Calculus and linear algebra for biomedical engineering
Week 14: Review of selected topics

Hartmut Führ
fuehr@matha.rwth-aachen.de

Lehrstuhl A für Mathematik, RWTH Aachen

February 2, 2009

Overview

Hesse normal form: Definition

The Hesse normal form is a convenient way to describe

- lines in $\mathbb{R}^2$, or
- planes in $\mathbb{R}^3$.

Theorem. Let $d \in \{2, 3\}$, let $\mathbb{S} \subset \mathbb{R}^d$ be a line (for $d = 2$) or plane (for $d = 3$). Then there exist unique $\mathbf{n} \in \mathbb{R}^d$ with $|n| = 1$ and $r \geq 0$ such that

$$\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n} = r\} .$$

The vector $\mathbf{n}$ is called normal vector of $\mathbb{S}$.

Only exception to uniqueness: If $r = 0$, then both $\mathbf{n}$ and $-\mathbf{n}$ give the same set $\mathbb{S}$.

Computing Hesse normal form: Line case

Given a line $\mathbb{L} \subset \mathbb{R}^2$ in parametric form

$$\mathbb{L} = \{\mathbf{c} = \mathbf{a} + s\mathbf{b} : s \in \mathbb{R}\} \quad ,$$

with $\mathbf{b} = (b_1, b_2)^T$, we compute the Hesse normal form by the following procedure:

- Determine the two possible candidates for the normal vector, namely
$$\mathbf{n}_{\pm} = \pm\frac{(b_2, -b1)^T}{|\mathbf{b}|} \quad .$$
- Pick $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{a} \geq 0$.
- Let $r = \mathbf{n} \cdot \mathbf{a}$.

Computing Hesse normal form: Line case

Given a line $\mathbb{L} \subset \mathbb{R}^2$, defined implicitly via

$$\mathbb{L} = \{(x_1, x_2)^T \in \mathbb{R}^2 : b_1 x_1 + b_2 x_2 = c\} \quad ,$$

we compute the Hesse normal form by the following procedure:

- Determine the two possible candidates for $\mathbf{n}$, namely

$$\mathbf{n}_{\pm} = \pm \frac{1}{|(b_1, b_2)^T|}(b_1, b_2)^T \quad .$$

- Pick $\mathbf{n}_+$ if $c \geq 0$, otherwise $\mathbf{n}_-$.
- Let $r = |c|/|b|$.

Computing Hesse normal form: Plane case

Given a plane $\mathbb{P} \subset \mathbb{R}^3$ in parametric form

$$\mathbb{P} = \{\mathbf{x} = \mathbf{z} + s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\} \quad ,$$

we compute the Hesse normal form by the following procedure:

- Determine the possible candidates for $\mathbf{n}$ via the normalized cross product of $\mathbf{a}$ and ,

$$\mathbf{n} = \pm \frac{1}{|\mathbf{a} \times \mathbf{b}|} \mathbf{a} \times \mathbf{b}$$

- Pick $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{z} \geq 0$.
- Let $r = \mathbf{n} \cdot \mathbf{z}$.

Computing Hesse normal form: Plane case

Given a plane $\mathbb{P} \subset \mathbb{R}^3$, defined implicitly via

$$\mathbb{P} = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : b_1 x_1 + b_2 x_2 + b_3 x_3 = c\} \quad ,$$

we compute the Hesse normal form by the following procedure:

- Determine the two possible candidates for $\mathbf{n}$, namely

$$\mathbf{n}_\pm = \pm \frac{1}{|(b_1, b_2, b_3)^T|}(b_1, b_2, b_3)^T \quad .$$

- Pick $\mathbf{n}_+$ if $c \geq 0$, otherwise $\mathbf{n}_-$.
- Let $r = |c|/|b|$.

Application of HNF: Computing distances

Given a set $\mathbb{S} \subset \mathbb{R}^d$ $(d = 2, 3)$ in HNF,

$$\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n} = r\} \ .$$

and a point $\mathbf{x} \in \mathbb{R}^d$, the shortest distance of $\mathbf{x}$ to $\mathbb{S}$ is computed as

$$\mathrm{dist}(\mathbf{x}, \mathbb{S}) = |\mathbf{x} \cdot \mathbf{n} - c| \ .$$

Application of HNF: Intersection of lines or planes

Let $\mathbb{S}_1, \mathbb{S}_2 \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, be given in HNF with

$$\mathbb{S}_i = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n}_i = r_i\} \ .$$

Then

- $\mathbb{S}_1 = \mathbb{S}_2$ if and only if either $r_1 = r_2 \neq 0$ and $\mathbf{n}_1 = \mathbf{n}_2$, or if $r_1 = r_2 = 0$, and $\mathbf{n}_1 = \pm\mathbf{n}$.
- $\mathbb{S}_1 \cap \mathbb{S}_2$ is a single point (for $d = 1$) or a line (for $d = 2$) if and only if $\mathbf{n}_1 \neq \pm\mathbf{n}_2$.
- In all remaining cases, $\mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset$.

Necessary convergence criteria

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of real numbers.

- Cauchy criterion:
  $(x_k)_{k\in\mathbb{N}}$ converges if and only if
  $$|x_k - x_n| \to 0 \quad \text{as} \quad \min(k,n) \to \infty$$
- Boundedness:
  If $(x_k)_{k\in\mathbb{N}}$ converges, it is bounded.
- The boundedness criterion follows from the Cauchy criterion.
- Violation of necessary criteria for convergence is a sufficient criterion for divergence.
- A bounded sequence that is not Cauchy (and thus divergent):
  $$x_k = (-1)^k \ .$$

Sufficient conditions for convergence

- Cauchy criterion
- If $(x_k)_{k\in\mathbb{N}}$ is bounded and monotonic, it converges.
- Convergence and algebraic operations: Sums, products, differences, quotients (if defined) of convergent sequences are convergent again, and the limits are obtained by the same operation: E.g., if

$$x = \lim_{k\to\infty} x_k \ , \ y = \lim_{k\to\infty} y_k$$

  then

$$x + y = \lim_{k\to\infty} x_k + y_k \ , \ xy = \lim_{k\to\infty} x_k y_k$$

  etc.
- Continuity:
  If $(x_k)_{k\to\infty} \subset D$ with $\lim_{k\to\infty} x_k \in D$, and $f : D \to \mathbb{R}$ is continuous, then

$$\lim_{k\to\infty} f(x_k) = f(x) \ .$$

Known limits

- For $\alpha \in \mathbb{R}$,

$$\lim_{k\to\infty} k^{\alpha} = \begin{cases} \infty & \alpha > 0 \\ 1 & \alpha = 0 \\ 0 & \alpha < 0 \end{cases}$$

- $\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e$.
- Given polynomials P, Q, one can determine $\lim_{k\to\infty} \frac{P(k)}{Q(k)}$ by comparing the degrees of the polynomials.
- $\lim_{k\to\infty} \frac{k^{\alpha}}{c^k} = 0$, for all $c > 1$ and all $\alpha \in \mathbb{R}$.
- $\lim_{k\to\infty} \frac{c^k}{k!} = 0$, for all $c \in \mathbb{R}$.
- $\lim_{k\to\infty} \sqrt[k]{c} = 1$, for all $c > 0$.
- $\lim_{k\to\infty} \sqrt[k]{k} = 1$.

Convergence criteria for series

- Necessary Criterion: If $\sum_{k=0}^{\infty} x_k = x \in \mathbb{R}$, then

$$\lim_{k\to\infty} x_k = 0$$

- This criterion is not sufficient:

$$\lim_{k\to\infty} k^{-1} = 0 \text{ , but } \sum_{k=1}^{\infty} k^{-1} = \infty \text{ .}$$

Sufficient criteria for series convergence

- Absolute convergence: If $\sum_{k=1}^{\infty} |x_k| < \infty$, then $\sum_{k=1}^{\infty} x_k$ converges, with $|\sum_{k=1}^{\infty} x_k| \leq \sum_{k=1}^{\infty} |x_k|$.
- Majorant criterion: If $\sum_{k=0}^{\infty} z_k$ is absolutely convergent with $|x_k| < |z_k|$, then $(x_k)_{k\in\mathbb{N}}$ converges absolutely.
- Quotient criterion: If there exists c with $0 < c < 1$ and $M > 0$, such that for all $n > M$, $\left|\frac{x_{n+1}}{x_n}\right| < c$, then $\sum_{n=0}^{\infty} x_n$ converges absolutely.
- Leibniz criterion: Suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to zero, and fulfills $|x_{n+1}| < |x_n|$ for all n, as well as $x_{n+1} \cdot x_n < 0$. Then $\sum_n x_n$ converges.
- Algebraic operations: Let $x_k = y_k + sz_k$, for all $k \in \mathbb{N}$), with convergent series $\sum_{k=1}^{\infty} y_k$ and $\sum_{k=1}^{\infty} z_k$, as well as $s \in \mathbb{R}$.

$$\text{Then } \sum_{k=1}^{\infty} x_k = \left(\sum_{k=1}^{\infty} y_k\right) + s\left(\sum_{k=1}^{\infty} z_k\right)$$

Known series

- Geometric series: For $|q| < 1$,
$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$
- The series
$$\sum_{k=1}^{\infty} k^{\alpha}$$
converges precisely for $\alpha < -1$. The divergent series corresponding to $\alpha = -1$ is called harmonic series.
- The exponential series is given by
$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \, ,$$
where $x \in \mathbb{R}$, convergent for every choice of x. (For further examples, see lecture on power series.)