Calculus and linear algebra for biomedical engineering
Week 0: Sets and numbers

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Overview

Mathematical propositions

Propositions are assertions about (usually mathematical) entities, which can be meaningfully assigned a truth value, "true" or "false".
Examples of propositions:

- Yesterday it rained in Aachen.
- Equations: For all real numbers a, b:
  $(a+b)^2 = a^2 + 2ab + b^2$. (This is a true proposition.)
- Inequalities: For all real numbers a, b: $(a+b)^2 > a^2 + b^2$. (This is a false proposition.)

Further examples

Sentences that are not propositions:

- Today it is going to rain. (Truth values cannot be assigned to prognoses.)
- I hope it does not rain again.
- The number π is more important than the number $\sqrt{2}$. ("Importance" is not a meaningful property of numbers.)
- Does π^2 equal 1? (This is not an assertion.)

Operations on mathematical propositions

Mathematical propositions can be combined to yield new statements. Suppose that A, B are mathematical propositions.

- Negation: $\neg A$ is true precisely when A is false.
- Conjunction: $A \wedge B$ (read: "A and B") is true precisely when both A and B are true.
- Disjunction: $A \vee B$ (read: "A or B") is true precisely when at least one of the statements A, B is true.
- Implication: $A \Rightarrow B$ (read "A implies B) is true precisely when the truth of A implies the truth of B. Formally, $A \Rightarrow B$ is true precisely when $(\neg A) \vee B$ is true.
- Equivalence: $A \Leftrightarrow B$ (read "A is equivalent to B") is true precisely when both $A \Rightarrow B$ and $B \Rightarrow A$ are true.

Examples

- The implication
  There are 2 € in my right pocket $\Rightarrow$ *I have at least 2 € on me*
  is true.
- Conversely, the implication
  I have at least 2 € on me $\Rightarrow$ *There are 2 € in my right pocket*
  is false.

Note: The validity of the implications need not depend on the truth of the isolated statements.

Sets

A set is a collection of well-defined, distinct objects. The objects that are contained in a set M are called the elements of M.
How to write down a set:

- Listing all the elements of the set: $M = \{a, b, c, d\}$ is the set containing the elements a, b, c and d.
- Describing the elements: $M = \{x : A(x) \text{ is true }\}$, where $A(x)$ is a proposition depending on x.

Examples:

- $M = \{2, 4, 6, 8\}$
- $N = \{x : x \text{ is an even natural number with } x < 10\}$

Subsets and inclusion

If A and B are sets, A is called a subset of B if every element of A is contained in B. We then write $A \subset B$, or $B \supset A$.

$$A \subset B \Leftrightarrow (\text{ for all } x \in A : x \in B)$$

Two sets are equal if they have the same elements. Hence

$$A = B \Leftrightarrow (A \subset B \wedge B \subset A)$$

Example:

$$\{2, 4, 6, 8\} = \{x : x \text{ is an even natural number with } x < 10\}$$

Operations on sets: Intersection and union

Given sets A and B,

- the union of A and B is the set of all elements contained in either one:

$$A \cup B = \{x : x \in A \vee x \in B\} \quad ;$$

- the intersection of A and B is the set of all elements contained in both:

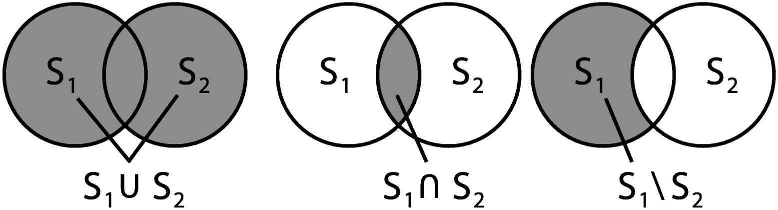
$$A \cap B = \{x : x \in A \wedge x \in B\} \quad .$$

- the difference of A and B is the set of all elements contained in A, but not in B:

$$A \setminus B = \{x : x \in A \wedge x \notin B\} \quad .$$

Visualization of set operations

From left to right: Union, intersection, difference



The most important sets: Number domains

- The empty set: The set containing no elements is denoted $\emptyset$.
- Natural numbers: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, \ldots\} = \mathbb{N} \cup \{0\}$.
- Integer numbers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.
- Rational numbers: The set of fractions $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q > 0\}$.
- Real numbers: $\mathbb{R} =$ set of all decimal expansions

$$x = n.a_1 a_2 a_3 \ldots \quad , n, a_1, \ldots a_n \in \mathbb{N}_0 \ , \ 0 \leq a_i \leq 9 \quad .$$

Examples:

- $\frac{1}{4} = 0.25$
- $\frac{1}{7} = 0.142857142857... = 0.\overline{142857}$
- The circumference of a circle with diameter 1 is given by $\pi = 3.1415926....$ (irrational number)

The purpose of number domains

Depending on the operations one wishes to perform on numbers, there is a hierarchy of number domains:

- Natural numbers: Useful for elementary tasks like counting objects. Sums of natural numbers are natural numbers. Taking differences of natural numbers leads to
- Integers: Integers are natural numbers with a sign. Taking quotients of integers leads to
- Rational numbers: Rational numbers are closed under taking differences and quotients. Computers and calculators use rational numbers. The necessity of taking roots (and other useful operations, like exponentiating) leads to
- Real numbers: Most importantly, real numbers and their properties are the basis of calculus.

Inclusions between number domains

The following chain of inclusions holds:

$$\emptyset \subset \mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

The first inclusion is true be default: The empty set is contained in every set.
For the last inclusion recall: A real number

$$x = n.a_1a_2a_3\ldots \quad , n, a_1, \ldots a_n \in \mathbb{N}_0 \ , \ 0 \leq a_i \leq 9 \quad .$$

is rational if and only if its decimal expansion breaks off or is periodic.

Adding and subtracting real numbers

Real numbers can be added and subtracted: For each pair (x, y) of real numbers there are unique numbers $x + y, x - y \in \mathbb{R}$ such that the following axioms:

- Neutral element: For all $x \in \mathbb{R}$: $x + 0 = x$.
- Associativity: $(x + y) + z = x + (y + z)$.
  Thus, we can omit brackets in this setting:
  $x + y + z := (x + y) + z$.
- Commutativity: $x + y = y + x$.
- Subtraction and addition are inverse operations:
  $y - y = 0$, and thus $x + y - y = x + 0 = x$.

Instead of $0 - y$ one writes $-y$. Hence $x - y = x + (-y)$. In particular, addition and subtraction commute.

Multiplying and dividing real numbers

Real numbers can be multiplied and divided:For each pair (x, y) of real numbers there are unique numbers $x \cdot y, x/y \in \mathbb{R}$ (with x/y only defined if $y \neq 0$!) such that the following axioms are fulfilled:

- Multiplication by zero: For all $x \in \mathbb{R}$: $x \cdot 0 = 0$.
- Neutral element: For all $x \in \mathbb{R}$: $x \cdot 1 = x$.
- Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
  Thus, we can omit brackets in this setting: $xyz = (x \cdot y) \cdot z$.
- Commutativity: $x \cdot y = y \cdot x$.
- Multiplication and division are inverse operations:
  For $y \in \mathbb{R}$, different from 0, $y/y = 1$, and thus
  $(xy)/y = x \cdot 1 = x$.

We y^{-1} instead of $1/y$ and $\frac{x}{y}$ instead of x/y. Then $\frac{x}{y} = x \cdot y^{-1}$.

Combining addition and multiplication

For $n \in \mathbb{N}_0, x \in \mathbb{R}$: $\underbrace{x + x + \ldots + x}_{n \text{ occurrences}} = n \cdot x$

Furthermore, one has distributive rules: For $x, y, z \in \mathbb{R}$,

- $x \cdot (y + z) = (xy) + (xz)$
- $(y + z)/x = (y/x) + (z/x)$, for $x \neq 0$
- $x \cdot (y - z) = (xy) - (xz)$
- $(y - z)/x = (y/x) - (z/x)$, for $x \neq 0$

To avoid cluttered notation, multiplication/division are always assumed to be performed before addition/subtraction. Hence:

$$(xy) + z = xy + z \quad , \quad x(y + z) = (xy) + (xz) = xy + xz$$

Ordering and comparing real numbers

Every real number $x \in \mathbb{R}$ fulfills precisely one of the following:

$$x < 0 \quad , \quad x = 0 \quad , \quad x > 0 \quad .$$

$x > 0$ is called positive, $x < 0$ is negative.
One writes $x < y$ if $x - y < 0$. This ordering fulfills the following axioms, for all $x, y, z \in \mathbb{R}$

1. $x < y$ and $y < z \Rightarrow x < z$.
2. $x < y \Rightarrow x + z < y + z$
3. $x < y \Rightarrow -y < -x$
4. $z > 0$ and $x < y \Rightarrow zx < zy$
5. $z < 0$ and $x < y \Rightarrow zx > zy$

Alternative ordering: $\leq$, $>$ etc.

One defines

$$y > x :\Leftrightarrow x < y$$

and

$$x \leq y :\Leftrightarrow (x < y) \vee (x = y) \quad .$$

Also, $y \geq x$ is the same as $x \leq y$. The rules derived for "$<$" on the previous slide are easily adapted to "$>, \leq, \geq$". An equivalence used in many proofs is

$$x = y \Leftrightarrow (x \leq y) \wedge (y \leq x) \quad .$$

It is also customary to write chains of inequalities:

$$x < y \leq z \Leftrightarrow (x < y) \wedge (y \leq z) \quad .$$

Intervals

Definition.

For $a, b \in \mathbb{R}$, with $a < b$, we define

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (open interval)
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ (half-open interval)
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ (half-open interval)
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval)
- $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$ and $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$
- $(a, \infty) = \{x \in \mathbb{R} : a < x\}$ and $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$

Absolute value

For every $y \in \mathbb{R}$, either $y \geq 0$ or $-y \geq 0$. We let

$$|y| = \begin{cases} y & \text{for} \quad y \geq 0 \\ -y & \text{for} \quad y < 0 \end{cases} ,$$

which is called absolute value or modulus of y.

Rules for the absolute value: Let $x, y \in \mathbb{R}$

- $|x| \geq 0$, and $|x| = 0 \Leftrightarrow x = 0$.
- $|xy| = |x| \; |y|$.
- $|x| = |-x|$.
- $|x + y| \leq |x| + |y|$.

The last property is known as the triangle inequality. Useful reformulations are

$$\big|\, |x| - |y| \,\big| \leq |x + y| \leq |x| + |y| \quad .$$

Powers of real numbers

We next want to make sense of the expression x^y, with $x, y \in \mathbb{R}$. This takes several steps. We start out by considering $y = n \in \mathbb{N}_0$:

Multiplying n times the same number $x \in \mathbb{R}$ gives the nth power of x

$$\underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ occurrences}} = x^n \quad .$$

Powers are assumed to be calculated before multiplication:
For example, $xy^n + z = (x(y^n)) + z$.

Rules for powers

Let $x, y \in \mathbb{R}$ and $m, n \in \mathbb{N}$.

1. $x^0 = 1$, for all $x \in \mathbb{R}$. (In particular: $0^0 = 1$.)
2. $x^n x^m = x^{n+m}$
3. $x^n y^n = (xy)^n$
4. $(x^n)^m = x^{nm}$

Negative powers: One writes

$$x^{-n} = (x^{-1})^n = 1/(x^n) \quad .$$

Application: Monotonicity of powers

Squares are positive: For $x \in \mathbb{R}$, $x \neq 0$:

$$x^2 > 0 \quad .$$

Indeed,

$$\begin{aligned} &\text{if } x < 0 \quad \Rightarrow \quad x \cdot x > 0 \cdot x = 0 \text{ (see slide 15, rule 5)} \\ &\text{if } x > 0 \quad \Rightarrow \quad x \cdot x > 0 \cdot x = 0 \text{ (see slide 15, rule 4)} \end{aligned}$$

Monotonicity of powers: For $n \in \mathbb{N}$ and $0 < x < y$,

$$0 < x^n < y^n \quad .$$

This rule is obtained by application of the order axioms:

$$0 < x < y \Rightarrow x^2 = x \cdot x < x \cdot y < y \cdot y = y^2 \quad ,$$

and so on. (Mathematically rigourous method: Proof by induction.)

Roots and fractional powers

Let $n \in \mathbb{N}$ and $x > 0$. Then there is a unique $y > 0$ such that

$$y^n = x \quad .$$

One defines

$$x^{1/n} := y \quad ,$$

and calls y the nth root of x. Alternative notation: $\sqrt[n]{x} := x^{1/n}$.

By definition of $x^{1/n}$, one has

$$(x^{1/n})^n = x = x^1 = x^{n/n} \quad .$$

Hence it makes sense to define x^y, for $y = m/n \in \mathbb{Q}$, by letting

$$x^y = (x^{1/n})^m.$$

Rules for fractional powers

The rules for integer powers carry over to fractional powers:
Let $x, y \in \mathbb{R}$ be positive, and $p, q \in \mathbb{Q}$.

1. $x^0 = 1$, for all $x \in \mathbb{R}$. (In particular: $0^0 = 1$.)
2. $x^p x^q = x^{p+q}$.
3. $x^p y^p = (xy)^p$
4. $(x^p)^q = x^{pq}$

Note: Do not forget the restriction $x > 0$!
We noted previously for every $y \in \mathbb{R}$, that $y^2 > 0$.
Hence the equation $y^2 = -1$ cannot be solved in $\mathbb{R}$, i.e., there is no real number $y = \sqrt{-1}$

Arbitrary powers

The expression x^y can now be extended to $x > 0$ and $y \in \mathbb{R}$ arbitrary, using that y can be arbitrarily well be approximated by $y' \in \mathbb{Q}$.
(Note: A more detailed explanation already requires notions from calculus.)

The rules for fractional powers carry over to arbitrary powers:
Let $x, y \in \mathbb{R}$ be positive, and $s, t \in \mathbb{R}$.

1. $x^0 = 1$, for all $x \in \mathbb{R}$. (In particular: $0^0 = 1$.)
2. $x^s x^t = x^{s+t}$.
3. $x^s y^s = (xy)^s$
4. $(x^s)^t = x^{st}$

Equations

Interesting quantities are often given as solutions of equations.
Several questions arise: Does a solution exist in a given set? Is it unique?
These questions are usually answered by determining the set $\mathbb{S}$ of all solutions.
Examples:

- Consider the equation $3 + 2x = 5 - 2x$. This can be easily solved for x, yielding $x = 0.5$. Hence the set of solutions is $\mathbb{S} = \{0.5\}$.
- The equation $(5x)^2 = 25x^2$ is true for every $x \in \mathbb{R}$. Hence we obtain $\mathbb{S} = \mathbb{R}$ as set of all solutions.
- We know for all $x \in \mathbb{R}$ that $x^2 > 0$. In particular, the equation $x^2 = -1$ has no solution in $\mathbb{R}$, and $\mathbb{S} = \emptyset$ in this case.

Further Examples

- The equation $x^2 = 2$ has no solutions in $\mathbb{Z}$. This is easily seen, since $0^2 = 0 \neq 2$, $(\pm 1)^2 = 1 \neq 2$, and $n^2 > 2$ for all $n \in \mathbb{Z}$, $|n| > 1$.
- It is true (but harder to show) that $x^2 = 2$ has no solution in $\mathbb{Q}$.
- The equation $x^2 = 2$ has two real solutions, $\mathbb{S} = \{\pm\sqrt{2}\}$. (Note that we defined $\sqrt{2}$ as the positive solution of this equation.)
- More generally, the equation $x^2 + ax + b = 0$, with fixed $a, b \in \mathbb{R}$ has the solutions

$$x_{1,2} = \frac{a \pm \sqrt{a^2 - 4b}}{2} \quad ,$$

provided that $a^2 - 4b \geq 0$. Hence there exist two solutions in $\mathbb{R}$ if $a^2 - 4b > 0$, one solution if $a^2 - 4b = 0$, and no solutions if $a^2 - 4b < 0$.

One more class of examples: Linear equations

A linear equation has the form $ax + b = 0$, with $a, b \in \mathbf{R}$ and variable x. Existence and numbers of solutions depend on a and b:

- If $a \neq 0$, we can solve directly for x

  $$ax + b = 0 \Leftrightarrow ax = -b \Leftrightarrow x = -\frac{b}{a} \quad ,$$

  showing that there exists precisely one solution.
- If $a = 0$, the equation becomes $b = 0$. Hence, if $b = 0$, then $\mathbb{S} = \mathbb{R}$, otherwise $\mathbb{S} = \emptyset$.

Summary

- Mathematics generally proceeds by the following steps.
  - Define objects (Propositions, sets, numbers).
  - Define operations on objects (e.g., disjunctions, unions, sums).
  - Fix rules or axioms that the operations must obey.
  - Derive true mathematical statements by applying the axioms.
- Most important object: The number domain $\mathbb{R}$
  - Algebraic operations on $\mathbb{R}$ and their properties
  - Extensions of the algebraic operations: Powers, roots
  - Ordering on $\mathbb{R}$ and its properties
  - The properties of $\mathbb{R}$ are the basis of calculus.