Calculus and linear algebra for biomedical engineering
Week 10: Power series.

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Overview

Motivation

For arbitrary $n+1$ times differentiable functions f, the Taylor polynomial

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

is only assumed to be an accurate approximation of $f(x)$ for $x \approx 0$. The reasoning is that the remainder term

$$R_{n,0}(x) = \frac{f^{(n+1)}(y)}{(n+1)!} x^{n+1}$$

with suitable y between 0 and x, is small because x is small (and x^{n+1} is even smaller).

Motivation

However, the Taylor polynomial will also provide a good approximation if x is not too big, and instead,

$$\frac{f^{(n+1)}(y)}{(n+1)!} \approx 0 \ .$$

I.e., if the derivative does not grow too fast on the interval between 0 and x, the Taylor approximation is accurate on larger intervals.

Thus, at least for certain functions f, summing over more terms of the Taylor series should approximate f on larger sets.

Second motivation

For arbitrary $x, y \in \mathbb{R}$, with $x > 0$, what is x^y?

Using standard operations (products, roots), we can evaluate x^y only for rational numbers y: If $y = \frac{n}{m}$, then

$$x^y = (x^n)^{1/m} = \sqrt[m]{x^n} \ .$$

For irrational y, something else is needed.

Solution: For base $e = 2.7182...$, we define e^y via a power series

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \ .$$

For other bases x, we define x^y from this function and the natural logarithm.

Power series

Definition. An expression of the sort

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is called a power series in x.

Remarks

- If $\sum_{k=0}^{\infty} |a_k| r^k < \infty$, for some $r > 0$, then $f(x)$ is well-defined for all x with $|x - x_0| < r$. Moreover, f is infinitely differentiable in $(-r, r)$.
- If a function f has a power series, this series is the Taylor series of f around x_0.

Taylor series

Definition. Let $f : D \to \mathbb{R}$ denote an infinitely differentiable function, with $x_0 \in D$. Then its Taylor series at x_0 is defined as the series

$$T_{\infty,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \ ,$$

Note: The Taylor series need not converge. Even when it does, $T_{\infty,x_0}(x)$ need not coincide with $f(x)$. However, for certain functions f, one finds that

$$R_{n,x_0}(x) \to 0 \ , \ \text{as } n \to \infty$$

and thus $T_{\infty,x_0}(x) = f(x)$.

Radius of convergence

Theorem. Consider a power series

$$(*)\ f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k\ .$$

Suppose that one of the two cases holds:

- $c = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.
  In this case, let $r = 1/c$. If $c = 0$, let $r = \infty$.
- $r = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}$ exists.

Then $(*)$ converges if $|x - x_0| < r$, and diverges if $|x - x_0| > r$.
If both limits exist, the two parts give the same value for r.

The number r in the Theorem is called radius of convergence. The interval $(x_0 - r, x_0 + r)$ is called interval of convergence.

Radius of convergence of the derivative

Theorem. Consider a power series

$$(*)\ f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \ .$$

with radius of convergence $r > 0$ (possibly $r = \infty$).

- f is differentiable, and f' is given by the power series of f'

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1} \ ,$$

  with radius convergence equal to r.
- Applying the previous observations repeatedly, we obtain

$$f^{(n)}(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k \ ,$$

  where $b_k = \frac{(k+n)!}{k!} a_{k+n}$.

Example: Cosine function

Let $f(x) = \cos(x)$. Then, using $\cos' = -\sin$ and $\sin' = \cos$, we can compute all higher derivatives as

$$f^{(n)} = \begin{cases} (-1)^{k+1}\sin(x) & n = 2k+1 \\ (-1)^{k+1}\cos(x) & n = 2k \end{cases}$$

Hence, plugging $\sin(0) = 0$, $\cos(0) = 1$ into the Taylor polynomial, we obtain

$$T_{2n,0}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Residual of the cosine function

The nth residual of the cosine function is estimated as

$$|R_{n,0}(x)| = \left| \frac{\cos^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \left| \frac{x^{n+1}}{(n+1)!} \right|$$

We want to find a range for x such that the Taylor approximation for $f(x)$ is accurate up to precision 0.1. Taking the $n+1$st root,

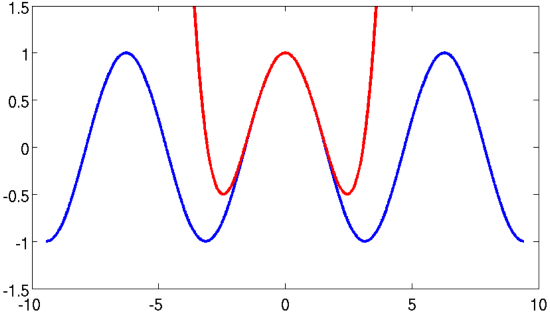
$$\left| \frac{x^{n+1}}{(n+1)!} \right| < 0.1 \Leftrightarrow |x| < \left(\frac{(n+1)!}{10} \right)^{(n+1)^{-1}}$$

This last inequality is fulfilled for instance,

- if $n = 4$ and $|x| < 1.64$;
- or if $n = 12$ and $|x| < 4.74$;
- or if $n = 18$ and $|x| < 7.02$.

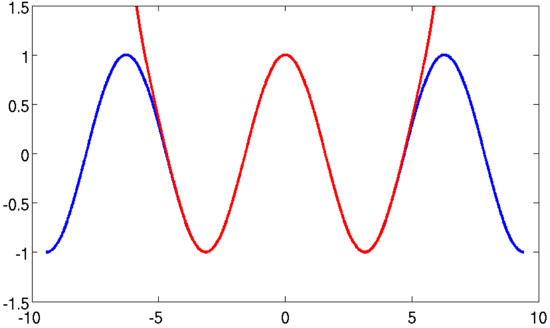
Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{4,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
Accurate up to 0.1 for $|x| < 1.64$



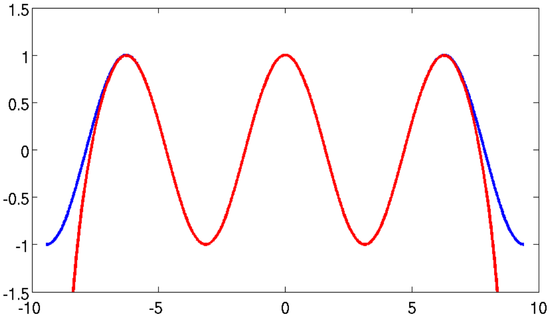
Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{12,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$
Accurate up to 0.1 for $|x| < 4.74$



Approximation of the cosine function

Blue: $\cos(x)$, Red: $T_{18,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{16}}{16!} - \frac{x^{18}}{18!}$
Accurate up to 0.1 for $|x| < 7.02$



Power series for cos, sin

We compute the radius of convergence for the coefficients given by

$$a_n = \begin{cases} 0 & n = 2k+1 \\ \dfrac{(-1)^k}{(2k)!} & n = 2k \end{cases}$$

Now Stirling's formula allows to show that

$$\sqrt[n]{|a_n|} \to 0 \text{ as } n \to \infty$$

and thus $r = \infty$. The same argument works for sin, hence:

Theorem. For all $x \in \mathbb{R}$

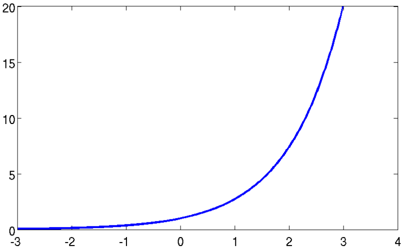
$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

The exponential function

Definition. The function $\exp : \mathbb{R} \to \mathbb{R}$ defined by the series

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is called exponential function.



Properties of the exponential function

Theorem.

1. $\exp : \mathbb{R} \to \mathbb{R}$ is continuous and strictly positive.
2. $\exp$ translates addition to multiplication:
   For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x)\exp(y)$.
3. $\exp$ is differentiable, with $\exp' = \exp$. In particular, $\exp$ is strictly increasing.
4. $\lim_{x\to-\infty} \exp(x) = 0$ and $\lim_{x\to\infty} \exp(x) = \infty$.
5. $\exp : \mathbb{R} \to (0, \infty)$ is bijective.

exp and e^x

- An alternative formula for exp is

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{1/n} .$$

- In particular, $\exp(1) = e$ (Euler's constant).
- Using multiplicativity of exp, one can show for $n \in \mathbb{Z}, m \in \mathbb{N}$ that

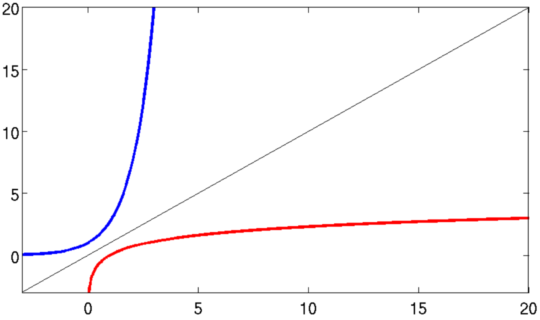
$$\exp(n/m) = e^{n/m}$$

  Hence $\exp(x) = e^x$ for rational x.
- We then define for arbitrary $x \in \mathbb{R}$:

$$e^x = \exp(x) .$$

The natural logarithm

Recall: $\exp : \mathbb{R} \to (0, \infty)$ is bijective. The inverse function is denoted as $\ln : (0, \infty) \to \mathbb{R}$, the natural logarithm.
Blue: exp, red: ln



Properties of ln

Theorem.

1. $\ln : (0, \infty) \to \mathbb{R}$ is continuous, bijective, and strictly increasing.
2. ln translates multiplication to addition:
   For all $x, y \in (0, \infty)$, $\ln(xy) = \ln(x) + \ln(y)$.
3. ln is differentiable on $(0, \infty)$, with
   $$\ln'(x) = \frac{1}{x} .$$
4. $\lim_{x\to 0} \ln(x) = -\infty$ and $\lim_{x\to\infty} \exp(x) = \infty$.
5. $\ln : (0, \infty) \to \mathbb{R}$ is bijective.

Arbitrary exponentials

We define x^y for arbitrary $x > 0$ and $y \in \mathbb{R}$.

$$x^y = e^{\ln(x)y} .$$

Then $f(x) = x^y$ fulfills

1. $f : \mathbb{R} \to (0, \infty)$ is bijective.
2. f translates addition to multiplication:
   For all $s, t \in \mathbb{R}$, $x^{s+t} = x^s x^t$.
3. f is differentiable, with $f' = \ln(x)f$.
4. Multiplication of exponents becomes exponentiation:
   For all $s, t \in \mathbb{R}$, $x^{st} = (x^s)^t$.

Arbitrary logarithms

The function $f(y) = x^y$ has an inverse function, called base x logarithm, denoted by $\log_x$. The function is computed as

$$\log_x(y) = \frac{\ln(y)}{\ln(x)}$$

Often used bases, besides e, are

- 10 (common logarithm $= \log = \log_{10}$)
- 2 (dyadic logarithm $= \log_2$)

Derivatives of logarithms:

$$\frac{d\log_x}{dy}(y_0) = \frac{1}{\ln(x)y_0} .$$

Application: Radioactive decay

If a quantity A of a radioactive substance is given at time $t = 0$, the remaining amount at time $t > 0$ is described by

$$f(t) = Ae^{-\lambda t} \ .$$

Here $\lambda > 0$ is the decay rate of the substance. λ is usually determined by measuring the half-life of the substance, i.e., the time $t_{1/2} > 0$ for which

$$f(t_{1/2}) = \frac{f(0)}{2} = \frac{A}{2} \ .$$

λ can be computed from $t_{1/2}$, and vice versa, because:

$$2 = \frac{f(0)}{f(t_{1/2})} = \frac{A}{Ae^{-\lambda t_{1/2}}} = e^{\lambda t_{1/2}} \Leftrightarrow \lambda t_{1/2} = \ln(2) \ .$$

Complex exponential and Euler's formula

Observation: The series

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

converges for every $z \in \mathbb{C}$. The result is a function

$$\exp : \mathbb{C} \to \mathbb{C}$$

with many interesting properties, in particular,

$$\exp(z + w) = \exp(z)\exp(w) \ .$$

Sorting the real and imaginary parts of $\exp(i\varphi)$ results in Euler's formula for $\alpha \in \mathbb{R}$

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) \ .$$

A proof of Euler's formula

To prove Euler's formula, write

$$e^{i\alpha} = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k \alpha^k}{k!} \ .$$

Using

$$i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i^1, \ldots$$

we find that the series splits into a real part (corresponding to even k) and a purely imaginary part (corresponding to odd k), yielding

$$\begin{aligned} e^{i\alpha} &= \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1}}{(2k+1)!} \\ &= \cos(\alpha) + i \sin(\alpha) \ . \end{aligned}$$

An application of Euler's formula

Addition theorems: Given $\alpha, \beta \in \mathbb{R}$, we compute $e^{i(\alpha+\beta)}$ in two different ways:

$$(*) \quad e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)\ ,$$

or, using $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$,

$$\begin{aligned} e^{i(\alpha+\beta)} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ &+ i(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta))\ . \end{aligned}$$

A comparison of the last expression with the right-hand side of $(*)$ yields the addition theorems for $\sin, \cos$:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) \end{aligned}$$

Summary

- Power series and radius of convergence
- Power series representation of $\sin, \cos$
- The exponential function exp and its properties
- Natural logarithms, arbitrary powers and logarithms
- Derivatives of powers and logarithms
- Rules for powers and logarithms
- Complex exponential and Euler's formula