# Calculus and linear algebra for biomedical engineering
# Week 12: Integration techniques

Hartmut Führ
fuehr@matha.rwth-aachen.de

Lehrstuhl A für Mathematik, RWTH Aachen

January 22, 2009

# Overview

1. Polynomials
2. Integration by parts
3. Substitution
4. Integrating rational functions using partial fraction decomposition

# Motivation

Recall from last week: An integral

$$\int_a^b f(x)dx$$

can be computed in two steps:

- Determine a primitive $F$ of $f$;
- Evaluate at the boundaries: $\int_a^b f(x)dx = F|_a^b = F(b) - F(a)$.

Unfortunately, there is no simple general procedure for the computation of primitives.

Methods for the simplification of integrals are obtained by reading differentiation rules backwards.

## Integrating polynomials

Recall: Monomials $f(x) = x^n$ are easily differentiated: $f'(x) = nx^{n-1}$. Conversely, a primitive of $f$ is obtained as $F(x) = \frac{x^{n+1}}{n+1}$. As a consequence, a primitive of a polynomial

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0$$

is obtained as

$$F(x) = \frac{a_k}{k+1} x^{k+1} + \frac{a_{k-1}}{k} x^k + \ldots + a_0 x + c\ ,$$

where $c \in \mathbb{R}$ is chosen arbitrarily.

Note: The function $F(x) = \frac{x^{s+1}}{s+1}$ is in fact a primitive for $f(x) = x^s$, if $s \in \mathbb{R} \setminus \{-1\}$. The primitive of $f(x) = x^{-1}$ is $F(x) = \ln(|x|)$.

# Product rule and integration by parts

Recall: The product rule for derivatives is

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \ .$$

We use this for the treatment of integrands of the form $f'g$:

$$\begin{aligned}\int_a^b f'(x)g(x)dx &= \int_a^b (fg)'(x)dx - \int_a^b f(x)g'(x)dx \\ &= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx\end{aligned}$$

For indefinite integrals, the rule becomes

$$\int f'(x)g(x)dx = fg - \int f(x)g'(x)dx \ .$$

Rule of thumb: Integration by parts is useful whenever $fg'$ is simpler to integrate than $f'g$.

# Example for integration by parts

Example: Using $f(x) = e^x$ and $g(x) = x^2$, we find

$$\begin{aligned}\int_0^1 e^x x^2 dx &= \int_0^1 f'(x)g(x)dx \\ &= x^2 e^x|_0^1 - \int_0^1 2xe^x dx \\ &= e - 2\int_0^1 xe^x dx\end{aligned}$$

We apply integration by parts again, this time with $f(x) = e^x$ and $g(x) = x$, to obtain

$$\begin{aligned}e - 2\int_0^1 xe^x dx &= e - 2(xe^x)|_0^1 + 2\int_0^1 e^x dx \\ &= e - 2(1 \cdot e^1 - 0e^0) + 2e^x|_0^1 = e - 2\end{aligned}$$

# Further example for integration by parts

Example: We want to determine a primitive for $\ln(x)$, by evaluating the integral

$$F(y) = \int_1^y \ln(x)dx \ .$$

Integration by parts of $1 \cdot \ln(x)$ yields

$$\begin{aligned}
\int_1^y \ln(x)dx &= x\ln(x)|_1^y - \int_1^y x\frac{1}{x}dx \\
&= y\ln(y) - \int_1^y dx \\
&= y\ln(y) - y + 1
\end{aligned}$$

## Chain rule and substitution

Recall: The chain rule for derivatives states that

$$(f \circ g)'(x) = f'(g(x))g'(x) \ .$$

This translates to the following integration rule:
Substitution rule. Suppose that $g : [a, b] \to \mathbb{R}$ is continuously differentiable, and that $f : g([a, b]) \to \mathbb{R}$ is integrable. Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy \ .$$

Proof: If $F$ is a primitive of $f$, then $H(x) = F(g(x))$ is a primitive of $f(g(x))g'(x)$. Therefore

$$\int_a^b f(g(x))g'(x)dx = H(b)-H(a) = F(g(b))-F(g(a)) = \int_{g(a)}^{g(b)} f(y)dy$$

## Substitution and change of variables

It is customary to think of $g(x)$ as a new variable $y$ replacing $x$. $y$ ranges from $g(a)$ to $g(b)$ as $x$ ranges from $a$ to $b$. Moreover,

$$\frac{dy}{dx} = g'(x) \ , \ \text{hence formally } dy = \frac{dy}{dx}dx = g'(x)dx$$

which results in the formula

$$\int_a^b f(y)dy = \int_{y(a)}^{y(b)} f(x)dx \ .$$

Rule of thumb: Substitution is useful, whenever the integrand can be written as $g'(x) \cdot H(x)$, where $g$ and $H$ are suitable functions, and $H(x)$ can be expressed in terms of $g(x)$.

# Examples for substitution

First example: We wish to compute the integral

$$\int_0^2 x\sin(x^2)\ dx = \frac{1}{2}\int_0^2 \sin(x^2)2xdx = \frac{1}{2}\int_0^2 f(g(x))g'(x)dx\ ,$$

with $f(y) = \sin(y)$ and $g(x) = x^2$. Hence $dy = 2xdx$, and

$$\frac{1}{2}\int_0^2 \sin(x^2)2xdx = \frac{1}{2}\int_{0^2}^{2^2} \sin(y)dy = \frac{1}{2}(1-\cos(4))$$

Second example: Let $f(x) = \frac{g'(x)}{g(x)}$, with $g$ continuously differentiable and non-vanishing on $[a,b]$. Then

$$\begin{aligned}\int_a^b \frac{g'(x)}{g(x)}dx &= \int_a^b \frac{1}{g(x)}g'(x)dx \\ &= \int_{g(a)}^{g(b)} \frac{1}{y}dy = \ln(|g(b)|) - \ln(|g(a)|)\end{aligned}$$

# Example: Substitution for an indefinite integral

We want to determine $F = \int (x+2)\sin(x^2+4x-6)dx$.
Substituting

$$y = x^2 + 4x - 6\ ,\ dy = (2x+4)dx\ ,\ (x+2)dx = \frac{dy}{2}$$

we find

$$F(x) = \int f(x)dx = \int \sin(y)\frac{dy}{2} = -\frac{\cos(y)}{2} = -\frac{\cos(x^2+4x-6)}{2}\ .$$

Remark: The new variable $y$ serves as a reminder that we must carry out the substitution before evaluating the integral.

# Integration of rational functions

Aim of the following: A recipe for the integration of functions of the type

$$f(x) = \frac{P(x)}{Q(x)} = \frac{s_m x^m + s_{m-1}x^{m-1} + \ldots + s_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0}$$

Note: One can always write

$$f(x) = c_\ell x^\ell + \ldots + c_0 + \frac{a_k x^k + a_{k-1}x^{k-1} + \ldots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0} ,$$

with $k < n$. We already know how to integrate the polynomial part.

Strategy:

- Write $f$ as a sum of manageable pieces;
- devise a method to integrate the manageable pieces.

# Decomposition into manageable pieces

Theorem. Let

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0} .$$

Then $Q$ has a unique factorization

$$Q(x) = C(x-\xi_1)^{k_1} \cdots (x-\xi_s)^{k_s} (x^2+\beta_1 x+\gamma_1)^{l_1} \cdots (x^2+\beta_t x+\gamma_t)^{l_t} .$$

with suitable numbers $s, t, k_i, l_i \in \mathbb{N}, \xi_i, \beta_i, \gamma_i \in \mathbb{R}$, satisfying in addition

$$4\gamma_i - \beta_i^2 > 0 \ (i = 1, \ldots, t) .$$

This condition is equivalent to requiring that $x^2 + \beta_i x + \gamma_i \neq 0$, for all $x \in \mathbb{R}$ and all $i = 1, \ldots, t$.

# Decomposition into manageable pieces continued

Let $f, P, Q$ be as on the previous slide, with $k < n$. Then there exist unique coefficients $A_{i,j}, B_{i,j}, C_{i,j}$ such that

$$
\begin{aligned}
f(x) &= \frac{A_{1,1}}{(x-\xi_1)^1} + \frac{A_{1,2}}{(x-\xi_1)^2} + \dots + \frac{A_{1,k_1}}{(x-\xi_1)^{k_1}} \\
&+ \frac{A_{2,1}}{(x-\xi_2)^1} + \frac{A_{2,2}}{(x-\xi_2)^2} + \dots + \frac{A_{2,k_2}}{(x-\xi_2)^{k_2}} \\
&+ \dots \\
&+ \frac{A_{s,1}}{(x-\xi_s)^1} + \frac{A_{s,2}}{(x-\xi_s)^2} + \dots + \frac{A_{s,k_s}}{(x-\xi_s)^{k_s}} \\
&+ \frac{B_{1,1}x + C_{1,1}}{(x^2+\beta_1 x+\gamma_1)^1} + \dots + \frac{B_{1,l_1}x + C_{1,l_1}}{(x^2+\beta_1 x+\gamma_1)^{l_1}} \\
&+ \dots \\
&+ \frac{B_{t,1}x + C_{t,1}}{(x^2+\beta_t x+\gamma_t)^1} + \dots + \frac{B_{t,l_t}x + C_{t,l_t}}{(x^2+\beta_t x+\gamma_t)^{l_t}}
\end{aligned}
$$

This sum is called partial fraction decomposition of $f$.

# Example

Suppose that

$$f(x) = \frac{1+x^2}{(x+1)^3(x^2+x+1)^2}$$

Then the partial fraction decomposition of $f$ is of the form

$$\begin{aligned} f(x) &= \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3} \\ &+ \frac{B_1x+C_1}{x^2+x+1} + \frac{B_2x+C_2}{(x^2+x+1)^2} \end{aligned}$$

Hence we need to determine 7 coefficients, $A_1, \ldots, C_2$.

Note: The enumerator does not influence the form of the partial fraction decomposition. It is needed to determine the coefficients $A_1, A_2, \ldots$.

# Primitives for manageable pieces

We still need primitives for the partial fractions:

$$\frac{A}{(x-\xi)^n}\,, \quad \frac{Bx+C}{(x^2+\beta x+\gamma)^n}$$

with $4\gamma - \beta^2 > 0$.

- The function

$$f(x) = \frac{1}{x-\xi} \text{ has the primitive } F(x) = \ln(|x-\xi|)\ .$$

- For $n > 1$, the function

$$f(x) = \frac{1}{(x-\xi)^n} \text{ has the primitive } F(x) = -\frac{1}{(n-1)(x-\xi)^{n-1}}\ .$$

# Primitives for manageable pieces, continued

Suppose that $4\gamma - \beta^2 > 0$. Then $f(x) = \frac{Bx+C}{x^2+\beta x+\gamma}$ has the primitive

$$F(x) = \frac{B}{2}\ln(|x^2 + \beta x + \gamma|) + \frac{2C - B\beta}{\sqrt{4\gamma - \beta^2}} \arctan\left(\frac{2x + \beta}{\sqrt{4\gamma - \beta^2}}\right)$$

The case $f(x) = (Bx + C)(x^2 + \beta x + \gamma)^{-n}$, with $n > 1$, is more complicated. We first simplify the denominator:

$$\int \frac{Bx + C}{(x^2 + \beta x + \gamma)^n} dx = \int \frac{B'y + C'}{(y^2 + 1)^n} dy$$

where

$$\lambda = \sqrt{\gamma - \beta^2/4}\ , \ y = \frac{x + \beta/2}{\lambda}\ , \ B' = \frac{B}{\lambda^{2n-2}}\ , \ C' = \frac{C - B\beta/2}{\lambda^{2n-1}}$$

# Primitives for manageable pieces, finished

We compute

$$\begin{aligned}\int \frac{By+C}{(y^2+1)^n}dy &= \frac{B}{2}\int \frac{2y}{(y^2+1)^n}dy + C\int \frac{1}{(y^2+1)^n}dy \\ &= -\frac{B}{2(n-1)(y^2+1)^{n-1}} + C\int \frac{1}{(y^2+1)^n}dy\ .\end{aligned}$$

For the remaining integral, we observe that

$$\begin{aligned}\int \frac{1}{(y^2+1)^n}dy &= \int \frac{y^2+1}{(y^2+1)^n} - \frac{y^2}{(y^2+1)^n}dy \\ &= \int \frac{1}{(y^2+1)^{n-1}}dy - \int \frac{y^2}{(y^2+1)^n}dy\end{aligned}$$

Furthermore, using integration by parts on the second integral:

$$\int \frac{y^2}{(y^2+1)^n}dy = -\frac{y}{2(n-1)(y^2+1)^{n-1}} + \frac{1}{2(n-1)}\int \frac{1}{(y^2+1)^{n-1}}dy$$

# Summary: Primitives for manageable pieces

The chief difficulty in computing

$$\int \frac{By + C}{(y^2+1)^n}\,dy \text{ is computing } \int \frac{1}{(y^2+1)^n}\,dy\ .$$

For $n > 1$, this is not achieved by a simple formula, but by repeating the same simplification step $n-1$ times:

$$\begin{aligned}
\int \frac{1}{(y^2+1)^1}\,dy &= \arctan(y) \\
\int \frac{1}{(y^2+1)^n}\,dy &= \frac{y}{2(n-1)(y^2+1)^{n-1}} \\
&\quad + \left(1 - \frac{1}{2(n-1)}\right) \int \frac{1}{(y^2+1)^{n-1}}\,dy
\end{aligned}$$

# Summary: Integrating rational functions via partial fractions

General procedure for the integration of $f(x) = \frac{P(x)}{Q(x)}$, $k < n$.

- Determine factorization of the denominator

$$Q(x) = C(x-\xi_1)^{k_1}\cdots(x-\xi_s)^{k_s}(x^2+\beta_1 x+\gamma_1)^{l_1}\cdots(x^2+\beta_t x+\gamma_t)^{l_t}$$

- Determine the coefficients $A_{i,j}, B_{i,j}, C_{i,j}$ in the partial fraction decomposition of $f$. (Comparison of coefficients $\rightsquigarrow$ solve linear equations; see examples)
- Integrate each term in the partial fraction decomposition separately:
  - Use a change of coordinates to simplify the denominator into $(y^2+1)^n$
  - The term $By(y^2+1)^{-n}$ can be integrated directly
  - The term $C(y^2+1)^{-n}$ can be integrated iteratively

## First example

Consider the function $f(x) = \dfrac{4x}{x^2 + 2x - 3}$.

Factorizing the denominator: We compute the roots $x_1 = -3$ and $x_2 = 1$. Hence $x^2 + 2x - 3 = (x-1)(x+3)$.

Partial fraction decomposition: We must determine $A, B$ such that for all $x$,

$$f(x) = \frac{A}{x-1} + \frac{B}{x+3} = \frac{A(x+3) + B(x-1)}{(x-1)(x+3)}$$

Comparing enumerators, this leads to a system of linear equations

$$4x = x(A+B) + 3A - B \Leftrightarrow 4 = A + B \ , \ 0 = 3A - B.$$

# First example

This system of equations is solved by $A = 1$ and $B = 3$. Thus

$$f(x) = \frac{1}{x-1} + \frac{3}{x+3}$$

Integrating the partial fractions yields

$$F(x) = \ln(|x-1|) + 3\ln(|x+3|)$$

# Second example

Consider the function

$$f(x) = \frac{4}{x^3 + x^2 - x - 1} \ .$$

Factorizing the denominator: Since

$$x^3 + x^2 = x^2(x+1) \ , \quad - x - 1 = -1(x+1) \ ,$$

the denominator simplifies to

$$\begin{aligned} x^3 + x^2 - x - 1 &= (x^2 - 1)(x+1) = (x+1)(x+1)(x-1) \\ &= (x+1)^2(x-1) \end{aligned}$$

## Second example

Partial fraction decomposition: We need $A, B, C$ with

$$f(x) = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

Multiplying by the denominator of $f$, we obtain the equation

$$\begin{aligned} 4 &= A(x+1)(x-1) + B(x-1) + C(x+1)^2 \\ &= x^2(A+C) + x(B+2C) - A - B + C \ . \end{aligned}$$

Comparing the coefficients for $x^2$, $x$ and 1, we obtain the equations

$$0 = A + C \ , \ 0 = B + 2C \ , \ 4 = -A - B + C.$$

This system has the solution

$$A = -1 \ , \ B = -2 \ , \ C = 1 \ .$$

# Second example

Therefore,

$$f(x) = \frac{-1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{x-1}$$

is the partial fractions decomposition of $f$.

Integrating the partial fractions yields

$$F(x) = -\ln(|x+1|) - \frac{2}{x+1} + \ln(|x-1|)\ .$$

# Third example

Consider the function

$$f(x) = \frac{3x+2}{(x^2+2x+5)^2}$$

$f$ itself is a partial fraction, hence we directly proceed to compute its primitive.

Substitution simplifies the denominator:

$$\int \frac{3x+2}{(x^2+2x+5)^2} dx = \int \frac{B'y+C'}{y^2+1} dy$$

where

$$y = \frac{x+1}{2}\ ,\ \lambda = \sqrt{5-2^2/4} = 2\ ,\ B' = \frac{3}{4}\ ,\ C' = \frac{-1}{8}$$

# Third example

Using the previously derived formulas, we compute

$$\begin{aligned}
\int \frac{B'y+C'}{(y^2+1)^2}dy &= -\frac{B'}{2(y^2+1)} + C'\int \frac{1}{(y^2+1)^2}dy \\
&= -\frac{B'}{2(y^2+1)} + C'\left(\frac{y}{2(y^2+1)} + \frac{1}{2}\int \frac{1}{(y^2+1)^1}dy\right) \\
&= -\frac{B'}{2(y^2+1)} + C'\left(\frac{y}{2(y^2+1)} + \frac{1}{2}\arctan(y)\right)
\end{aligned}$$

Substituting the expressions for $y, B', C'$ and simplifying yields

$$\int \frac{3x+2}{(x^2+2x+5)^2}dx = -\frac{13+x}{8(x^2+2x+5)} + \frac{\arctan\left(\frac{x+1}{2}\right)}{16}$$

## Summary

- Simple looking integrands may be very hard (or impossible) to integrate. There is no generally applicable integration method, there are only techniques.
- The most important integration techniques:
  - Integration by parts
  - Substitution
  - Partial fractions (for rational integrands)
  - Educated guess and verification by differentiation