Calculus and linear algebra for biomedical engineering
Week 7: Continuous functions

Hartmut Führ
fuehr@matha.rwth-aachen.de

Lehrstuhl A für Mathematik, RWTH Aachen

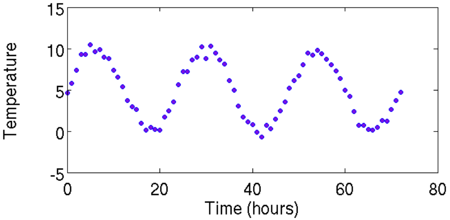
November 27, 2008

Overview

1. Definition of continuity
2. Uniform continuity
3. Computing with continuous functions
4. Continuity on intervals

Motivation: Temperature Measurements (again)

Recall last week's setup: We have a sequence $y_0, y_1, y_2, \ldots$ of temperature measurements at times $t = 0, 1, 2, \ldots$. (in hours)



Do these measurements allow to determine the temperature after 12.7 hours?

A mathematical formulation

Let $f : [0, M] \to \mathbb{R}$ denote the temperature function.

Measured data:
Measurements $f(t)$ at times $t_0, t_1, t_2, t_3 \ldots, t_N \in [0, M]$.

Challenge: Given $s \in [0, M]$, determine $f(s)$ approximately from $f(t_0), \ldots, f(t_N)$.

Plausible answer: Find t_i closest to s, then hopefully $f(t_i) \approx f(s)$.

Question: Given target precision $\epsilon > 0$, what do we need to know about f and $t_0, \ldots, t_N$ to ensure that $|f(t_i) - f(s)| < \epsilon$, for any s?
This leads to the notion of continuity.

Limit of a function

Definition.
Let $f : D \to \mathbb{R}$ be a function, with $D \subset \mathbb{R}^n$, and let $\mathbf{x}_0 \in \mathbb{R}^n$. For $a \in \mathbb{R}$, we write

$$a = \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$$

if the following two conditions are fulfilled:

- There exists a sequence $(\mathbf{x}_k)_{k \in \mathbb{N}} \subset D$ satisfying $\mathbf{x} \neq \mathbf{x}_k$, for all $k \in \mathbb{N}$, but $\mathbf{x} = \lim_{k \to \infty} \mathbf{x}_k$.
- For all sequences $(\mathbf{x}_k)_{k \in \mathbb{N}} \subset D$ satisfying $\mathbf{x} = \lim_{k \to \infty} \mathbf{x}_k$,

$$a = \lim_{k \to \infty} f(\mathbf{x}_k) \ .$$

Continuous functions

Definition.
Let $f : D \to \mathbb{R}$ be a function, with $D \subset \mathbb{R}^n$.

- Let $\mathbf{x}_0 \in D$. f is called continuous at $\mathbf{x}_0$ if

$$f(\mathbf{x}_0) = \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \ .$$

- f is called continuous on D if it is continuous at all $\mathbf{x} \in D$.

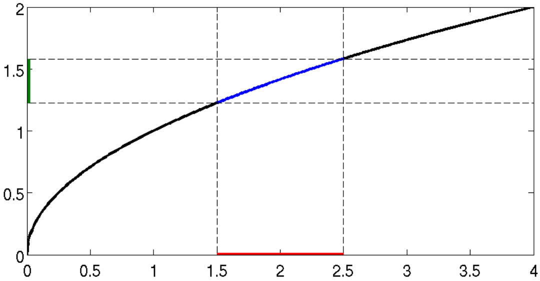
Theorem 1. (ϵ-δ-criterion)
$f : D \to \mathbb{R}$ is continuous at $\mathbf{x}_0$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall \mathbf{y} \in D \ : \ |\mathbf{x}_0 - \mathbf{y}| < \delta \Rightarrow |f(\mathbf{x}_0) - f(\mathbf{y})| < \epsilon$$

Note: δ may depend on $\mathbf{x}_0$ and ϵ.

Illustration of the ϵ-δ-criterion

Continuity of $x \mapsto \sqrt{x}$ at $x_0 = 2$: Fix $\epsilon = 0.2$.
By monotonicity of the square root:
For all x with $|x - x_0| < 0.5$ (red set), $\sqrt{1.5} < \sqrt{x} < \sqrt{2.5}$ (green set).
Since $\sqrt{2} - \sqrt{1.5}, \sqrt{2.5} - \sqrt{2} < \epsilon$, choosing $\delta = 0.5$ is sufficient.



(Note: To prove continuity, we must be able to do this for any $\epsilon > 0$.)

Continuous mappings

Definition.
Let $D \subset \mathbb{R}^n$, and $f : D \to \mathbb{R}^m$. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T \ ,$$

with suitable functions $f_1, f_2, \ldots, f_m : D \to \mathbb{R}$. We say that f is continuous at $\mathbf{x}_0 \in D$ if $f_1, f_2, \ldots, f_m$ are all continuous at $\mathbf{x}_0$.

Theorem 2. (ϵ-δ-criterion for mappings)
$f : D \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall \mathbf{y} \in D \ : \ |\mathbf{x}_0 - \mathbf{y}| < \delta \Rightarrow |f(\mathbf{x}_0) - f(\mathbf{y})| < \epsilon$$

Uniform continuity

Definition.
Let $f : D \to \mathbb{R}$ be a function, with $D \subset \mathbb{R}^n$. Then f is called uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall \mathbf{x}, \mathbf{y} \in D \ : \ |\mathbf{x} - \mathbf{y}| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$$

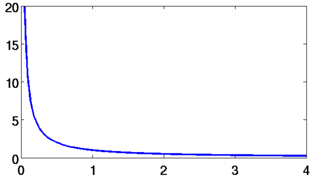
Note: δ only depends on ϵ!

Theorem 3.
Let $f : D \to \mathbb{R}$, with $D \subset \mathbb{R}^n$.

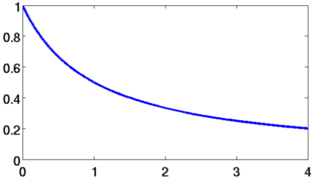
- If f is uniformly continuous, f is continuous.
- Assume $n = 1$ and $D = [a, b]$, with $a, b \in \mathbb{R}$. If f is continuous, then f is uniformly continuous and bounded.

Examples

The functions $f : (0,4] \to \mathbb{R}$, with $f(x) = x^{-1}$, and $g : [0,4] \to \mathbb{R}$, with $g(x) = (1+x)^{-1}$. Both functions are continuous, but f is unbounded, and not uniformly continuous, whereas g is uniformly continuous.



,



Uniform continuity and temperature measurements

Let $f : [0, M] \to \mathbb{R}$ describe the temperature during the time interval $[0, M]$. Assuming that f is continuous, we know by Theorem 2 that f is uniformly continuous.

Hence, given target precision $\epsilon > 0$, we find $\delta > 0$ such that

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$$

Hence, by measuring temperature at $t_0 = 0, t_1 = \delta, t_2 = 2\delta, \ldots,$ we ensure that each point $s \in [0, M]$ has distance at most δ to one point t_i. Accordingly,

$$|f(t_i) - f(s)| < \epsilon \ ,$$

as desired.

Conclusions from the estimate

Positive conclusion: By increasing the density of measurements, we can obtain approximations of any desired precision.

Drawback: We have no method of determining δ explicitly, if we don't know f.

Classes of continuous functions

- $f : \mathbb{R}^n \to \mathbb{R}$, defined by $f(x) = |x|$ is continuous
- Polynomials $f : \mathbb{R} \to \mathbb{R}$ are continuous. This includes affine functions of the form $f(x) = ax + b$.
- Trigonometric functions: $\sin, \cos, \tan$ are continuous on their domains.
- Exponential functions $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = c^x$ (for fixed $c > 0$) are continuous.
- The function $\min : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \min(x, y)$ is continuous. The same holds for max.
- The function $+ : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x + y$, is continuous. Similarly, $(x, y) \mapsto xy$ is continuous.

Further examples

- If A is an $m \times n$-matrix, the mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with $f(\mathbf{x}) = A\mathbf{x}$ is continuous.
- Vector addition is continuous, if we identify pairs $(\mathbf{x}, \mathbf{y})$ of vectors in $\mathbf{R}^n$ with vectors $(x_1, \ldots, x_n, y_1, \ldots, y_n)^T \in \mathbb{R}^{2n}$:

$$+ : \mathbb{R}^{2n} \to \mathbb{R}^n \ , \ (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y}$$

  Similarly, scalar multiplication is continuous

$$\cdot \mathbf{R}^{n+1} \to \mathbf{R}^n \ , \ (r, x_1, \ldots, x_n)^T \mapsto (rx_1, \ldots, rx_n)^T \ .$$

- Let $f : D \to \mathbb{R}$ with

$$D = \{(x, y) \in \mathbb{R} : y \neq 0\}$$

  and $f(x, y) = \frac{x}{y}$. Then f is continuous.

Continuity criteria

Composition of continuous functions yields continuous functions:

Theorem 3. Let $f : D \to \mathbb{R}^m$, $g : E \to \mathbb{R}^n$, with $E \subset \mathbb{R}^k$, and assume that $g(E) \subset D$.

1. Let $\mathbf{x}_0 \in E$. If
   - g is continuous at $\mathbf{x}_0 \in E$; and
   - f is continuous at $g(\mathbf{x}_0)$;

   then $f \circ g$ is continuous at $\mathbf{x}_0$.
2. If g is continuous on E and f is continuous on D, then $f \circ g$ is continuous on E.

This criterion is very useful for showing continuity.

Operations on continuous functions

Theorem 4. Let $f, g : D \to \mathbb{R}$, with $D \subset \mathbb{R}^n$, and $\mathbf{x}_0 \in D$.

- If f, g are continuous at $\mathbf{x}_0$, then so are

$$f \cdot g \ , \ rf + sg \ .$$

- If f, g are continuous at $\mathbf{x}_0$ and $g(\mathbf{x}_0) \neq 0$, then $\frac{f}{g}$ is continuous at $\mathbf{x}_0$.

Remark: These statements follow by concatenating known continuous functions.
E.g., if f, g are continuous, then the mapping $F : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (f(x), g(x))^T$ is continuous. Also, we know that $m : \mathbb{R}^2 \to \mathbb{R}$, where $m(x, y) = xy$, is continuous.
But then $f \cdot g = m \circ F$ is continuous.

Application: Computing limits

Assume we want to compute

$$y = \lim_{n\to\infty} \sin\left(\sqrt{\left(1+\frac{1}{n}\right)^n}\right) .$$

We know that

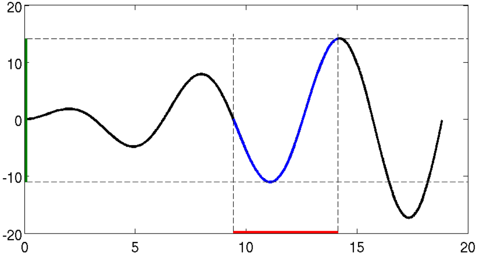
- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (Euler's constant)
- $\sqrt{\cdot} : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous, hence $\lim_{n\to\infty}\sqrt{\left(1+\frac{1}{n}\right)^n} = \sqrt{e}$
- $\sin : \mathbb{R} \to \mathbb{R}$ is continuous, hence

$$y = \sin(\sqrt{e})$$

Continuous images of closed and bounded interval

Theorem 5.
Let $f : D \to \mathbb{R}$, and suppose that $[a, b] \subset D$, for $a, b \in \mathbb{R}$. Then there exist $r, s \in \mathbb{R}$ such that $f([a, b]) = [r, s]$.

$\rightsquigarrow$ A closed and bounded interval (red) is mapped onto a closed and bounded interval (green)



,

Existence of extrema

Corollary 1. (Weierstrasse Extreme Value Theorem)
Let $f : D \to \mathbb{R}$ be continuous, and suppose that $[a, b] \subset D$, for $a, b \in \mathbb{R}$. Then there exist $x_{\min}, x_{\max} \in [a, b]$ such that for all $x \in [a, b]$,

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \ ,$$

or in other words,

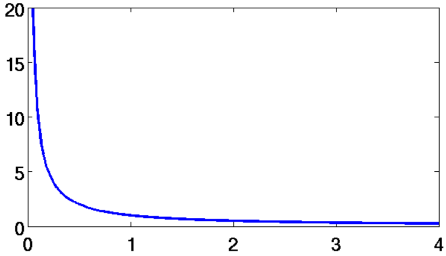
$$f(x_{\max}) = \max\{f(x) : x \in [a, b]\} \ , \ f(x_{\min}) = \min\{f(x) : x \in [a, b]\} \ .$$

The point $x_{\max}$ is called a maximum point with maximum $f(x_{\max})$. Likewise, $x_{\min}$ is called minimum point with minimum $f(x_{\min})$.

Caution

It is important that f is defined on the closed and bounded interval $[a, b]$: The function $f(x) = 1/x$, defined on $(0, 4]$, does not have a maximum. Likewise, no statements are possible for intervals $[a, \infty)$ or $(-\infty, b]$.

Standard example: $f(x) = 1/x$, defined on $(0, 4]$.



Intermediate value theorem

Corollary 2. (Intermediate value theorem)
Let $f : D \to \mathbb{R}$ be continuous, and suppose that $[a, b] \subset D$, for $a, b \in \mathbb{R}$. For every y between $f(a)$ and $f(b)$, there exists $x \in [a, b]$ with $f(x) = y$.

Corollary 3.(Existence of roots)
Let $f : D \to \mathbb{R}$ be continuous, and suppose that $[a, b] \subset D$, for $a, b \in \mathbb{R}$. If $f(a)f(b) < 0$, there exists $x \in [a, b]$ with $f(x) = 0$.

Remark. The condition $f(a)f(b) < 0$ means that $f(a)$ and $f(b)$ have different signs.

Application: Searching for roots

Corollary 3 can be employed to (approximately) find roots of a continuous function: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a)f(b) < 0$.

By Corollary 3, we there exists $x \in [a, b]$ with $f(x) = 0$. In general, we can only hope to find an approximation to x.

Pick $c \in (a, b)$. Then, either $f(c)f(b) < 0$ or $f(a)f(c) < 0$.
In the first case, Corollary 3 implies the existence of a root in $[c, b]$, in the second case, there must be a root in $[a, c]$.

In any case, we have narrowed the search down from the interval $[a, b]$ to either $[a, c]$ or $[c, b]$.

Illustration: Subdividing the interval

Sample function: $f(x) = 0.5 + x^{3/2}\cos(x)$, with $f(0) > 0 > f(4)$.

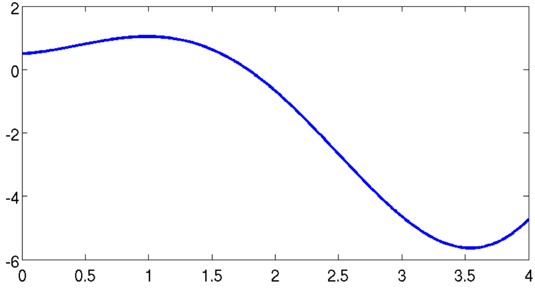
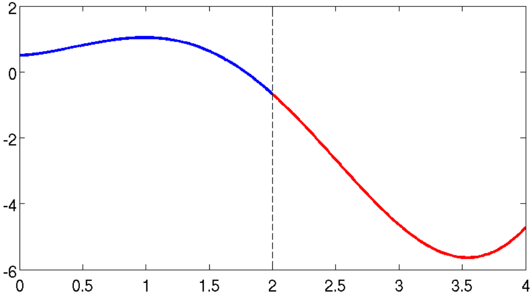


Illustration: Subdividing the interval

Introducing $c = 2$: Since $f(0)f(2) < 0$, we can restrict our search to the interval $[0, 2]$.



Comparison of continuous functions

A further simple application is the following: If the continuous function $f : [a, b] \to \mathbb{R}$ fulfills $f(x) \neq 0$ for all $x \in [a, b]$, then either $f(x) > 0$, for all $x \in [a, b]$, or $f(x) < 0$, for all $x \in [a, b]$.

Example: Solving inequalities.
We are given a continuous function $f : D \to \mathbb{R}$, where D is an interval (possibly unbounded). We need to determine the set

$$\mathbb{S} = \{x \in D : f(x) \leq 0\}$$

We assume that f has only finitely many roots, given by

$$-\infty < x_1 < x_2 < \ldots x_n < \infty$$

Comparison of continuous functions

We introduce $x_0 = -\infty$ and $x_{n+1} = \infty$. By assumption, each interval (x_i, x_{i+1}) contains no roots, hence the sign of f is constant. It can therefore be determined by evaluating $f(y_i)$ for some arbitrary $y_i \in (x_i, x_{i+1})$.

Hence we determine $\mathbb{S}$ as follows:

- For $i = 0, \ldots, n$: Pick an arbitrary $y_i \in (x_i, x_{i+1})$.
- $\mathbb{S} = \{x_i : i = 1, \ldots, n\} \cup \bigcup \{(x_i, x_{i+1}) : f(y_i) < 0\}$

Solving inequalities: Example

Our aim is to determine the set $\mathbb{S}$ of all $x \in \mathbb{R}$ satisfying

$$|x-1| + x \leq 5 \ .$$

Clearly, $f(x) = |x-1| + x - 5$ is continuous, and it has $x_0 = 3$ as its only root. Hence, for every closed interval $[a, b]$ contained in $(3, \infty)$ or $(-\infty, 3)$, the sign of f is constant on $[a, b]$.

Hence, we only need to check two intervals:

- We pick an arbitrary $x \in (3, \infty)$, say 4. Since $f(4) = 2 > 0$, we conclude that $(3, \infty) \cap \mathbb{S} = \emptyset$.
- We evaluate $f(0) = -4$ and conclude that $(-\infty, 3) \subset \mathbb{S}$.

$\Rightarrow$ The set of all solutions to the inequality is given by $\mathbb{S} = (-\infty, 3]$.

$\rightsquigarrow$ Only two evaluations are needed to obtain a complete solution!

Inverse of continuous functions

Theorem 6. Let I be an interval, and $f : I \to \mathbb{R}$ be continuous.

- f is injective if and only if f is strictly monotonic.
- If f is injective, then the inverse function $f^{-1} : f(I) \to I$ is again continuous.

Examples

- The root functions $x \mapsto \sqrt[k]{x}$ are continuous.
- The arctangent function is continuous.

Summary

- Important definitions: Limit of a function, continuity, uniform continuity.
- Application of continuity to limits of sequences.
- Known classes of continuous functions: Polynomials, absolute value, min,max, trigonometric functions, powers, roots
- Checking continuity: Continuity is preserved by composition, sums, products, inverse functions, etc.
- Properties of continuous functions: Intermediate value theorem, Extrema, etc.
- Application of the properties: Search for roots, solving inequalities