Calculus and linear algebra for biomedical engineering
Week 5: Sequences, series, and their limits

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Overview

1. Sequences
2. Limits
3. Computing limits
4. Series

Motivation

We want to study the growth of a culture of bacteria. We are given an initial population, consisting of N bacteria, and our aim is to predict the number of bacteria after one time unit.

Underlying assumption: At any given time, the reproduction rate equals one. That is, assuming that the population were constant over a time interval of length ϵ, the population size will have changed by $N \cdot \epsilon$.

However, the population size will not be constant over any time interval. In order to obtain a good approximation, we subdivide the time interval into n subintervals of equal length, introducing $t_0 = 0, t_1 = \frac{1}{n}, \ldots, t_n = 1$.

Motivation

We then obtain the following approximations of the population size after each subinterval:

$$\text{at time} \quad t_1 \quad : \; N \cdot (1 + \frac{1}{n}) \; , \; \text{at time } t_2 \; : \; N \cdot (1 + \frac{1}{n})^2 \; , \; \ldots \; ,$$

$$\text{at time} \quad t_n \; = 1 \; : \; N \cdot \left(1 + \frac{1}{n}\right)^n$$

Each step depends on the assumption that the population size is constant in the time between t_i and t_{i+1}.

This assumption should be increasingly accurate as the intervals become small (i.e., as n becomes large)

Motivation

We derived $N \cdot \left(1 + \frac{1}{n}\right)^n$ as an estimate of the population size at time 1. As $n \to \infty$, we expect the estimate to be arbitrarily close to the true value:

That is, we are interested in the limit of

$$x_n = N \cdot \left(1 + \frac{1}{n}\right)^n \quad ,$$

as $n \to \infty$.

A second example

Recall that calculators use rational approximations of real numbers. Thus we need a mechanism to compute such approximations. The following is a simple scheme to approximate $\sqrt{2}$:

- Start with $x_0 = 1$.
- Given a rational x_n, we define

$$x_{n+1} = \frac{x_n + 2/x_n}{2} \in \mathbb{Q} .$$

Then one can prove that for all $n \in \mathbb{N}_0$,

$$1 \leq x_n < x_{n+1} < \sqrt{2},$$

i.e., x_{n+1} is indeed closer to $\sqrt{2}$ than x_n. Moreover, one expects that for any predefined precision ϵ, sufficiently many repetitions yield a value that approximates $\sqrt{2}$ within ϵ.

Sequences

Definition. A sequence of numbers is a rule assigning each natural number n a real number $\mathbf{x}_n \in \mathbb{R}$. (Also called a mapping $\mathbb{N}_0 \to \mathbb{R}$). It is denoted as

$$(x_n)_{n\in\mathbb{N}_0}, \text{ or } x_0, x_1, \ldots,$$

Examples:

- Let $x_n = r$, for all $n \in \mathbb{N}$ and some fixed $r \in \mathbb{R}$. This defines a constant sequence.
- Letting $x_n = 2n + 1$, for $n \in \mathbb{N}_0$, one obtains the sequence $1, 3, 5, 7, \ldots$ of odd numbers, sorted in ascending order.
- $x_n = n^\alpha$, for $n \in \mathbb{N}_0$, and fixed α
- Example of a recursively defined series: Define $(x_n)_{n\in\mathbb{N}_0}$ by

$$x_0 = 1\ ,\ x_{n+1} = \frac{x_n + 2/x_n}{2} \text{ (for } n \in \mathbb{N}_0)$$

Properties of sequences

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. The sequence is called

1. (monotonically) decreasing if for all $n \in \mathbb{N}$, $x_{n+1} \leq x_n$;
2. (monotonically) increasing if for all $n \in \mathbb{N}$, $x_{n+1} \geq x_n$;
3. monotonic if it is either an increasing or a decreasing sequence;
4. bounded from below if for some $y \in \mathbb{R}$ and all $n \in \mathbb{N}$, $y \leq x_n$;
5. bounded from above if for some $y \in \mathbb{R}$ and all $n \in \mathbb{N}$, $y \geq x_n$;
6. bounded if it is both bounded from above and from below.

Moreover the sequence is called strictly decreasing (or increasing), if $x_{n+1} < x_n$ holds (resp. $x_{n+1} > x_n$) for all n.

Examples

- Obviously, a decreasing sequence is bounded from above (e.g., by $y = x_0$). Likewise, an increasing sequence is bounded from below.
- The sequences $x_n = 2n + 1$ $(n \in \mathbb{N}_0)$ and $y_n = n^2$ $(n \in \mathbb{N}_0)$ are bounded from below, strictly increasing and not bounded from above.
- The sequence $x_n = \frac{1}{n}$ $(n \in \mathbb{N})$ is strictly decreasing, and bounded both from above and below: $0 < x_n < 1$.
- The sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \left(1 + \frac{1}{n}\right)^n$, is increasing and bounded from above.
- The sequence $(x_n)_{n\in\mathbb{N}}$, defined by

$$x_0 = 1\ ,\ x_{n+1} = \frac{x_n + 2/x_n}{2}\ (n \in \mathbb{N}_0)$$

fulfills $1 < x_n < x_{n+1} < \sqrt{2}$. Hence it is monotonic and bounded.

Limit and convergence of sequences

Definition. Let $(x_n)_{n\in\mathbb{N}}$ denote a sequence, and $x \in \mathbb{R}$. Then $(x_n)_{n\in\mathbb{R}}$ converges to x if for all $\epsilon > 0$ there exists a natural number $N = N(\epsilon) \in \mathbb{R}$ such that,

$$\forall n > N(\epsilon) \ : \ |x_n - x| < \epsilon$$

In this case, we call x the limit of the sequence, also expressed as

$$x = \lim_{n\to\infty} x_n \quad ,$$

and the sequence is called convergent. A sequence that does not converge, diverges.

ϵ can be understood as "target precision". Convergence means that for all target precisions ϵ one can find an index $N(\epsilon)$ such that all sequence elements with index larger than $N(\epsilon)$ approximate x with error at most ϵ.

Cauchy criterion for sequences

Theorem 1. Let $(x_n)_{n\in\mathbb{R}}$ be a sequence. The sequence has a limit if and only if it satisfies the Cauchy criterion: For all $\epsilon > 0$ there exists an $M(\epsilon) \in \mathbb{R}$ such that for all

$$\forall m, n > M(\epsilon) \ : \ |x_n - x_m| < \epsilon$$

Observation: We do not need to know the limit to check this criterion.

Convergence vs. boundedness

Theorem 2.

Let $(x_n)_{n\in\mathbb{R}}$ be a sequence.

(a) The limit is unique, i.e., if $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} x_n$, then $x = y$.

(b) If the sequence converges, it is bounded.

(c) Assume that the sequence is monotonic. If it is bounded, the sequence converges to $x \in \mathbb{R}$. Otherwise, it converges to $\pm\infty$.

Example:

- The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ is increasing and bounded, hence converges. The limit

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

is called Euler number, $e \approx 2.7182...$

Indefinite convergence

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. Then

$$\lim_{n\to\infty} x_n = \infty$$

holds if for all $M \in \mathbb{R}$ there exists $N = N(M)$ such that

$$\forall n > N(M) \; : \; x_n > M \; .$$

We write

$$\lim_{n\to\infty} x_n = -\infty$$

if for all $M \in \mathbb{R}$ there exists $N = N(M)$ such that

$$\forall n > N(M) \; : \; x_n < M \; .$$

Further examples

- The constant sequence $x_n = r$ (for all $n \in \mathbb{N}$) converges to r.
- The sequence $x_n = 2n + 1$ (for $n \in \mathbb{N}$) is unbounded, hence divergent. Instead, $x_n \to \infty$.
- The sequence $x_n = n^\alpha$ (for $n \in \mathbb{N}$) converges to 0 if $\alpha < 0$, converges to 1 for $\alpha = 0$, and converges indefinitely for $\alpha > 0$.
- The alternating sequence $x_n = (-1)^n$ (for $n \in \mathbb{N}$), is bounded from below and above, yet divergent.
- The sequence $x_n = (-1)^n n$ has neither lower nor upper bound. In particular, it converges neither to $\pm\infty$ nor to any real number.

Asymptotic growth of sequences

Given two indefinitely converging sequences $x_n \to \infty$, $y_n \to \infty$, the convergence behaviour of $\frac{x_n}{y_n}$ allows to compare their growth for large n . Important examples are:

- For all $\alpha, \beta > 0$,

$$\lim_{n\to\infty} \frac{n^\alpha}{n^\beta} = \lim_{n\to\infty} n^{\alpha-\beta} = \begin{cases} \infty & \alpha > \beta \\ 1 & \alpha = \beta \\ 0 & \alpha < \beta \end{cases}$$

- For all $\alpha > 0, c > 1$,

$$\lim_{n\to\infty} \frac{n^\alpha}{c^n} = 0 \text{ , but also } \quad \lim_{n\to\infty} \frac{c^n}{n!} = 0$$

  where $n! = 1 \cdot 2 \cdot \ldots \cdot n$. I.e., as $n \to \infty$, $(n^\alpha)_{n\in\mathbb{N}}$ grows more slowly than $(c^n)_{n\in\mathbb{N}}$, which in turn grows more slowly than $(n!)_{n\in\mathbb{N}}$.

Computing with limits

Theorem 3. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences, and suppose that there exists N such that $x_n = y_n$ for all $n > N$. Then

$$x = \lim_{n\to\infty} x_n \Leftrightarrow x = \lim_{n\to\infty} y_n \quad .$$

Theorem 4. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences, and $r, s \in \mathbb{R}$. If

$$x = \lim_{n\to\infty} x_n \ , \ y = \lim_{n\to\infty} y_n$$

then

$$rx + sy = \lim_{n\to\infty} rx_n + sy_n \ , xy = \lim_{n\to\infty} x_n y_n . \tag{1}$$

Moreover, if $y \neq 0$, then there exists $N > 0$ such that $y_n \neq 0$ for all $n > N$, and

$$\frac{x}{y} = \lim_{n\to\infty} \frac{x_n}{y_n} \ .$$

Examples and applications

- Going back to the initial example: The population after one time unit
$$\lim_{n\to\infty} N \cdot \left(1+\frac{1}{n}\right)^n = Ne \ ,$$
where e is Euler's constant, and N is the initial population.
- We want to compute $\lim_{n\to\infty} \frac{n^2-3n+1}{n^2+1}$. Dividing both denominator and enumerator by n^2, we see that this limit equals $\lim_{n\to\infty} \frac{1-3n^{-1}+n^{-2}}{1+n^{-2}}$. Using that $n^{-\alpha} \to 0$, for $\alpha = 1, 2$, the theorem allows to compute
$$\lim_{n\to\infty} \frac{n^2-3n+1}{n^2+1} = \frac{\lim_{n\to\infty} 1 - 3n^{-1} + n^{-2}}{\lim_{n\to\infty} 1 + n^{-2}} = \frac{1}{1} = 1.$$

Generalizing the example

The argument employed for the previous example can be generalized to the ratio of polynomials:
Corollary. Let P, Q be polynomials, i.e.,

$$P(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0 \ , \ Q(x) = b_k x^k + b_{k-1} x^{k-1} + \ldots + b_0 \ ,$$

with $a_0, \ldots, a_m, b_0, \ldots, b_k \in \mathbb{R}$. Assume that $a_m \neq 0 \neq b_k$. Then

$$\lim_{n \to \infty} \frac{P(n)}{Q(n)} = \begin{cases} \infty & \text{if } m > k \\ \frac{a_m}{b_k} & \text{if } k = m \\ 0 & \text{if } m < k \end{cases}$$

Vector-valued sequences

Definition. A sequence of vectors is a rule assigning each $n \in \mathbb{N}$ a vector $\mathbf{x}_n \in \mathbb{R}^d$. Here the dimension d is independent of n. A vector $\mathbf{x} \in \mathbb{R}^d$ is called limit of the sequence if for all $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$\forall n > N(\epsilon) \ : \ |\mathbf{x}_n - \mathbf{x}| < \epsilon$$

Again, we write $\mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n$.

Theorem 5. Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^d$, and $\mathbf{x} \in \mathbb{R}^d$. Suppose that

$$\mathbf{x}_n = (x_n(1), \dots, x_n(d))^T \ , \ \mathbf{x} = (x(1), \dots, x(d))^T \ .$$

Then $\mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n$ if and only if

$$\forall j = 1, \dots, d \ : \ x(j) = \lim_{n\to\infty} x_n(j) \ .$$

Examples

- The sequence $\mathbf{x}_n = (r, 1/n)^T$ converges to $(r, 0)$.
- The sequence $\mathbf{x}_n = (2n+1, r)^T$ diverges, because the sequence $(2n+1)_{n\in\mathbb{N}}$ diverges.
- We fix an element of $\mathbb{C} = \mathbb{R}^2$, and consider the sequence $(z^n)_{n\in\mathbb{N}}$. Using $|z^n| = |z|^n$, one sees that this sequence
  - converges to 1 if $z = 1$;
  - converges to 0 if $|z| < 1$ (note that $|z^n - 0| = |z|^n \to 0$);
  - diverges in all other cases.

Series

Definition. Let $(x_n)_{n\in\mathbb{N}_0}$ be a sequence. The series $\sum_{n=0}^{\infty} x_n$ is the sequence $(y_n)_{n\in\mathbb{N}_0}$ of partial sums

$$y_n = \sum_{k=0}^{n} x_k = x_0 + x_1 + \ldots + x_n \ .$$

The series converges to $y \in \mathbb{R}$ if $y = \lim_{n\to\infty} y_n$, in which case we write

$$y = \sum_{n=0}^{\infty} x_n \ .$$

We say that the series $\sum_{n=0}^{\infty} x_n$ converges absolutely if $\sum_{n=0}^{\infty} |x_n|$ converges.

Examples

- Consider the series $\sum_{n=0}^{\infty} x_n$ for $x_n = r$, the constant sequence. The partial sum is computed as $y_n = (n+1)r$, which diverges unless $r = 0$.
- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- Consider the series $\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+2}$. We compute its partial sums:

$$y_0 = 1-\frac{1}{2}\ ,\ y_1 = y_0+\frac{1}{2}-\frac{1}{3} = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3} = 1-\frac{1}{3}\ ,\ \ldots,\ ,\ y_n = 1-\frac{1}{n+}$$

Thus

$$\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} = \lim_{n\to\infty} y_n = 1$$

Computing with series

Theorem 6.

- The limit of a series is unique.
- Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences, and $r, s \in \mathbb{R}$. If

$$x = \sum_{n=0}^{\infty} x_n \ , \ y = \sum_{n=0}^{\infty} y_n$$

then

$$rx + sy = \sum_{n=0}^{\infty} rx_n + sy_n \ . \tag{2}$$

Remark: There are no simple rules for products of series.

The geometric series

Let $q \in \mathbb{R}$. We want to determine the limit of $\sum_{n=0}^{\infty} q^n$, if it exists. We already know that $q = 1$ will not give a convergent series, hence $q \neq 1$. Let $y_n = \sum_{k=0}^{n} q^k$. Then we observe that

$$\begin{aligned} y_n \cdot (1-q) &= (1+q+q^2+\ldots+q^n)(1-q) \\ &= 1+q+q^2+\ldots+q^n-q-q^2-\ldots-q^n-.q^{n+1} \\ &= 1-q^{n+1} \quad . \end{aligned}$$

Thus

$$y_n = \frac{1-q^{n+1}}{1-q}$$

The geometric series

If $|q| > 1$, then

$$y_n = \frac{1 - q^{n+1}}{1 - q} \text{ does not converge } , \quad \text{as } |q|^{n+1} \to \infty$$

hence the sum diverges. In the other case, $q^{n+1} \to 0$ entails that

$$\sum_{n=0}^{\infty} q^n = \lim_{n \to \infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} \ .$$

We have thus proved:

Theorem 7. The sum $\sum_{n=0}^{\infty} q^n$ converges iff $|q| < 1$, with

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1 - q}$$

Convergence criteria

Theorem 8. Let $(x_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$.

(a) $\sum_{n=0}^{\infty} x_n$ converges if it converges absolutely.

(b) Necessary condition: If $\sum_{n=0}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

(c) Let $\alpha > 0$. Then $\sum_{n=1}^{\infty} n^{-\alpha}$ converges precisely for $\alpha > 1$.

Sufficient convergence criteria

Theorem 9. Let $(x_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$.

(a) Majorant criterion: Let $\sum_{n=0}^{\infty} z_n$ be an absolutely convergent series such that $|x_n| < |z_n|$. Then $(x_n)_{n\in\mathbb{R}}$ converges absolutely.

(b) Quotient criterion: If there exists a constant c with $0 < c < 1$, such that for all $n \in \mathbb{N}$, with $n > M$, $\left|\frac{x_{n+1}}{x_n}\right| < c$, then $\sum_{n=0}^{\infty} x_n$ converges absolutely.

(c) Leibniz criterion: Suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to zero, and fulfills $|x_{n+1}| < |x_n|$ as well as $x_{n+1} \cdot x_n \leq 0$. Then $\sum_{n=0}^{\infty} x_n$ converges.

Example: The series $\sum_{n=1}^{\infty} n^{-1}$ diverges (Theorem 6.c)). However, $\sum_{n=1}^{\infty}(-1)^n n^{-1}$ converges, as a consequence of the Leibniz criterion: $|(-1)^n n^{-1}| > |-1^{n+1}(n+1)^{-1}|$, and $(-1)^n n^{-1}(-1)^{n+1}(n+1)^{-1} = \frac{-1}{n(n+1)}$.

Examples and remarks

Example: An important application of the quotient criterion is that the exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. In fact, this series is related to Euler's constant by the equation

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \ .$$

Remarks:

- The quotient criterion follows from the convergence of the geometric series by applying the majorant criterion.

Summary

- Convergence of sequences and series, indefinite and absolute convergence
- Convergence criteria for sequences: Necessary (e.g., boundedness), sufficient (e.g., boundedness and monotonicity)
- Convergence criteria for series: Majorant criterion, quotient criterion
- Important examples: Harmonic and geometric series
- Rules for the computation of limits

Note: It can be easy to determine whether a series or sequence converges, and hard to find the limit.