# Calculus and linear algebra for biomedical engineering
# Week 1: Complex numbers and trigonometric functions

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# Overview

## Motivation

Recall from last week:

- Extensions of number domains (e.g., from $\mathbb{N}$ to $\mathbb{Z}$, from $\mathbb{Z}$ to $\mathbb{Q}$, from $\mathbb{Q}$ to $\mathbb{R}$) are motivated partly by the desire to extend operations (e.g., subtraction, division, powers).
- A negative real number does not have a square root in $\mathbb{R}$.

Definition. The set $\mathbb{C}$ of complex numbers is defined as

$$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\} .$$

where $(x, y)$ denotes an ordered pair of real numbers.
Note: Two ordered pairs $(a, b)$ and $(x, y)$ are equal if and only if $a = x$ and $b = y$. In particular, $(a, b) = (b, a)$ only if $a = b$.

## Operations on complex numbers

Definition. Let $(a, b), (x, y) \in \mathbb{C}$.

- The sum resp. difference is defined as

$$(a, b) + (x, y) = (a + x, b + y) \qquad (1)$$
$$(a, b) - (x, y) = (a - x, b - y) \ . \qquad (2)$$

- The product is defined as

$$(a, b) \cdot (x, y) = (ax - by, ay + bx) \ . \qquad (3)$$

## Notations for complex numbers

Pairs are useful for the rigourous definition of complex numbers. For carrying out computations with complex numbers, other notations are preferred.

- We identify $(x, 0) \in \mathbb{C}$ with $x \in \mathbb{R}$.
  Note that now, $x + y$ could mean the usual sum of real numbers, or the result of the addition $(x, 0) + (y, 0)$ in $\mathbb{C}$. However, the latter is $(x + y, 0)$, which we identify with $x + y$.
- We define the imaginary unit as $i := (0, 1) \in \mathbb{C}$. Note $i^2 = -1$.
- We can now write arbitrary complex numbers as

$$z = (x, y) = (x, 0) + (0, y) = x + (0, 1) \cdot y = x + iy \ .$$

- In the new notation, sum and product become

$$(a + ib) + (x + iy) = (a + x) + i(b + y) \qquad (4)$$
$$(a + ib) \cdot (x + iy) = (ax - by) + i(ay + bx) \ . \qquad (5)$$

## Axioms for sums and products

**Theorem.** All axioms regarding sums (differences) and products in $\mathbb{R}$ carry over to $\mathbb{C}$. In particular, the following properties can be verified directly.

- Addition is commutative and associative.
- Multiplication is commutative and associative.
- The distributive law relating addition and multiplication holds.
- Using the identifications from above, specifically $1 = (1, 0)$ and $0 = (0, 0)$, we find for arbitrary $z \in \mathbb{C}$

$$z = z \cdot 1 = z + 0 \ , \ z \cdot 0 = 0 \ \ .$$

# Complex conjugate, Real and Imaginary part, Modulus

Definition. For $z = x + iy \in \mathbb{C}$, we introduce the following notions:

$$\mathrm{Re}(z) = x \quad , \quad \text{the real part of } z \tag{6}$$

$$\mathrm{Im}(z) = y \quad , \quad \text{the imaginary part of } z \tag{7}$$

$$\overline{z} = x - iy \quad , \quad \text{the complex conjugate of } z \tag{8}$$

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2} \quad , \quad \text{the modulus or length of } z \tag{9}$$

Note that $x^2 + y^2 \geq 0$, hence $|z|$ is well-defined and positive. $|z|$ is also called absolute value of $z$.

Useful formulas:

$$z = \mathrm{Re}(z) + i\mathrm{Im}(z) \quad , \quad \mathrm{Re}(z) = \frac{z + \overline{z}}{2} \quad , \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2}$$

# Complex division

Theorem. For $z = x + iy \in \mathbb{C} \setminus \{0\}$, write

$$z^{-1} = |z|^{-2} \cdot \overline{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \quad.$$

Then

$$z \cdot z^{-1} = z \cdot \frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1 \quad. \tag{10}$$

Remarks: This allows to define division by $z \in \mathbb{C} \setminus \{0\}$, via

$$\frac{w}{z} = w \cdot z^{-1}$$

All properties known for division in $\mathbb{R}$ remain true in $\mathbb{C}$.

# Properties of the modulus

Note that for a real number $x = x + i0$, one computes $|x + i0| = \sqrt{x^2 + 0^2} = |x|$. Hence the modulus of a real number is the same, whether we regard $x$ as a real or complex number.

Theorem. (Rules for the absolute value:)
Let $w, z \in \mathbb{C}$

- $|z| \geq 0$, and $|z| = 0 \Leftrightarrow z = 0$.
- $|wz| = |w|\,|z|$.
- $|z| = |-z| = |\bar{z}|$
- $|w + z| \leq |w| + |z|$.

The last property is known as the triangle inequality. Useful reformulations are

$$|\,|w| - |z|\,| \leq |w + z| \leq |w| + |z|\ .$$

# Computing real and imaginary parts of a quotient

General procedure: Given a quotient $z = \dfrac{x + iy}{a + ib}$, multiply denominator and enumerator with the complex conjugate of the denominator,

$$z = \frac{x + iy}{a + ib} = \frac{(x + iy)(a - ib)}{(a + ib)(a - ib)} = \frac{(x + iy)(a - ib)}{a^2 + b^2} \ .$$

Now the denominator is a real number, and we only need to compute the enumerator using formula (5).

Example: Computing the real and imaginary part of $z = \frac{5+3i}{1-2i}$:

$$z = \frac{5 + 3i}{1 - 2i} = \frac{(5 + 3i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{-1 + 13i}{5} = -\frac{1}{5} + \frac{13}{5}i.$$

Hence $\mathrm{Re}(z) = -\frac{1}{5}$ and $\mathrm{Im}(z) = \frac{13}{5}$.

# Summary: Algebraic Properties of $\mathbb{C}$

So far, we have seen that the set $\mathbb{C}$ further expands our number domain:

$$\emptyset \subset \mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Algebraic operations, eg. taking sums, differences, products, quotients, are extended to $\mathbb{C}$, and the same computational rules as for $\mathbb{R}$ apply to $\mathbb{C}$ also. (Exception: monotonicity).
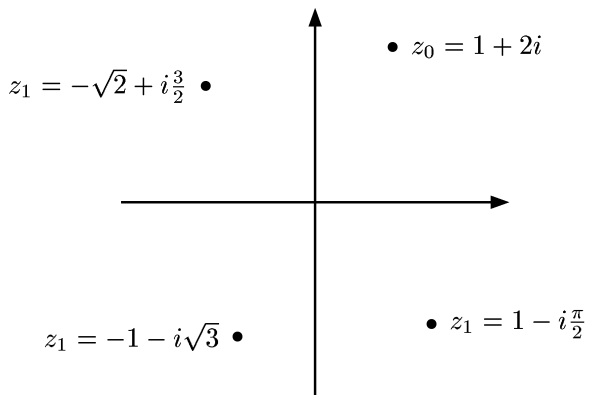
Note: We can now take the square root of $-1$:

$$i^2 = (0 + i \cdot 1) \cdot (0 + i \cdot 1) = 0^2 - 1^2 + 1 \cdot 0 + 0 \cdot 1 = -1$$

## Geometric interpretation of complex numbers

A complex number $a + ib$ is a pair of coordinates describing a point $(a, b)$ in the plane. The modulus is the distance to the origin. The $a$-axis is called real axis, the $b$-axis is the imaginary axis.

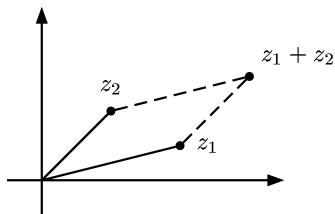## Some more examples



$z_1 = -\sqrt{2} + i\frac{3}{2}$ •

• $z_0 = 1 + 2i$

$z_1 = -1 - i\sqrt{3}$ •

• $z_1 = 1 - i\frac{\pi}{2}$
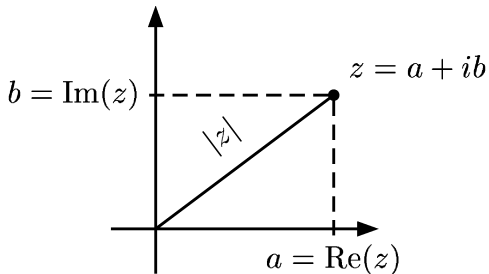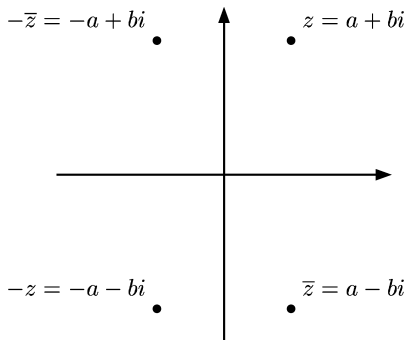
## Geometric interpretation of addition

The sum of two complex numbers $z_1, z_2$ corresponds to the diagonal of the parallelogram with sides $z_1$ and $z_2$.
Triangle inequality: The sum of the sidelengths is greater than or equal to the length of the diagonal.

# Geometric interpretation of complex conjugate

Taking complex conjugates or negatives of complex numbers amounts to <span style="color:red">reflection</span> about a coordinate axis or about the origin



$-\overline{z} = -a + bi$

$z = a + bi$

$-z = -a - bi$

$\overline{z} = a - bi$

## Trigonometric functions
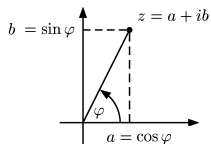
The geometric interpretation of the product requires polar coordinates and trigonometric functions.

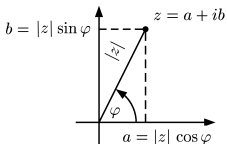Consider a complex number $z = a + ib$ with

$$1 = |z| = \sqrt{a^2 + b^2}$$

$z$ describes an angle $\varphi$ with the real axis, and $|z| = 1$ leads to the equations

$$a = \cos(\varphi) \quad , \quad b = \sin(\varphi)$$

## Polar coordinates

For a general nonzero complex number $z = a + ib$, we write $z = |z|w$, where $w$ has length 1. We then get the same picture, except that all lengths are multiplied by $|z|$:
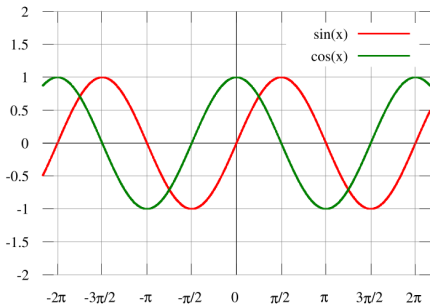


Thus, every $z \in \mathbb{C} \setminus \{0\}$, can be written (uniquely) as

$$z = r \left( \cos(\varphi) + i \sin(\varphi) \right) \text{, where } r > 0 \text{ and } -\pi < \varphi \leq \pi \text{ .}$$

$(r, \varphi)$ are the polar coordinates of $z$, with $r = |z|$, the length of $z$. $\varphi$ is called argument of $z$, denoted $\arg(z)$.

## Plots of sine and cosine



Useful values of sine and cosine:

| $\alpha$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $\sin(\alpha)$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 |
| $\cos(\alpha)$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-1$ |

# Important properties of sine and cosine

**Theorem.**

- Symmetry: For all angles $\alpha$
  $$\sin(-\alpha) = -\sin(\alpha) \ , \ \cos(-\alpha) = \cos(\alpha) \ , \ \sin(\alpha + \frac{\pi}{2}) = \cos(\alpha)$$

- Periodicity: For all angles $\alpha$
  $$\sin(\alpha) = \sin(\alpha + 2\pi) \ , \ \cos(\alpha) = \cos(\alpha + 2\pi) \ \ .$$

- For all angles $\alpha$
  $$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$

- Addition Theorem. For all $\alpha, \beta \in \mathbb{R}$,
  $$\sin(\alpha + \beta) \ = \ \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \qquad (11)$$
  $$\cos(\alpha + \beta) \ = \ \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha) \qquad (12)$$
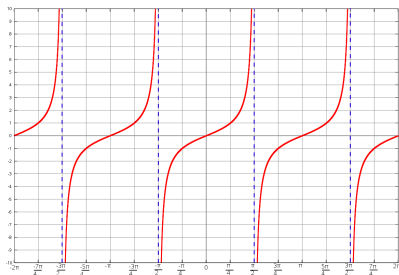
# Tangent and Arctangent

Definition. For $x \in \mathbb{R}$, we define the tangent function as

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Properties of the tangent function:

1. tan is $\pi$-periodic. It is undefined for all $x = k\pi + \pi/2$, with $k \in \mathbb{Z}$.

2. For all $x \in \mathbb{R}$, $\tan(-x) = -\tan(x)$.

3. For every $y \in \mathbb{R}$ there is a unique $x$ with $-\pi/2 < x < \pi/2$ such that $y = \tan(x)$.

4. Let $y \in \mathbb{R}$. Suppose that $-\pi/2 < x < \pi/2$ is the unique number with $\tan(x) = y$. We define $\arctan(y) = x$, the arctangent of $y$.

# Plot of tangent



Useful values of tan:

| $\alpha$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $\tan(\alpha)$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | n.d. | 0 |

## Computing polar coordinates

Given a complex number $z = a + ib$, its length is easily computed as

$$|z| = \sqrt{a^2 + b^2} \ .$$

The argument is defined as $-\pi < \alpha \leq \pi$ such that

$$|z|\cos(\alpha) = a \ , \ |z|\sin(\alpha) = b \ .$$

Hence $\tan(\alpha) = \frac{b}{a}$ is a necessary requirement.

General formula: The argument of $z = a + ib$ is given by

$$\alpha = \begin{cases} \arctan(b/a) & a > 0 \\ \pi - \arctan(b/a) & a < 0 < b \\ -\pi + \arctan(b/a) & a, b < 0 \\ \pi/2 & b > 0 \wedge a = 0 \\ -\pi/2 & b < 0 \wedge a = 0 \end{cases}$$

## Geometric interpretation of the product

Let $z_1 = r_1(\cos(\varphi_1) + i\sin(\varphi_1))$ and $w = r_2(\cos(\varphi_2) + i\sin(\varphi_2))$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2) \\ &+ i\left(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)\right)) \end{aligned}$$
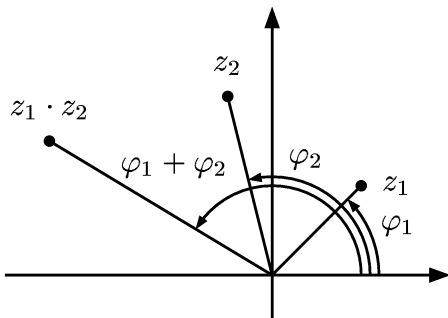
Now the addition theorem for trigonometric functions yields

$$z_1 z_2 = \underbrace{r_1 r_2}_{= |z_1 z_2|} \, (\cos(\underbrace{\varphi_1 + \varphi_2}_{\arg(z_1 z_2)}) + i\sin(\varphi_1 + \varphi_2)) \ ,$$

i.e., $z_1 z_2$ has polar coordinates $(r_1 r_2, \varphi_1 + \varphi_2)$. Hence, one computes the polar coordinates of $z_1 \cdot z_2$ by

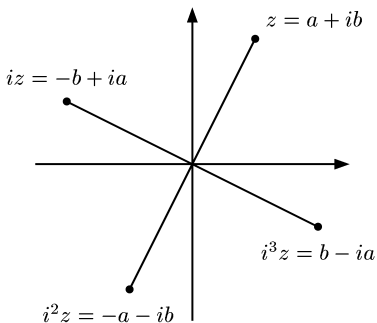- multiplying the lengths of $z_1$ and $z_2$, and
- adding the arguments of $z_1$ and $z_2$
  (possibly adding or subtracting $2\pi$ to remain in $(-\pi, \pi]$)

# Illustration of multiplication

## Example: Multiplication by powers of $i$

$i$ has polar coordinates $(0, \pi/2)$. Hence multiplying by $i$ is the same as rotating by 90 degrees counterclockwise:

## Application: Integer powers and inverse

**Theorem. (DeMoivre)**
Let $n \in \mathbb{N}$, and $z = r(\cos(\varphi) + i \sin(\varphi))$. Then

$$z^n = r^n(\cos(n\varphi) + i \sin(n\varphi))$$

i.e., $z^n$ has polar coordinates $(r^n, n\varphi)$.

**Inverse:** $1 = (1,0) \in \mathbb{C}$ has length 1 and argument $\alpha = 0$. Hence, for arbitrary $z \in \mathbb{C} \setminus \{0\}$

- $1 = |z \cdot z^{-1}| = |z|\, |z^{-1}|$, hence $|z^{-1}| = \frac{1}{|z|}$
- the arguments add up to zero, hence $z^{-1}$ has argument $-\varphi$

therefore the polar coordinates of $z^{-1}$ are $(r^{-1}, -\varphi)$.

$\Rightarrow$ DeMoivre's theorem holds for all $n \in \mathbb{Z}$

## Application: Taking roots

Let $n \in \mathbb{N}$ and $w \in \mathbb{C} \setminus \{0\}$, with polar coordinates $(r, \varphi)$. We want to find all solutions of the equation

$$z^n = w \quad .$$

By periodicity of $\sin, \cos$, there are precisely $n$ such complex numbers, having polar coordinates

$$\left( r^{1/n}, \frac{\varphi}{n} \right) \; , \; \left( r^{1/n}, \frac{\varphi + 2\pi}{n} \right) \; , \; \ldots \; , \left( r^{1/n}, \frac{\varphi + 2\pi(n-1)}{n} \right) \quad .$$

<span style="color:red">Note:</span> Some of the angles

$$\frac{\varphi}{n}, \frac{\varphi + 2\pi}{n}, \ldots, \frac{\varphi + 2\pi(n-1)}{n}$$

are greater than $\pi$. We subtract $2\pi$ from these angles to obtain angles in the prescribed interval $(-\pi, \pi]$.
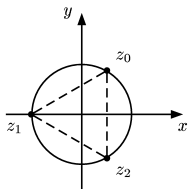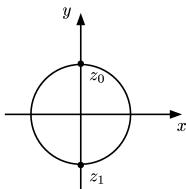
# Examples: Roots of $-1$

Example $w = -1$ has polar coordinates $(1, \pi)$. Hence its square roots have polar coordinates $(1, \frac{\pi}{2})$ and $(1, -\frac{\pi}{2})$, corresponding to $z_{0,1} = \pm i$.

The cubic roots of $-1$ have length 1 and arguments $\frac{\pi}{3}, \pi, \frac{-\pi}{3}$, yielding

$$z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \ , \ z_1 = -1 \ , \ z_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Left: Square roots, right: cubic roots of $-1$

## Further examples: Quadratic equation

Note: An arbitrary quadratic equation

$$z^2 + az + b = 0 \quad ,$$

with $a, b \in \mathbb{C}$, has at least one complex solution. Just as for real coefficients, we can derive the formula

$$z_{1,2} = \frac{a \pm \sqrt{a^2 - 4b}}{2} \quad ,$$

describing all possible solutions, and the root can now be evaluated for every choice of $a$ and $b$.

Remark: In fact, much more is true. Given any polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

with $a_{n-1}, \ldots, a_0 \in \mathbb{C}$, there exists $z \in \mathbb{C}$ with $f(z) = 0$.

## Summary

- Complex numbers and operations on them: Sums, products, inverses of complex numbers
- Imaginary and real parts, complex conjugates
- Polar coordinates: Modulus, argument and their uses
- Geometric interpretation of complex numbers and operations on them
- Computation of powers and roots of complex numbers