Calculus and Linear Algebra for Biomedical Engineering

Week 9: Power series: The exponential function, trigonometric functions

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Motivation

For arbitrary functions f, the Taylor polynomial

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}}{k!} x^k$$

is only assumed to be an accurate approximation of $f(x)$ for $x \approx 0$. The reasoning is that the remainder term

$$R_{n,0}(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

with suitable z between 0 and x, is small because x is small (and x^{n+1} is even smaller).

Motivation

However, the Taylor polynomial will also provide a good approximation if x is not too big, and instead,

$$\frac{f^{(n+1)}(z)}{(n+1)!} \approx 0 \ .$$

I.e., if the derivative does not grow too fast, the Taylor approximation is accurate on larger intervals.

Thus, at least for certain functions f, summing over more terms of the Taylor series should approximate f on larger sets.

Second motivation

For arbitrary $x, y \in \mathbb{R}$, with $x > 0$, what is x^y?

Using standard operations (products, roots), we can evaluate x^y only for rational numbers y: If $y = \frac{n}{m}$, then

$$x^y = (x^n)^{1/m} = \sqrt[m]{x^n} \ .$$

For irrational y, something else is needed.

Solution: For base $e = 2.7182...$, we define e^y via a power series

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \ .$$

For other bases x, we define x^y from this function and the natural logarithm.

Power series

Definition. An expression of the sort

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is called a power series in x.

Remarks

- If $\sum_{k=0}^{\infty} |a_k| r^k < \infty$, for some $r > 0$, then $f(x)$ is well-defined for all x with $|x - x_0| < r$. Moreover, f is infinitely differentiable in $(-r, r)$.
- If a function f has a power series, this series is the Taylor series of f around x_0.

Taylor series

Definition. Let $f : D \to \mathbb{R}$ denote an infinitely differentiable function, with $x_0 \in D$. Then its Taylor series at x_0 is defined as the series

$$T_{\infty,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \ ,$$

Note: The Taylor series need not converge. Even when it does, $T_{\infty,x_0}(x)$ need not coincide with $f(x)$. However, for certain functions f, one finds that

$$R_{n,x_0}(x) \to 0 \ , \text{ as } n \to \infty$$

and thus $T_{\infty,x_0}(x) = f(x)$.

Radius of convergence

Theorem. Consider a power series

$$(*)\ f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \ .$$

Suppose that one of the two cases holds:

- $c = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.
  In this case, let $r = 1/c$. If $c = 0$, let $r = \infty$.
- $r = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}$ exists.

Then $(*)$ converges if $|x - x_0| < r$, and diverges if $|x - x_0| > r$.
If both limits exist, the two parts give the same value for r.

The number r in the Theorem is called radius of convergence. The interval $(x_0 - r, x_0 + r)$ is called interval of convergence.

Example: Cosine function

Let $f(x) = \cos(x)$. Then, using $\cos' = \sin$ and $\sin' = \cos$, we can compute all higher derivatives as

$$f^{(n)} = \begin{cases} (-1)^{k+1}\sin(x) & n = 2k+1 \\ (-1)^{k+1}\cos(x) & n = 2k \end{cases}$$

Hence, plugging $\sin(0) = 0$, $\cos(0) = 1$ into the Taylor polynomial, we obtain

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Residual of the cosine function

The nth residual of the cosine function is estimated as

$$|R_{n,0}(x)| = \left|\frac{\cos^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \leq \left|\frac{x^{n+1}}{(n+1)!}\right|$$

We want to find a range for x such that the Taylor approximation for $f(x)$ is accurate up to precision 0.1. Taking the $n+1$st root,

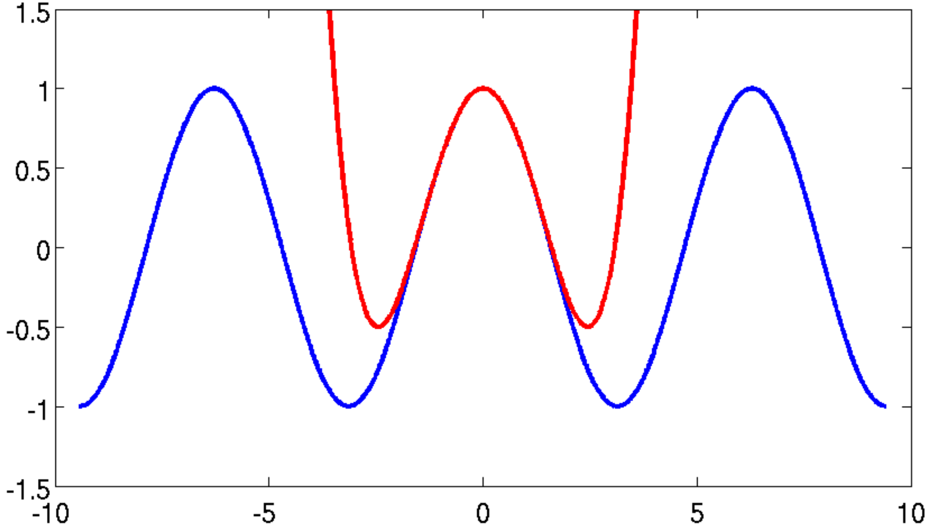
$$\left|\frac{x^{n+1}}{(n+1)!}\right| < 0.1 \Leftrightarrow |x| < \left(\frac{(n+1)!}{10}\right)^{(n+1)^{-1}}$$

This last inequality is fulfilled for instance,

- if $n = 4$ and $|x| < 1.64$;
- or if $n = 12$ and $|x| < 4.74$;
- or if $n = 18$ and $|x| < 7.02$.

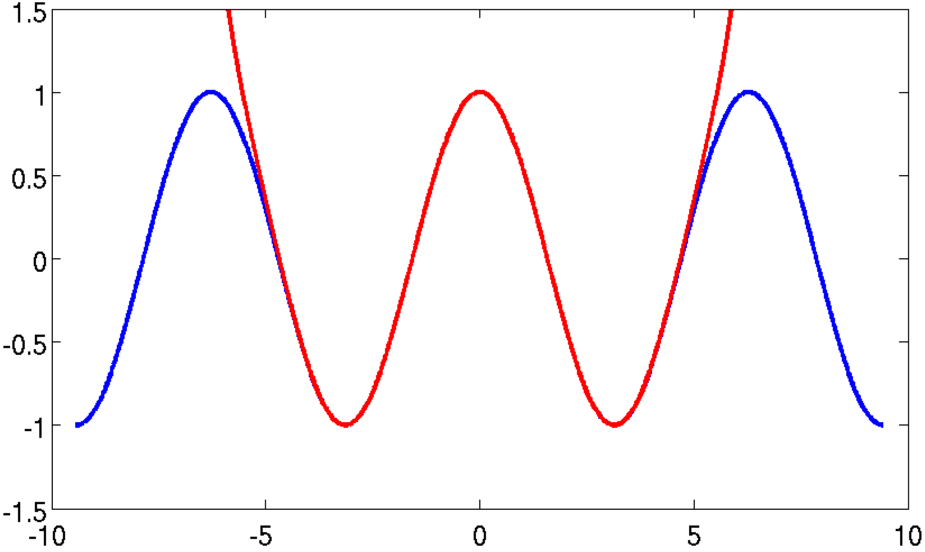
Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{4,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
Accurate up to 0.1 for $|x| < 1.64$



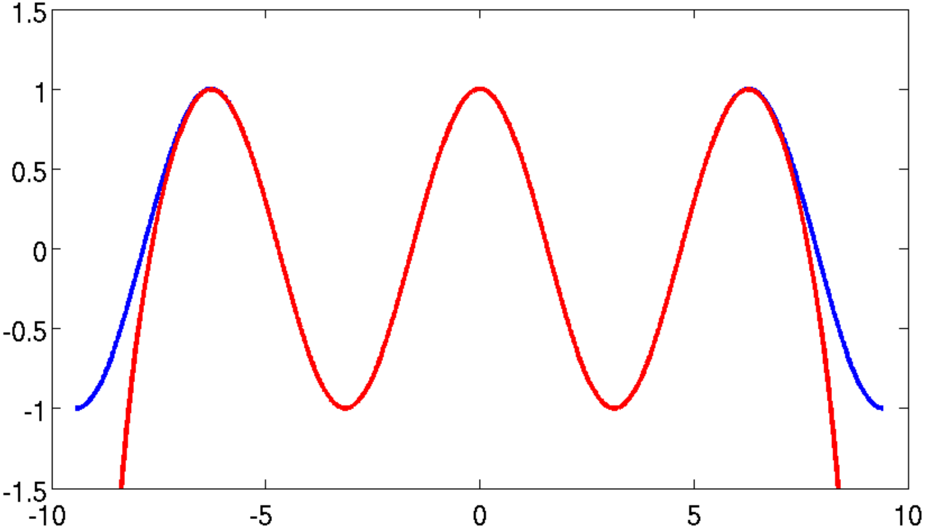
Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{12,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$
Accurate up to 0.1 for $|x| < 4.74$



Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{18,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{16}}{16!} - \frac{x^{18}}{18!}$
Accurate up to 0.1 for $|x| < 7.02$



Power series for $\cos, \sin$

We compute the radius of convergence for the coefficients given by

$$a_n = \begin{cases} 0 & n = 2k+1 \\ \dfrac{(-1)^k}{(2k)!} & n = 2k \end{cases}$$

Now Stirling's formula allows to show that

$$\sqrt[n]{|a_n|} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and thus $r = \infty$. The same argument works for $\sin$, hence:

Theorem. For all $x \in \mathbb{R}$

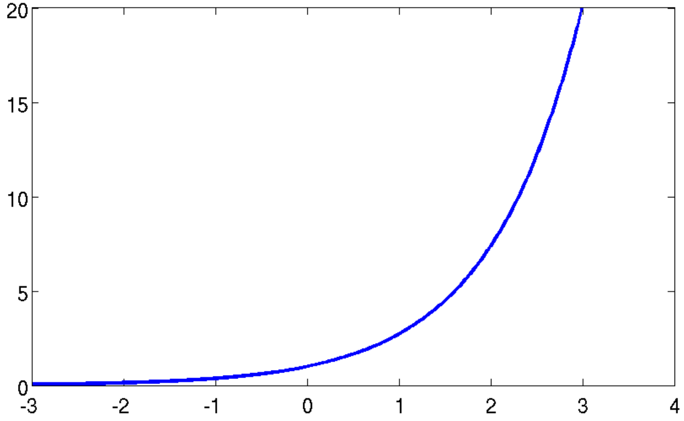
$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\,, \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

The exponential function

Definition. The function $\exp : \mathbb{R} \to \mathbb{R}$ defined by the series

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is called exponential function.



Properties of the exponential function

Theorem.

1. $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and strictly positive.
2. $\exp$ translates addition to multiplication:
   For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x)\exp(y)$.
3. $\exp$ is differentiable, with $\exp' = \exp$. In particular, $\exp$ is strictly increasing.
4. $\lim_{x \to -\infty} \exp(x) = 0$ and $\lim_{x \to \infty} \exp(x) = \infty$.
5. $\exp : \mathbb{R} \longrightarrow (0, \infty)$ is bijective.

$\exp$ and e^x

- An alternative formula for $\exp$ is

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{1/n} .$$

- In particular, $\exp(1) = e$ (Euler's constant).
- Using multiplicativity of $\exp$, one can show for $n \in \mathbb{Z}, m \in \mathbb{N}$ that

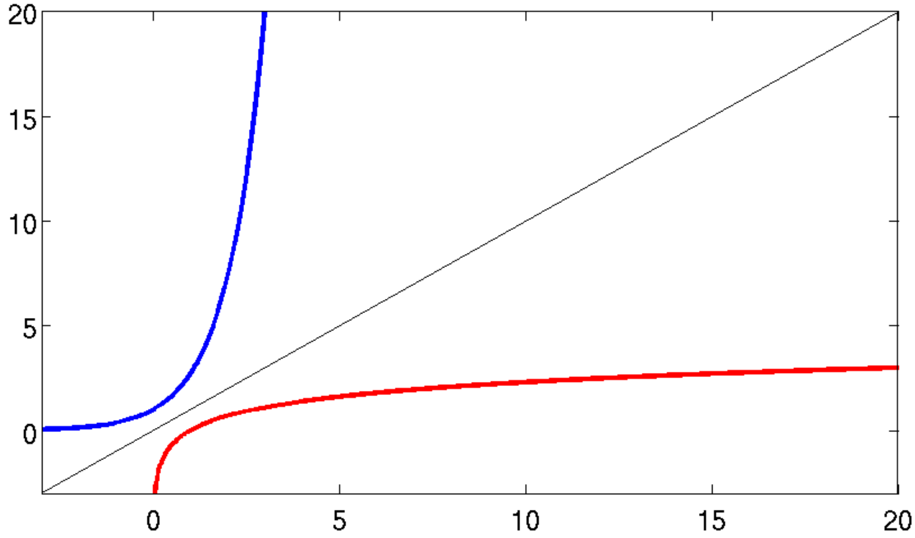
$$\exp(n/m) = e^{n/m}$$

  Hence $\exp(x) = e^x$ for rational x.
- We then define for arbitrary $x \in \mathbb{R}$:

$$e^x = \exp(x) .$$

The natural logarithm

Recall: $\exp : \mathbb{R} \to (0, \infty)$ is bijective. The inverse function is denoted as $\ln : (0, \infty) \to \mathbb{R}$, the natural logarithm.
Blue: $\exp$, red: $\ln$



Properties of $\ln$

Theorem.

1. $\ln : (0, \infty) \to \mathbb{R}$ is continuous, bijective, and strictly increasing.
2. $\ln$ translates multiplication to addition:
   For all $x, y \in (0, \infty)$, $\ln(xy) = \ln(x) + \ln(y)$.
3. $\ln$ is differentiable on $(0, \infty)$, with
$$\ln'(x) = \frac{1}{x} \, .$$
4. $\lim_{x\to 0} \ln(x) = -\infty$ and $\lim_{x\to\infty} \exp(x) = \infty$.
5. $\ln : (0, \infty) \to \mathbb{R}$ is bijective.

Arbitrary exponentials

We define x^y for arbitrary $x > 0$ and $y \in \mathbb{R}$.

$$x^y = e^{\ln(x)y} \ .$$

Then $f(x) = x^y$ fulfills

1. $f : \mathbb{R} \longrightarrow (0, \infty)$ is bijective.
2. f translates addition to multiplication:
   For all $s, t \in \mathbb{R}$, $x^{s+t} = x^s x^t$.
3. f is differentiable, with $f' = \ln(x) f$.
4. Multiplication of exponents becomes exponentiation:
   For all $s, t \in \mathbb{R}$, $x^{st} = (x^s)^t$.

Arbitrary logarithms

The function $f(y) = x^y$ has an inverse function, called base x logarithm, denoted by $\log_x$. The function is computed as

$$\log_x(y) = \frac{\ln(y)}{\ln(x)}$$

Often used bases, besides e, are

- 10 (common logarithm = $\log = \log_{10}$)
- 2 (dyadic logarithm = $\log_2$)

Derivatives of logarithms:

$$\frac{d\log_x}{dy}(y_0) = \frac{1}{\ln(x)y_0}\ .$$

Application: Radioactive decay

If a quantity A of a radioactive substance is given at time $t = 0$, the remaining amount at time $t > 0$ is described by

$$f(t) = Ae^{-\lambda t} \ .$$

Here $\lambda > 0$ is the decay rate of the substance. λ is usually determined by measuring the half-life of the substance, i.e., the time $t_{1/2} > 0$ for which

$$f(t_{1/2}) = \frac{f(0)}{2} = \frac{A}{2} \ .$$

λ can be computed from $t_{1/2}$, and vice versa, because:

$$2 = \frac{f(0)}{f(t_{1/2})} = \frac{A}{Ae^{-\lambda t_{1/2}}} = e^{\lambda t_{1/2}} \Leftrightarrow \lambda t_{1/2} = \ln(2) \ .$$

Complex exponential and Euler's formula

Observation: The series

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges for every $z \in \mathbb{C}$. The result is a function

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}$$

with many interesting properties, in particular,

$$\exp(z + w) = \exp(z)\exp(w) \ .$$

Sorting the real and imaginary parts of $\exp(i\varphi)$ results in Euler's formula for $\alpha \in \mathbb{R}$

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) \ .$$

An application of Euler's formula

Addition theorems: Given $\alpha, \beta \in \mathbb{R}$, we compute $e^{i(\alpha+\beta)}$ in two different ways:

$$(*) \quad e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) \ ,$$

or, using $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$,

$$\begin{aligned} e^{i(\alpha+\beta)} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ &+ i(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)) \ . \end{aligned}$$

A comparison of the last expression with the right-hand side of $(*)$ yields the addition theorems for $\sin, \cos$:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) \end{aligned}$$

Summary

- Power series and radius of convergence
- Power series representation of $\sin, \cos$
- The exponential function $\exp$ and its properties
- Natural logarithms, arbitrary powers and logarithms
- Derivatives of powers and logarithms
- Rules for powers and logarithms
- Complex exponential and Euler's formula