Week 9: Power series: The exponential function, trigonometric functions
Motivation

For arbitrary functions $f$, the Taylor polynomial

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} x^k$$

is only assumed to be an accurate approximation of $f(x)$ for $x \approx 0$. The reasoning is that the remainder term

$$R_{n,0}(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

with suitable $z$ between 0 and $x$, is small because $x$ is small (and $x^{n+1}$ is even smaller).
Motivation

However, the Taylor polynomial will also provide a good approximation if $x$ is not too big, and instead,

$$\frac{f^{(n+1)}(z)}{(n + 1)!} \approx 0.$$  

I.e., if the derivative does not grow too fast, the Taylor approximation is accurate on larger intervals.

Thus, at least for certain functions $f$, summing over more terms of the Taylor series should approximate $f$ on larger sets.
Second motivation

For arbitrary $x, y \in \mathbb{R}$, with $x > 0$, what is $x^y$?

Using standard operations (products, roots), we can evaluate $x^y$ only for rational numbers $y$: If $y = \frac{n}{m}$, then

$$x^y = (x^n)^{1/m} = \sqrt[m]{x^n}.$$ 

For irrational $y$, something else is needed.

**Solution:** For base $e = 2.7182\ldots$, we define $e^y$ via a power series

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}.$$ 

For other bases $x$, we define $x^y$ from this function and the natural logarithm.
Power series

Definition. An expression of the sort

\[ f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \]

is called a power series in \( x \).

Remarks

- If \( \sum_{k=0}^{\infty} |a_k| r^k < \infty \), for some \( r > 0 \), then \( f(x) \) is well-defined for all \( x \) with \( |x - x_0| < r \). Moreover, \( f \) is infinitely differentiable in \((-r, r)\).

- If a function \( f \) has a power series, this series is the Taylor series of \( f \) around \( x_0 \).
**Taylor series**

**Definition.** Let $f : D \to \mathbb{R}$ denote an infinitely differentiable function, with $x_0 \in D$. Then its **Taylor series** at $x_0$ is defined as the series

$$T_{\infty,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

**Note:** The Taylor series need not converge. Even when it does, $T_{\infty,x_0}(x)$ need not coincide with $f(x)$. However, for certain functions $f$, one finds that

$$R_{n,x_0}(x) \to 0, \text{ as } n \to \infty$$

and thus $T_{\infty,x_0}(x) = f(x)$. 
Radius of convergence

Theorem. Consider a power series

\[(\ast) \quad f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k.\]

Suppose that one of the two cases holds:

1. \(c = \lim_{n \to \infty} n^{\sqrt{|a_n|}}\) exists.
   
   In this case, let \(r = 1/c\). If \(c = 0\), let \(r = \infty\).

2. \(r = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}\) exists.

Then \((\ast)\) converges if \(|x - x_0| < r\), and diverges if \(|x - x_0| > r\).

If both limits exist, the two parts give the same value for \(r\).

The number \(r\) in the Theorem is called radius of convergence. The interval \((x_0 - r, x_0 + r)\) is called interval of convergence.
Example: Cosine function

Let $f(x) = \cos(x)$. Then, using $\cos' = \sin$ and $\sin' = \cos$, we can compute all higher derivatives as

$$f^{(n)} = \begin{cases} (-1)^{k+1} \sin(x) & n = 2k + 1 \\ (-1)^{k+1} \cos(x) & n = 2k \end{cases}$$

Hence, plugging $\sin(0) = 0$, $\cos(0) = 1$ into the Taylor polynomial, we obtain

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{(-1)^{k}x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$
The $n$th residual of the cosine function is estimated as

$$|R_{n,0}(x)| = \left| \cos^{(n+1)}(z) x^{n+1} \right| \leq \left| \frac{x^{n+1}}{(n+1)!} \right|$$

We want to find a range for $x$ such that the Taylor approximation for $f(x)$ is accurate up to precision $0.1$. Taking the $n+1$st root,

$$\left| \frac{x^{n+1}}{(n+1)!} \right| < 0.1 \Leftrightarrow |x| < \left( \frac{(n+1)!}{10} \right)^{(n+1)^{-1}}$$

This last inequality is fulfilled for instance,

- if $n = 4$ and $|x| < 1.64$;
- or if $n = 12$ and $|x| < 4.74$;
- or if $n = 18$ and $|x| < 7.02$. 
Approximation of the cosine function

Blue: \( \cos(x) \), Red: \( T_{4,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \)

Accurate up to 0.1 for \( |x| < 1.64 \)
Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{12,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$

Accurate up to 0.1 for $|x| < 4.74$
Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{18,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{16}}{16!} - \frac{x^{18}}{18!}$

Accurate up to 0.1 for $|x| < 7.02$
Power series for $\cos, \sin$

We compute the radius of convergence for the coefficients given by

$$a_n = \begin{cases} 
0 & n = 2k + 1 \\
\frac{(-1)^k}{(2k)!} & n = 2k
\end{cases}$$

Now Stirling's formula allows to show that

$$\sqrt[n]{|a_n|} \to 0 \text{ as } n \to \infty$$

and thus $r = \infty$. The same argument works for $\sin$, hence:

**Theorem.** For all $x \in \mathbb{R}$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!}$$
The exponential function

Definition. The function \( \exp : \mathbb{R} \to \mathbb{R} \) defined by the series

\[
\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}
\]

is called exponential function.
Theorem.

1. \( \exp : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and strictly positive.

2. \( \exp \) translates addition to multiplication:
   
   For all \( x, y \in \mathbb{R} \), \( \exp(x + y) = \exp(x) \exp(y) \).

3. \( \exp \) is differentiable, with \( \exp' = \exp \). In particular, \( \exp \) is strictly increasing.

4. \( \lim_{x \rightarrow -\infty} \exp(x) = 0 \) and \( \lim_{x \rightarrow \infty} \exp(x) = \infty \).

5. \( \exp : \mathbb{R} \rightarrow (0, \infty) \) is bijective.
An alternative formula for \( \exp \) is

\[
\exp(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{1/n}.
\]

In particular, \( \exp(1) = e \) (Euler’s constant).

Using multiplicativity of \( \exp \), one can show for \( n \in \mathbb{Z}, m \in \mathbb{N} \) that

\[
\exp(n/m) = e^{n/m}
\]

Hence \( \exp(x) = e^x \) for rational \( x \).

We then define for arbitrary \( x \in \mathbb{R} \):

\[
e^x = \exp(x) .
\]
Recall: \( \exp : \mathbb{R} \rightarrow (0, \infty) \) is bijective. The inverse function is denoted as \( \ln : (0, \infty) \rightarrow \mathbb{R} \), the natural logarithm.

Blue: \( \exp \), red: \( \ln \)
Properties of $\ln$

Theorem.

1. $\ln : (0, \infty) \rightarrow \mathbb{R}$ is continuous, bijective, and strictly increasing.

2. $\ln$ translates multiplication to addition: 
   For all $x, y \in (0, \infty)$, $\ln(xy) = \ln(x) + \ln(y)$.

3. $\ln$ is differentiable on $(0, \infty)$, with
   $$\ln'(x) = \frac{1}{x}.$$  

4. $\lim_{x \to 0} \ln(x) = -\infty$ and $\lim_{x \to \infty} \exp(x) = \infty$.

5. $\ln : (0, \infty) \rightarrow \mathbb{R}$ is bijective.
**Arbitrary exponentials**

We define $x^y$ for arbitrary $x > 0$ and $y \in \mathbb{R}$.

$$x^y = e^{\ln(x)y}.$$  

Then $f(x) = x^y$ fulfills

1. $f : \mathbb{R} \to (0, \infty)$ is bijective.

2. $f$ translates addition to multiplication:
   For all $s, t \in \mathbb{R}$, $x^{s+t} = x^s x^t$.

3. $f$ is differentiable, with $f' = \ln(x)f$.

4. Multiplication of exponents becomes exponentiation:
   For all $s, t \in \mathbb{R}$, $x^{st} = (x^s)^t$.
Arbitrary logarithms

The function $f(y) = x^y$ has an inverse function, called base $x$ logarithm, denoted by $\log_x$. The function is computed as

$$\log_x(y) = \frac{\ln(y)}{\ln(x)}$$

Often used bases, besides $e$, are

- $10$ (common logarithm = $\log = \log_{10}$)
- $2$ (dyadic logarithm = $\log_2$)

Derivatives of logarithms:

$$\frac{d}{dy} \log_x(y_0) = \frac{1}{\ln(x)y_0}.$$
Application: Radioactive decay

If a quantity $A$ of a radioactive substance is given at time $t = 0$, the remaining amount at time $t > 0$ is described by

$$f(t) = Ae^{-\lambda t}.$$ 

Here $\lambda > 0$ is the decay rate of the substance. $\lambda$ is usually determined by measuring the half-life of the substance, i.e., the time $t_{1/2} > 0$ for which

$$f(t_{1/2}) = \frac{f(0)}{2} = \frac{A}{2}.$$ 

$\lambda$ can be computed from $t_{1/2}$, and vice versa, because:

$$2 = \frac{f(0)}{f(t_{1/2})} = \frac{A}{Ae^{-\lambda t_{1/2}}} = e^{\lambda t_{1/2}} \iff \lambda t_{1/2} = \ln(2).$$
Complex exponential and Euler’s formula

Observation: The series

\[ \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \]

converges for every \( z \in \mathbb{C} \). The result is a function

\[ \exp : \mathbb{C} \to \mathbb{C} \]

with many interesting properties, in particular,

\[ \exp(z + w) = \exp(z) \exp(w) \, . \]

Sorting the real and imaginary parts of \( \exp(i\varphi) \) results in Euler’s formula for \( \alpha \in \mathbb{R} \)

\[ e^{i\alpha} = \cos(\alpha) + i \sin(\alpha) \, . \]
An application of Euler’s formula

Addition theorems: Given $\alpha, \beta \in \mathbb{R}$, we compute $e^{i(\alpha + \beta)}$ in two different ways:

\begin{align*}
(*) \quad e^{i(\alpha + \beta)} &= \cos(\alpha + \beta) + i \sin(\alpha + \beta), \\
&= \cos\alpha \cos\beta - \sin\alpha \sin\beta + i(\cos\alpha \sin\beta + \sin\alpha \cos\beta).
\end{align*}

A comparison of the last expression with the right-hand side of $(*)$ yields the addition theorems for $\sin, \cos$:

\begin{align*}
\cos(\alpha + \beta) &= \cos\alpha \cos\beta - \sin\alpha \sin\beta, \\
\sin(\alpha + \beta) &= \cos\alpha \sin\beta + \sin\alpha \cos\beta.
\end{align*}
Summary

- Power series and radius of convergence
- Power series representation of $\sin, \cos$
- The exponential function $\exp$ and its properties
- Natural logarithms, arbitrary powers and logarithms
- Derivatives of powers and logarithms
- Rules for powers and logarithms
- Complex exponential and Euler’s formula