Calculus and linear algebra for biomedical engineering
Week 2: Linear algebra and analytic geometry

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Overview

Tuples or row vectors

Definition.
For $n \in \mathbb{N}$, and $x_1, \dots, x_n \in \mathbb{R}$, we denote the associated n-tuple or row vector by $(x_1, \dots, x_n)$. Two row vectors $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are equal precisely when

$$x_1 = y_1 \ \wedge \ x_2 = y_2 \ \wedge \ \dots \ \wedge \ x_n = y_n \ .$$

Remarks

- Note the difference of tuples to sets: $\{1, 2, 4\} = \{4, 2, 1\}$, but $(1, 2, 4) \neq (4, 2, 1)$.
- Clearly, we can identify $\mathbb{R}^1$ with $\mathbb{R}$. Observe also that $\mathbb{R}^2 = \mathbb{C}$, by our original definition.

Interpretation of tuples

Tuples are ordered collections of data. For instance, suppose you want to record, for a group of people,

- shoe size (german units),
- height (in cm), and
- weight (in kg).

This amounts to recording a 3-tuple (or triple) of numbers for each person, e.g., in the order shoe size, height, weight.

Here it is clear that the tuples $(43, 180, 75)$ and $(75, 180, 43)$ are vastly different.

Column vectors. The set $\mathbb{R}^n$

Definition.
For $n \in \mathbb{N}$, and $x_1, \ldots, x_n \in \mathbb{R}$, we denote the associated column vector by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ or } (x_1, \ldots, x_n)^T \ .$$

We define the *n*-dimensional Euclidian space as the set of column vectors

$$\mathbb{R}^n = \{(x_1, \ldots, x_n)^T \ : \ x_1, \ldots, x_n \in \mathbb{R}\}$$

Elements of $\mathbb{R}^n$ are denoted as $\mathbf{x} = (x_1, \ldots, x_n)^T$. The origin is the vector $\mathbf{0} = (0, \ldots, 0)^T \in \mathbf{R}^n$.

Vector space operations

Definition.
Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, and $s \in \mathbb{R}$.
Vector addition/subtraction: The sum of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, \ldots, x_n+y_n)^T \ , \ \mathbf{x}-\mathbf{y} = (x_1-y_1, \ldots, x_n-y_n)^T \quad (1)$$

Multiplying a vector with a scalar: Scalar multiplication of $s \in \mathbb{R}$ with the vector $\mathbf{x}$ is defined as

$$s \cdot (x_1, \ldots, x_n)^T = (sx_1, \ldots, sx_n)^T \quad . \qquad (2)$$

The $\cdot$ is often omitted.

Remarks: The definition of the addition generalizes addition in $\mathbb{R} = \mathbb{R}^1$ and in $\mathbb{C} = \mathbb{R}^2$.

Vector space axioms

Theorem.

$\mathbb{R}^n$ fulfills the vector space axioms: Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ be arbitrary vectors, and $s, t \in \mathbb{R}$.

V.1 $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

V.2 $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

V.3 $\mathbf{0} = (\mathbf{a} - \mathbf{a}) = 0 \cdot \mathbf{a}$

V.4 $s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}$.

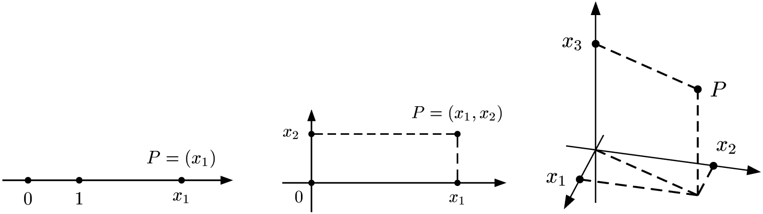
V.5 $(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}$

V.6 $\mathbf{a} = 1\mathbf{a} = \mathbf{0} + \mathbf{a}$

Geometric interpretation of vectors

For $n = 1, 2, 3$, we can think of $\mathbb{R}^n$ as a straight line, a plane and as three-dimensional space, respectively. Elements are visualized both as points or as arrows connecting the points with the origin $\mathbf{0}$.

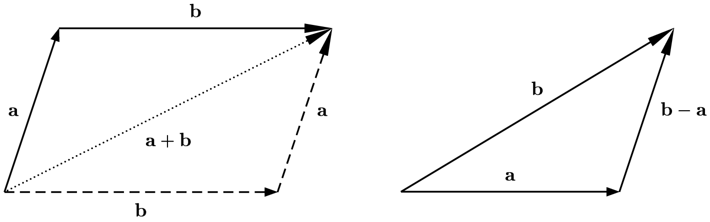
Points in n-dimensional space, $n = 1, 2, 3$



Geometric interpretation of addition

The sum of two vectors **a**, **b** corresponds to the diagonal of the parallelogram with sides **a** and **b**.

Illustration of sum and difference



Scalar product and length

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$.

1. The scalar product of $\mathbf{a}$ and $\mathbf{b}$ is defined as
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \ldots + a_n b_n \quad .$$
2. $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ is called length or Euclidian norm of $\mathbf{a}$. The distance between two vectors $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} - \mathbf{b}|$.
3. If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are called orthogonal, and we write $\mathbf{a} \perp \mathbf{b}$.

Remarks:

- Warning: Do not confuse scalar product with scalar multiplication!
  In scalar products, the $\cdot$ is not omitted.
- The length of a vector generalizes the length of complex numbers.

Properties of scalar product and length

Theorem.

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $s \in \mathbb{R}$.

(i) $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$

(ii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(iii) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

(iv) $|\mathbf{a}| \geq 0$, with $|\mathbf{a}| = 0$ only for $\mathbf{a} = 0$

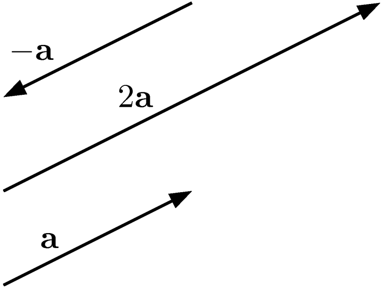
(v) $|s\mathbf{a}| = |s|\ |\mathbf{a}|$

(vi) Cauchy-Schwarz inequality: $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\ |\mathbf{b}|$

(vii) Triangle inequality: $\big|\, |\mathbf{a}| - |\mathbf{b}| \,\big| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

Geometric interpretation of scalar multiplication

As a consequence of part (v) of the theorem: Multiplication by a scalar $s > 0$ amounts to multiplying the length with s. Multiplication by $s = -1$ results in a vector pointing in the opposite direction.



Angle and projection

Definition.

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

(i) The orthogonal projection of $\mathbf{a}$ onto $\mathbf{b}$ is defined as

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

(ii) The angle between $\mathbf{a}$ and $\mathbf{b}$ is defined as the unique $\alpha \in [0, \pi)$ satisfying

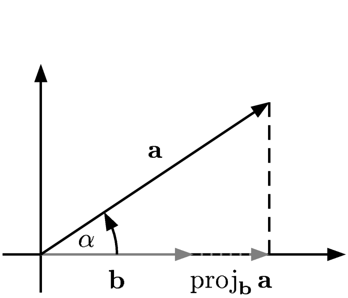
$$\cos(\alpha) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} \quad .$$

Note that if $|\mathbf{b}| = 1$, the two notions are related via

$$|\mathrm{proj}_{\mathbf{b}}\mathbf{a}| = |\mathbf{a}| \cos(\alpha)$$

Illustration of angle and projection

Orthogonal projection (left) and angle (right). Projection amounts to dropping a perpendicular from **a** onto **b**.



Properties of angle and projection

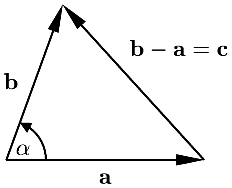
Theorem. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with angle α, and $\mathbf{c} = \mathbf{a} - \mathbf{b}$.

(i) Cosine Theorem:

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\alpha) \quad . \tag{3}$$

(ii) Pythagoras' Theorem: If $\mathbf{a} \perp \mathbf{b}$, then

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \quad . \tag{4}$$



Straight lines

Definition. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, with $\mathbf{b} \neq 0$. The straight line through $\mathbf{a}$ with direction $\mathbf{b}$ is the set

$$\mathbb{L} = \{\mathbf{c} = \mathbf{a} + s\mathbf{b} : s \in \mathbb{R}\} \quad . \tag{5}$$

The description (5) is called parametric form of $\mathbb{L}$, $\mathbf{b}$ is called its direction vector.

Remark: The line $\mathbb{L}$ does not change, if we replace

- $\mathbf{a}$ by $\mathbf{a}' = \mathbf{a} + s\mathbf{b}$, and
- $\mathbf{b}$ by $\mathbf{b}' = r\mathbf{b}$ (for $r \neq 0$).

I.e., the line does not depend on the length of the direction vector.

Parallel lines

Definition. Let $\mathbb{L}, \mathbb{L}' \subset \mathbb{R}^n$ be straight lines, such that

1. $\mathbb{L}$ is the straight line through $\mathbf{a}$ with direction $\mathbf{b}$;
2. $\mathbb{L}'$ is the straight line through $\mathbf{a}'$ with direction $\mathbf{b}'$; and
3. there exists real number s such that $\mathbf{b}' = s\mathbf{b}$.

Then $\mathbb{L}$ and $\mathbb{L}'$ are called parallel.

Straight lines in $\mathbb{R}^2$

Theorem. Let $\mathbb{L}, \mathbb{L}' \subset \mathbb{R}^2$ denote straight lines. Then precisely one of the following three cases can occur:

1. $\mathbb{L} = \mathbb{L}'$;
2. $\mathbb{L}$ and $\mathbb{L}'$ are parallel, with $\mathbb{L} \cap \mathbb{L}' = \emptyset$;
3. $\mathbb{L} \cap \mathbb{L}' = \{\mathbf{x}\}$, for a suitable $\mathbf{x} \in \mathbb{R}^2$.

Hence, the intersection of two straight lines consists either of zero, one or infinitely many points.
How does one decide which case applies? And how does one compute the intersection?

Alternative descriptions of lines

Theorem. Let $\mathbb{L} \subset \mathbb{R}^n$ be a straight line

(i) $\mathbb{L}$ is uniquely determined by two points $\mathbf{x}, \mathbf{y} \in \mathbb{L}$, with $\mathbf{x} \neq \mathbf{y}$: Defining $\mathbf{b} = \mathbf{x} - \mathbf{y}$, one has

$$\mathbb{L} = \{\mathbf{c} = \mathbf{x} + s\mathbf{b} : s \in \mathbb{R}\} \quad . \tag{6}$$

(ii) Assume that $n = 2$. There exists a vector $\mathbf{n} \in \mathbb{R}^2$ with $|\mathbf{n}| = 1$, and $r \geq 0$ such that

$$\mathbb{L} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{n} = r\} \quad . \tag{7}$$

Part (i) is very convenient for defining lines, whereas part (ii) will turn out useful for calculations. The equation (7) is called Hesse's normal form of the line $\mathbb{L}$, and $\mathbf{n}$ is the normal vector of $\mathbb{L}$.

Applications of Hesse's normal form

Theorem

Consider two straight lines $\mathbb{L}, \mathbb{M} \subset \mathbb{R}^2$, given by the equations

$$\mathbb{L} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{n} = r\} \text{ and } \mathbb{M} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{m} = s\} \,, \qquad (8)$$

with $\mathbf{n}, \mathbf{m}$ of length 1, and $r, s > 0$. Then the following statements are true:

(i) $\mathbb{L}$ is uniquely defined by $\mathbf{n}$ and $r > 0$: $\mathbb{L} = \mathbb{M}$ if and only if $\mathbf{n} = \mathbf{m}$ and $r = s$.

(ii) $\mathbb{L} \cap \mathbb{M} = \emptyset$ if and only if $(\mathbf{n} = \mathbf{m}$ and $r \neq s)$ or $(\mathbf{n} = -\mathbf{m})$.

Computation of Hesse's normal form

Given a line $\mathbb{L} \subset \mathbb{R}^2$ in parametric form

$$\mathbb{L} = \{\mathbf{c} = \mathbf{a} + s\mathbf{b} : s \in \mathbb{R}\} \quad ,$$

how does one compute its Hesse normal form? We are looking for $\mathbf{n}$ and $r > 0$ such that, in particular,

$$\begin{aligned} \mathbf{n} \cdot \mathbf{a} &= r \\ \mathbf{n} \cdot (\mathbf{a} + \mathbf{b}) &= r \quad . \end{aligned}$$

Subtracting the upper from the lower equation, we obtain

$$\mathbf{n} \cdot \mathbf{b} = 0 \text{ , which means } \mathbf{n} \perp \mathbf{b}, \text{ or } n_1 b_1 + n_2 b_2 = 0 \quad .$$

We also want $|\mathbf{n}| = 1$, which leaves two possible solutions

$$\mathbf{n}_{1,2} = \pm \frac{(b_2, -b1)}{|\mathbf{b}|} \quad .$$

Among these, pick $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{a} \geq 0$, and let $r = \mathbf{n} \cdot \mathbf{a}$.

Minimal distance of a point to a line

Given a line $\mathbb{L} \subset \mathbb{R}^2$ and a point $\mathbf{x} \in \mathbb{R}^2$, we define the distance of $\mathbf{x}$ to $\mathbb{L}$ as

$$\mathrm{dist}(\mathbf{x}, \mathbb{L}) = \min_{\mathbf{c} \in \mathbb{L}} |\mathbf{c} - \mathbf{x}|$$

Theorem. Let $\mathbb{L}$ be given in Hesse normal form,

$$\mathbb{L} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{n} = r\}$$

Then the distance of a point $\mathbf{x}$ to line $\mathbb{L}$ is computed as

$$\mathrm{dist}(\mathbf{x}, \mathbb{L}) = |\mathbf{x} \cdot \mathbf{n} - r| \quad .$$

The point on $\mathbb{L}$ with smallest distance to $\mathbf{x}$ is computed as

$$\mathbf{c}_0 = \mathbf{x} + (r - \mathbf{x} \cdot \mathbf{n})\mathbf{n} \ .$$

Linear combinations and linear independence

Definition. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$.

1. $\mathbf{b} \in \mathbb{R}^n$ is called linear combination of $\mathbf{a}_1, \ldots, \mathbf{a_m}$ if there exist coefficients $s_1, \ldots, s_m \in \mathbb{R}$ such that

$$\mathbf{b} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \ldots + s_m\mathbf{a}_m \quad .$$

2. The system $(\mathbf{a}_j)_{j=1,\ldots,m}$ is called linearly dependent if, for some index $1 \leq i \leq m$, the vector $\mathbf{a}_i$ is a linear combination of $(\mathbf{a}_j)_{j\neq i}$. Otherwise, it is called linearly independent.

Theorem. (Uniqueness of coefficients) Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be linearly independent, and assume that

$$\begin{aligned} \mathbf{b} &= s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \ldots + s_m\mathbf{a}_m \\ &= t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \ldots + t_m\mathbf{a}_m \end{aligned}$$

Then $t_1 = s_1,\ t_2 = s_2, \ldots,\ t_m = s_m$.

Example for linear independence

Consider the vectors $\mathbf{a} = (1, 0), \mathbf{b} = (0, 1)^T$ and $\mathbf{c} = (1, 1)^T$. Then $\mathbf{a}, \mathbf{b}$ are linearly independent:

$$\text{For all } s \in \mathbb{R} : s\mathbf{a} = (s, 0)^T \neq (0, 1)^T = \mathbf{b} \quad .$$

One shows similarly that $\mathbf{a}, \mathbf{c}$ are linearly independent, as well as $\mathbf{b}, \mathbf{c}$.

But: The system $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is linearly dependent: $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

The span of a system of vectors

Definition. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. The span of these vectors is the set

$$\begin{aligned} &\mathrm{span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m) = \\ &\quad = \{\mathbf{b} \in \mathbf{R}^n : \mathbf{b} \text{ is a linear combination of } \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\} \subset \mathbf{R}^n \end{aligned}$$

Theorem. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$, and let $U = \mathrm{span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m)$. Then U is a subspace, i.e. it fulfills

S.1 $\mathbf{0} \in U$.

S.2 If $\mathbf{b}, \mathbf{c} \in U$, then $\mathbf{b} + \mathbf{c} \in U$.

S.3 If $s \in \mathbb{R}$, and $\mathbf{b} \in U$, then $s\mathbf{b} \in U$.

Remark: The theorem expresses that U is closed under vector addition and scalar multiplication.

Definition of a plane

Definition. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, with $\mathbf{a}$, $\mathbf{b}$ linearly independent. The plane through $\mathbf{c}$ spanned by $\mathbf{a}$ and $\mathbf{b}$ is the set

$$\begin{aligned} \mathbb{P} &= \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} = \mathbf{c} + \mathbf{y} \ , \quad \text{with } \mathbf{y} \in \operatorname{span}(\mathbf{a}, \mathbf{b})\} \\ &= \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} = \mathbf{c} + s\mathbf{a} + t\mathbf{b} \ , \ s, t \in \mathbb{R}\} \end{aligned}$$

Examples

- The set $\{(r, s, 1)^T : r, s \in \mathbb{R}\}$ is the plane through $(0, 0, 1)^T$ spanned by the vectors $(1, 0, 0)^T$ and $(0, 1, 0)^T$.
- $\mathbb{R}^2$ is the only plane contained in $\mathbb{R}^2$: If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ are linearly independent, $\operatorname{span}(\mathbf{a}, \mathbf{b}) = \mathbb{R}^2$. As a consequence, every plane $\mathbb{P} \subset \mathbb{R}^2$ fulfills $\mathbb{P} = \mathbb{R}^2$.

Alternative definitions of planes

Theorem. Let $\mathbb{P} \subset \mathbb{R}^n$ be a plane

(i) $\mathbb{P}$ is uniquely determined by three points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{P}$, subject to the condition that

$$\mathbf{a} = \mathbf{x} - \mathbf{z} \text{ and } \mathbf{b} = \mathbf{y} - \mathbf{z}$$

are linearly independent. In this case,

$$\mathbb{P} = \{\mathbf{c} = \mathbf{z} + s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\} \quad . \tag{9}$$

(ii) Assume that $n = 3$. There exists a vector $\mathbf{n} \in \mathbb{R}^3$ with $|\mathbf{n}| = 1$, and $r \geq 0$ such that

$$\mathbb{P} = \{\mathbf{c} \in \mathbb{R}^3 : \mathbf{c} \cdot \mathbf{n} = r\} \quad . \tag{10}$$

The equation (10) is called Hesse's normal form of the plane $\mathbb{P}$, and $\mathbf{n}$ is called the normal vector of the plane.

Planes in $\mathbb{R}^3$

Theorem. Let $\mathbb{P}, \mathbb{P}' \subset \mathbb{R}^3$ denote planes. Then precisely one of the following three cases can occur:

1. $\mathbb{P} = \mathbb{P}'$;
2. $\mathbb{P} \cap \mathbb{P}'$ is a straight line;
3. $\mathbb{P} \cap \mathbb{P}' = \emptyset$.

In the first or third case, $\mathbb{P}$ and $\mathbb{P}'$ are called parallel

As before, the different cases are best sorted out using the Hesse normal forms of the two planes.

Applications of Hesse's normal form for planes

Theorem

Consider planes $\mathbb{P}, \mathbb{M} \subset \mathbb{R}^3$, given by the equations

$$\mathbb{P} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{n} = r\} \text{ and } \mathbb{M} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot \mathbf{m} = s\} \ , \qquad (11)$$

with $\mathbf{n}, \mathbf{m}$ of length 1, and $r, s > 0$. Then the following statements are true:

(i) $\mathbb{P}$ is uniquely defined by $\mathbf{n}$ and $r > 0$: $\mathbb{P} = \mathbb{M}$ if and only if $\mathbf{n} = \mathbf{m}$ and $r = s$.

(ii) $\mathbb{P} \cap \mathbb{M} = \emptyset$ if and only if $(\mathbf{n} = \mathbf{m}$ and $r \neq s)$ or $(\mathbf{n} = -\mathbf{m})$. Hence: $\mathbb{P}$ and $\mathbb{M}$ are parallel iff their normal vectors coincide up to a sign.

(iii) $\mathbb{P} \cap \mathbb{M}$ is a straight line if and only if $\mathbf{n} \neq \pm\mathbf{m}$.

Remaining question: How to compute the HNF of a plane.

Cross product of two vectors

Definition. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the cross product $\mathbf{a} \times \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} . \tag{12}$$

Theorem: Properties of the cross product.
For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $r \in \mathbb{R}$,

(i) $r(\mathbf{a} \times \mathbf{b}) = (r\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (r\mathbf{b})$, as well as $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.

(ii) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, in particular $\mathbf{a} \times \mathbf{a} = 0$.

(iii) $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

(iv) If $\mathbf{a}, \mathbf{b}$ have angle α, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\alpha)$ (area of the parallelogram with sides $\mathbf{a}, \mathbf{b}$).

Computation of Hesse's normal form for planes

Given a plane $\mathbb{P} \subset \mathbb{R}^2$ in parametric form

$$\mathbb{P} = \{\mathbf{x} = \mathbf{z} + s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\} \quad ,$$

we are looking for $\mathbf{n}$ with $|\mathbf{n}| = 1$ and $r > 0$, meeting the requirements

$$\begin{aligned} \mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{b} &= 0 \\ \mathbf{n} \cdot \mathbf{z} &= r \quad . \end{aligned}$$

Using the properties of the cross product, we see that

$$\mathbf{n} = \pm \frac{1}{|\mathbf{a}|\ |\mathbf{b}|\ \sin(\alpha)} \mathbf{a} \times \mathbf{b}$$

is the suitable candidate. The sign of the right hand side is chosen to guarantee that $\mathbf{n} \cdot \mathbf{z} \geq 0$, and let $r = \mathbf{n} \cdot \mathbf{z}$.

Minimal distance of a point to a plane

Given a plane $\mathbb{P} \subset \mathbb{R}^3$ and a point $\mathbf{x} \in \mathbb{R}^3$, we define the distance of $\mathbf{x}$ to $\mathbb{P}$ as

$$\mathrm{dist}(\mathbf{x}, \mathbb{P}) = \min_{\mathbf{c} \in \mathbb{P}} |\mathbf{c} - \mathbf{x}|$$

Theorem. Let $\mathbb{P}$ be given in Hesse normal form,

$$\mathbb{P} = \{\mathbf{c} \in \mathbb{R}^3 : \mathbf{c} \cdot \mathbf{n} = r\}$$

Then the distance of a point $\mathbf{x}$ to $\mathbb{P}$ is computed as

$$\mathrm{dist}(\mathbf{x}, \mathbb{P}) = |\mathbf{x} \cdot \mathbf{n} - r| \quad .$$

The point in $\mathbb{P}$ with smallest distance to $\mathbf{x}$ is computed as

$$\mathbf{c}_0 = \mathbf{x} + (r - \mathbf{x} \cdot \mathbf{n})\mathbf{n} \ .$$

Summary

Important notions

- The vector space $\mathbb{R}^n$ and operations defined on it.
- Geometric interpretations of vector addition and scalar multiplication
- Scalar product and Euclidian length
- Linear combinations and linear independence
- Subspaces, lines and planes
- Hesse's normal form for lines in $\mathbb{R}^2$ and for planes in $\mathbb{R}^3$.
  - How to compute it, and
  - how to use it (e.g., computing distances)