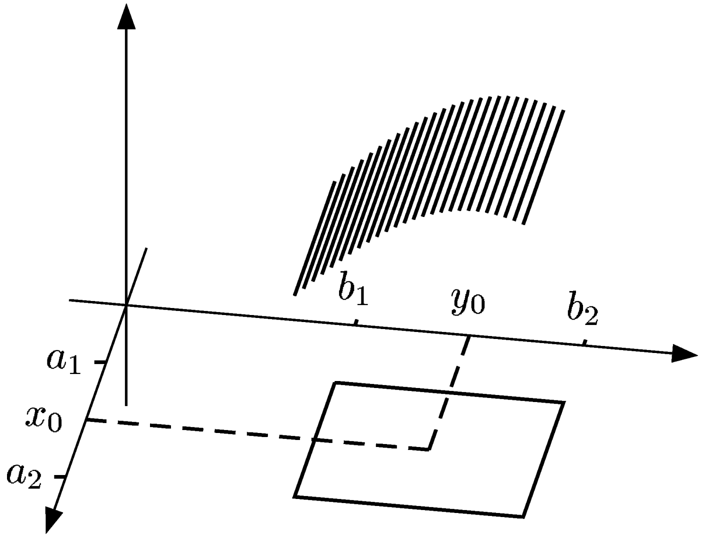
A curve is not a graph if it contains two distinct points with the same x-coordinate.



Graphs as surfaces

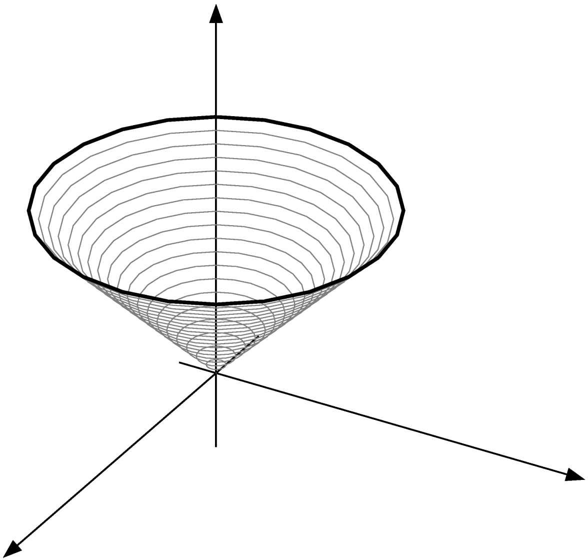
If $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}^2$, the graph can be interpreted as a two-dimensional surface in $\mathbb{R}^3$

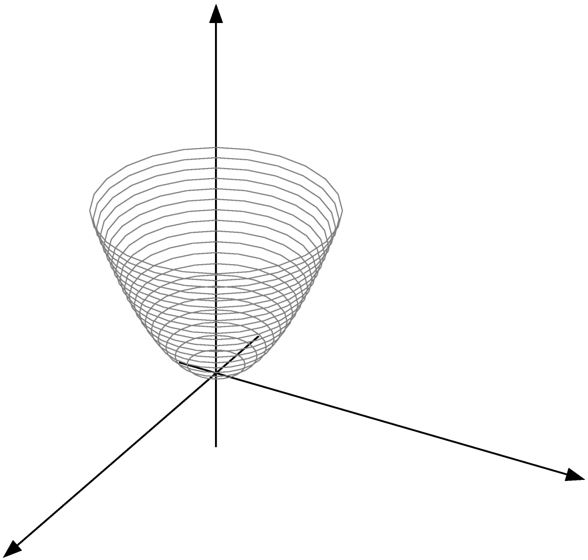
Sketch: G_f for $f : D \to \mathbb{R}$, with $D = [a_1, a_2] \times [b_1, b_2]$



Graphs as surfaces: Examples

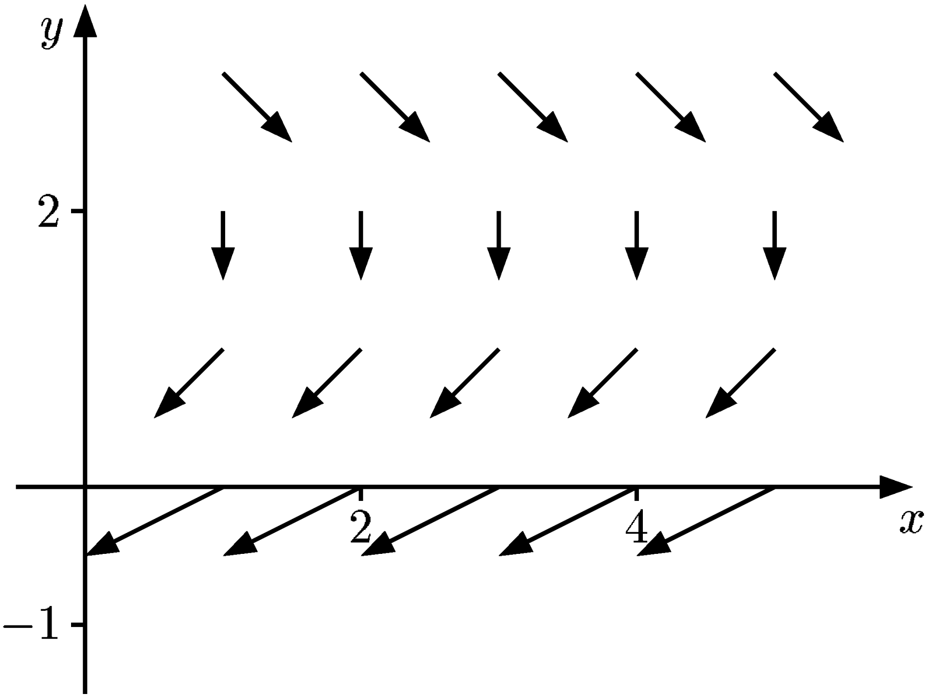
Left: $f(\mathbf{x}) = |\mathbf{x}|$ Right: $f(\mathbf{x}) = |\mathbf{x}|^2$





Graphs and vector fields

For mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$, the graph G_f is visualized as a vector field



Maximal domain of definition

An expression $y = f(x)$ defines a mapping only if

- $f(x)$ is well-defined and
- $f(x)$ is unambiguous.

Definition. Given an expression $\mathbf{y} = f(\mathbf{x})$, the (maximal) domain of definition $D(f)$ is the set of all $\mathbf{x}$, for which $f(\mathbf{x})$ is well-defined.

Examples:

- The function $f(x) = \frac{x-1}{x^2-1}$ has domain $D(f) = \mathbb{R} \setminus \{-1, 1\}$.
- The function $f(x) = \sqrt{1+x}$ has domain $D(f) = [-1, \infty)$.
- The mapping $f(x_1, x_2) = \frac{x_1^2+x_2^2}{x_1-x_2}$ has domain $D(f) = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \neq x_2\}$.

Operations on mappings

Definition.
Let $D \subset \mathbb{R}^n$, and $f, g : D \to \mathbb{R}^m$. If $r, s \in \mathbb{R}$, we define $rf + sg : D \to \mathbb{R}^m$ as

$$(rf + sg)(\mathbf{x}) = rf(\mathbf{x}) + sg(\mathbf{x}) \ , \ (fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x}) \ (\text{ for } m = 1)$$

Furthermore, letting $E = \{\mathbf{x} \in D : g(\mathbf{x}) \neq 0\}$, we define $\frac{f}{g} : E \to \mathbb{R}$ as $\frac{f}{g}(\mathbf{x}) = \frac{f(\mathbf{x})}{g(\mathbf{x})}$

Definition. (Concatenation of mappings)
Let $f : D \to \mathbb{R}^m$, $g : E \to \mathbb{R}^n$, with $E \subset \mathbb{R}^k$. If $g(E) \subset D$, we define

$$f \circ g : E \to \mathbb{R}^m \ , \ (f \circ g)(\mathbf{x}) = f(g(\mathbf{x}))$$

Inverse mapping

Definition.
Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$.

(a) f is called one-to-one or injective on D if for all $\mathbf{x}, \mathbf{y} \in D$ with $\mathbf{x} \neq \mathbf{y}$, $f(\mathbf{x}) \neq f(\mathbf{y})$.

(b) Let $E \subset \mathbb{R}^m$. We write $f : D \to E$ if $f(D) \subset E$. The mapping $f : D \to E$ is called onto or surjective if $f(D) = E$.

(c) $f : D \to E$ is called bijective if it is one-to-one and onto.

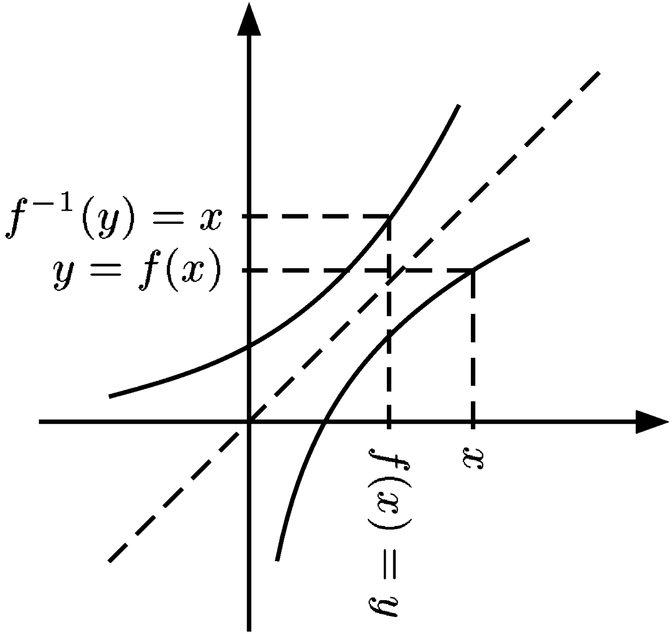
Proposition and Definition. If $f : D \to E$ is bijective, there exists a unique mapping $g : E \to D$ such that, for all $\mathbf{x} \in D, \mathbf{y} \in E$

$$(f \circ g)(\mathbf{x}) = \mathbf{x}\ ,\ \ (g \circ f)(\mathbf{y}) = \mathbf{y}$$

We write $g = f^{-1}$, and call g the inverse mapping of f.

Visualization of the inverse function

The graph of f^{-1} is obtained by exchanging x- and y- coordinates in the graph of f. This amounts to a reflection at the diagonal.

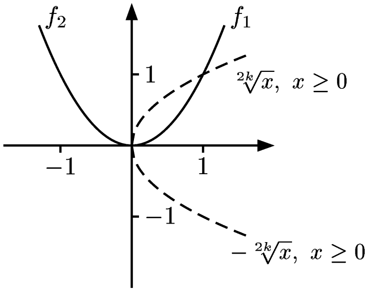


Example: The square root revisited

One can show: The function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $f(x) = x^2$, is bijective. (The proof uses continuity, see next week.)

Note that we defined f only on $D = \mathbb{R}_0^+$ (instead of $\mathbb{R}$) in order to make f injective.

As a result, f has an inverse function g.
By definition, $g(y) =$ the unique number $x > 0$ satisfying $x^2 = y$.
A sketch (k=1):

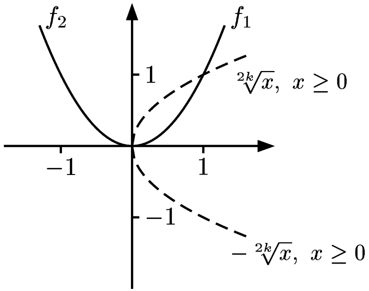


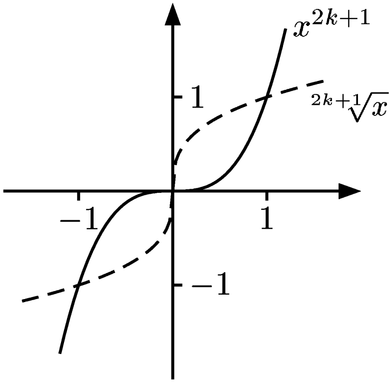
A further example: Higher order roots

Let $d \in \mathbb{N}$. In order to define the root function $g(x) = \sqrt[d]{x}$ as inverse function of the function $f(x) = x^d$, we distinguish two cases:

- $d = 2k$, with $k \in \mathbb{N}$. The map $f(x) = x^{2k}$ is bijective only if we define $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$, hence $g : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$.
- $d = 2k+1$, with $k \in \mathbb{N}$. Then $f : \mathbb{R} \longrightarrow \mathbb{R}$ is bijective, and we obtain an inverse function $g = f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$.

Even case (left), odd case (right)

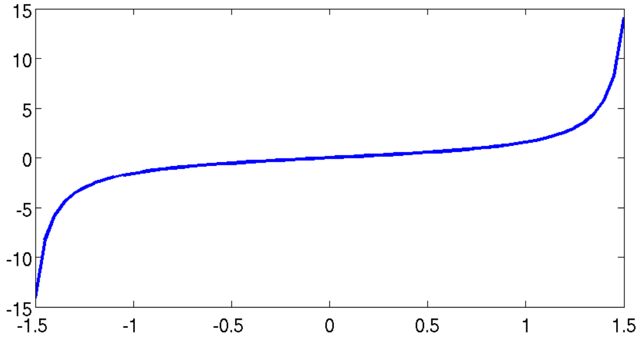




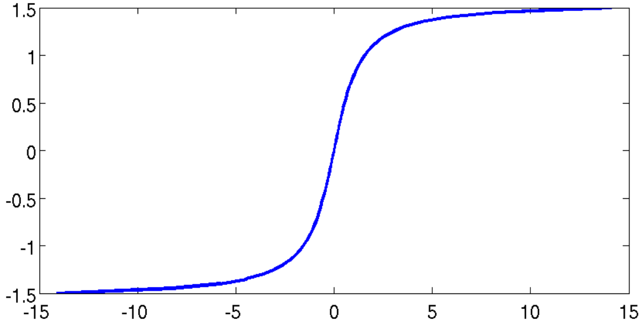
A further example: Arcus tangent

Recall that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is bijective, with $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ as inverse function

Plot of $\tan$:



Plot of $\arctan$:



Function classes

Important classes of functions are

- Trigonometric functions: $\sin, \cos, \tan$, and their inverse functions, e.g., $\arctan$.
- Polynomial functions: A function $P : \mathbb{R} \to \mathbb{R}$ is called a polynomial if there exist coefficients $a_0, \ldots, a_n \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \; .$$

  If $a_n \neq 0$, then n is called the degree of P.
- Rational functions: A function $f : D \to \mathbb{R}$ is called a rational function if there exist polynomials P, Q such that for all $x \in D$,

  $$f(x) = \frac{P(x)}{Q(x)} \; .$$

  Here the maximal domain $D(f)$ is given by $D(f) = \mathbb{R} \setminus \{x \in \mathbb{R} : Q(x) = 0\}$.

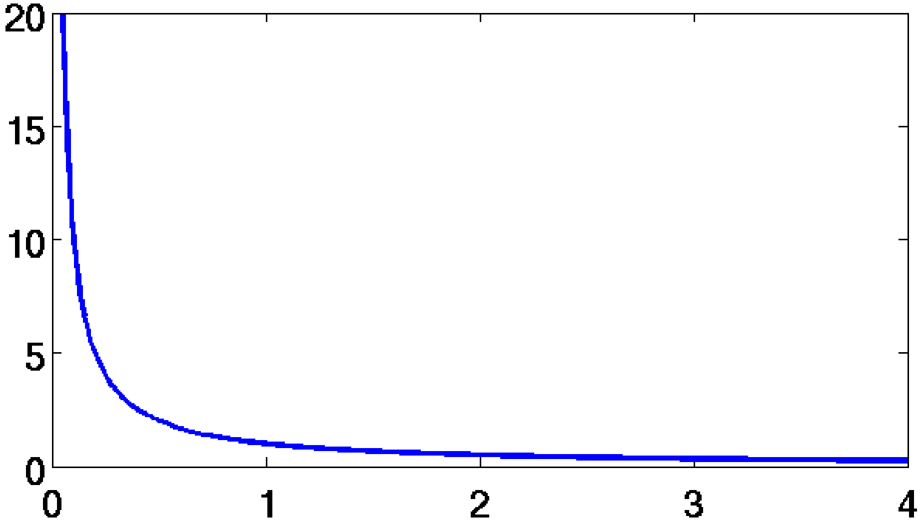
Definition.

Let $D \subset \mathbb{R}$ be an interval, $f : D \to \mathbb{R}$. f is called

- (monotonically) increasing if for all $x, y \in D$ with $x < y$: $f(x) \leq f(y)$.
- (monotonically) decreasing if for all $x, y \in D$ with $x < y$: $f(x) \geq f(y)$.
- strictly increasing if for all $x, y \in D$ with $x < y$: $f(x) < f(y)$.
- strictly increasing if for all $x, y \in D$ with $x < y$: $f(x) > f(y)$.
- (strictly) montonic if it is either (strictly) increasing or decreasing.
- bounded from above if there exists $M \in \mathbb{R}$ such that for all $x \in D$, $f(x) \leq M$.
- bounded from below if there exists $N \in \mathbb{R}$ such that for all $x \in D$, $x \geq N$.

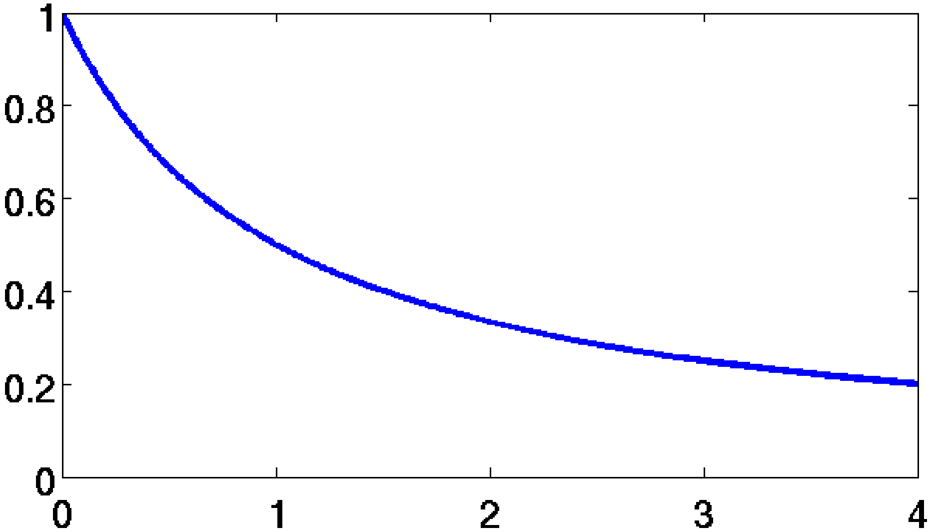
Examples

The function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with $f(x) = x^{-1}$, is strictly decreasing and unbounded.



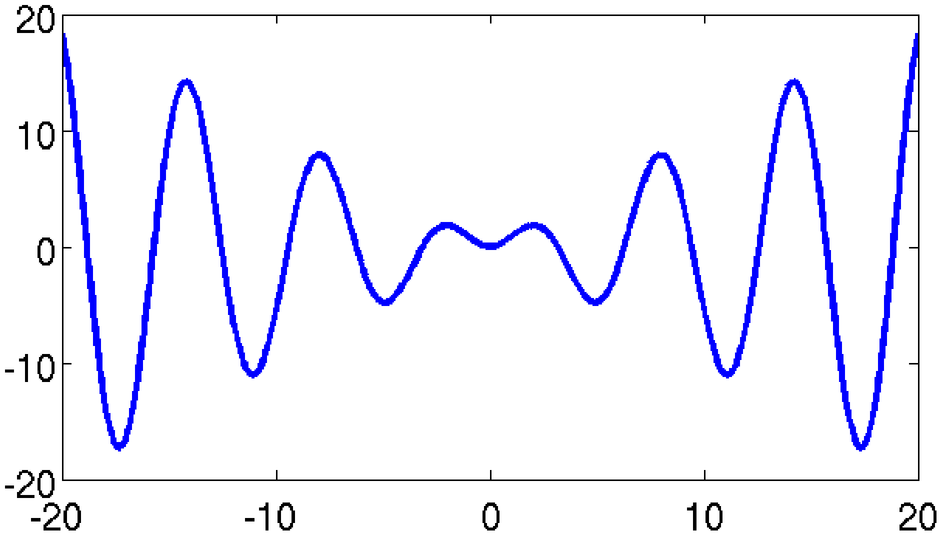
Examples

The function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with $f(x) = (1+x)^{-1}$, is strictly decreasing and bounded.



Examples

The function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = x\sin(x)$, is bounded neither from above nor from below.



Comparison of the different properties

Theorem.
Let $f : D \longrightarrow \mathbb{R}$.

- If $D = [a, b)$ or $D = [a, b]$ for $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$, and f is increasing (resp. decreasing), then f is bounded from below (resp. above).
- If $D = [a, b]$ or $D = (a, b]$ for $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R}$, and f is increasing (resp. decreasing), then f is bounded from above (resp. below).
- If f is a strictly monotonic function, it is one-to-one.

Summary

- Important notions: Mappings, functions, graphs
- Properties of functions: Monotonicity, boundedness, injectivity, surjectivity, bijectivity
- Inverse mappings
  - Geometric interpretation: G_f vs. $G_{f^{-1}}$
  - Special examples: Roots, $\arctan$
- Function classes: Trigonometric functions, polynomials, rational functions