Calculus and linear algebra for biomedical engineering
Week 9: Applications of differentiable functions

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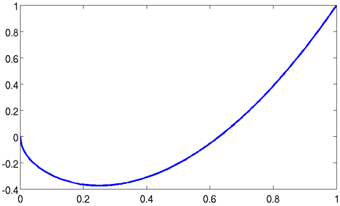
December 11, 2008

Overview

Motivation

Consider the function $f(x) = 2x^2 - \sqrt{x}$ on the interval $[0, 1]$.



f is continuous on $[0, \pi]$, hence we know that there exist $x_{\max}$ and $x_{\min} \in [0, \pi]$ such that

$$f(x_{\max}) = \max\{f(x) : 0 \leq x \leq \pi\}\ , \ f(x_{\min}) = \min(\{f(x) : 0 \leq x \leq \pi\}\ .$$

How do we find $x_{\max}, x_{\min}$? How do we determine monotonicity of f?

Monotonicity and the first derivative

Theorem 1.
Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b).

- f is increasing on $[a, b]$ iff $f'(x) \geq 0$, for all $x \in (a, b)$.
- f is strictly increasing on $[a, b]$ if $f'(x) > 0$, for all $x \in (a, b)$.
- f is decreasing on $[a, b]$ iff $f'(x) \leq 0$, for all $x \in (a, b)$.
- f is strictly decreasing on $[a, b]$ if $f'(x) < 0$, for all $x \in (a, b)$.

(Partial) Proof: Assume that $x, y \in (a, b)$ with $x < y$. By the mean value theorem,

$$\frac{f(y) - f(x)}{y - x} = f'(z) \ ,$$

for a suitable z between x and y. Since $y > x$, this equation implies that $f(y) - f(x) \geq 0$ iff $f'(z) \geq 0$.

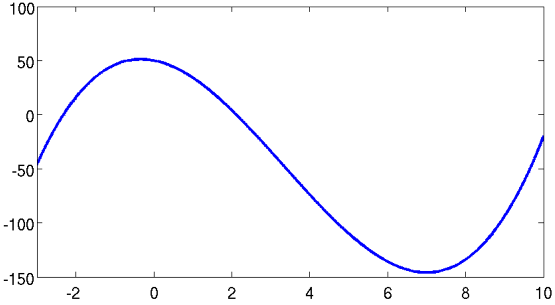
Determining monotonicity intervals of a function

Let f be continuously differentiable on (a, b), and suppose that f' has only finitely many roots in (a, b). Then the monotonicity behaviour of f is determined as follows:

- Compute f'.
- Compute all roots $x_0, \ldots, x_k$ of f' in (a, b).
- In each interval (x_i, x_{i+1}), determine the sign of f' by evaluating $f'(c_i)$, for suitable $c_i \in (x_i, x_{i+1})$.
- On $[x_i, x_{i+1}]$, f is strictly increasing, if $f'(c_i) > 0$; otherwise f is strictly decreasing.

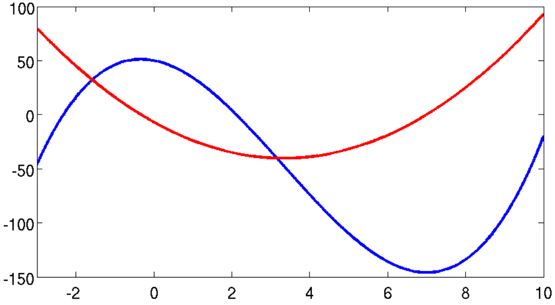
An example

Consider $f(x) = x^3 - 10x^2 - 7x + 50$



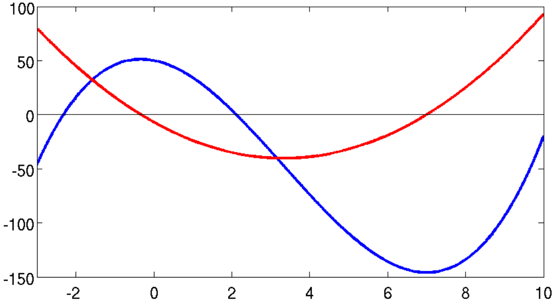
An example

Then $f'(x) = 3x^2 - 20x - 7 = (3x+1)(x-7)$



An example

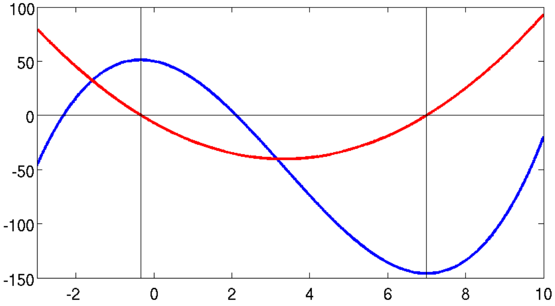
Hence, f' has roots $-1/3$ and 7



An example

f (blue) increases wherever f' (red) is positive. Hence:

- $f'(x) > 0$ for $x \in (-\infty, -1/3)$ and $x \in (7, \infty)$ implies: f is strictly increasing on $(-\infty, -1/3]$ and on $[7, \infty)$.
- $f'(x) < 0$ in $(-1/3, 7)$ implies: f is strictly decreasing on $[-1/3, 7]$.



Extreme values

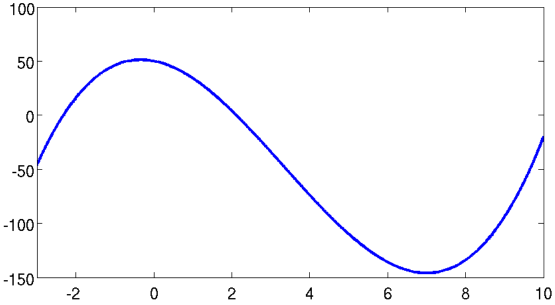
Definition: Let $f : [a, b] \to \mathbb{R}$, and $x_0 \in [a, b]$.

- x_0 is called local minimum point if for a suitable $\delta > 0$ and all $x \in (b - \delta, b + \delta) \cap [a, b]$, $f(x_0) \leq f(x)$
- x_0 is called local maximum point if for a suitable $\delta > 0$ and all $x \in (b - \delta, b + \delta) \cap [a, b]$, $f(x_0) \geq f(x)$
- x_0 is called global minimum point of f on $[a, b]$ if for all $x \in [a, b]$, $f(x_0) \leq f(x)$.
- x_0 is called global maximum point of f on $[a, b]$ if for all $x \in [a, b]$, $f(x_0) \geq f(x)$.
- The local (or global) minimum and maximum points are called local (or global) extrema.

Note: Global extrema are local extrema as well.

Illustration: Extreme values

The function $f(x) = x^3 - 10x^2 - 7x + 50$ has two local maximum and two local minimum points in the interval $[-3, 10]$ (which will be determined later).



Extreme values and monotonicity

Theorem 2

Let $f : [a, b] \to \mathbb{R}$, and $x \in [a, b]$.

- Suppose that f is decreasing in $(x - \delta, x] \cap [a, b]$, and increasing in $[x, x + \delta)$, for some $\delta > 0$. Then x is a local minimum point.
- Suppose that f is increasing in $(x - \delta, x] \cap [a, b]$, and decreasing in $[x, x + \delta)$, for some $\delta > 0$. Then x is a local maximum point.

Informally:

- If f increases to the left of x and decreases to the right of x, then x is a local maximum point.
- For the boundary points a, b, only one-sided behaviour must be considered.

Extreme values and the first derivative

Theorem 3.

Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b).

- If $x \in [a, b]$ is such that $f'(y) \leq 0$ for all $y \in (x - \delta, x) \cap [a, b]$, and $f'(y) \geq 0$ for all $y \in (x, x + \delta) \cap [a, b]$, then f is a local minimum point.
- If $x \in [a, b]$ is such that $f'(y) \geq 0$ for all $y \in (x - \delta, x) \cap [a, b]$, and $f'(y) \leq 0$ for all $y \in (x, x + \delta) \cap [a, b]$, then f is a local maximum point.
- If $x \in (a, b)$ is a local extremum point, then $f'(x) = 0$.

The analogous statements, with reversed inequalities, holds for local minimum points.

Note: The condition $f'(x) = 0$ (for an inner point) is only necessary, not sufficient. For sufficient criteria, we need higher derivatives.

Extreme values and higher derivatives

Theorem 4. Assume that $f : [a, b] \to \mathbb{R}$ is $2k$ times differentiable, for some $k \in \mathbb{N}$. Let $a < x_0 < y$ be such that

$$f'(x_0) = f''(x_0) = \ldots = f^{(2k-1)}(x_0) = 0 \ .$$

- If $f^{(2k)}(x_0) < 0$, then f has a local maximum at x_0.
- If $f^{(2k)}(x_0) > 0$, then f has a local minimum at x_0.
- If $f^{(2k)}(x_0) = 0$, and f is $2k + 1$ times differentiable with $f^{(2k+1)}(x_0) \neq 0$, then f has neither a local maximum nor a local minimum at x_0. (x_0 is a saddle point.)

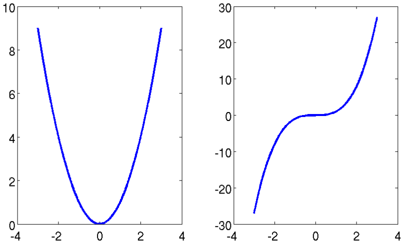
Example: Integer powers

Consider $f(x) = x^n$, with $n \in \mathbb{N}$. Then

$$f'(0) = f''(0) = \ldots = f^{(n-1)}(0) = 0 \, , \;\; f^{(n)}(0) = n! > 0 \, .$$

Hence

- If n is even, say $n = 2k$, then $x_0 = 0$ is a local minimum point.
- If n is odd, say $n = 2k+1$, there is no local extremum at $x_0 = 0$



Example: Determining local and global extrema

We are interested in local and global extrema of $f(x) = x^3 - 10x^2 - 7x + 50$ on the interval $[-3, 10]$. Recalling that

$$f'(x) = 3x^2 - 20x - 7 = (3x+1)(x-7)\ , f''(x) = 6x - 20$$

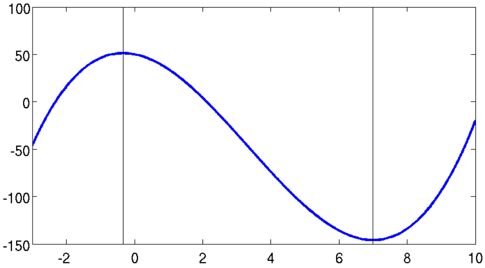
we determine the following possible candidates for local extrema:

- Left boundary: $x = -3$. Because of $f'(-3) = 80 > 0$, $x = -3$ is a local minimum, with $f(-3) = -46$.
- First root: $x = -1/3$. We have $f''(-1/3) = -22 < 0$, which makes $x = -1/3$ a local maximum.
- Second root: $x = 7$. Here $f''(7) = 22 > 0$, hence $x = 7$ is a local minimum with $f(7) = -146$.
- Right boundary: $x = 10$. Because of $f'(10) = 93 > 0$, $x = 10$ is a local maximum, with $f(x) = -20$.

Determining global extrema

Comparing local extrema, we find that

- $x_{\text{min}} = 7$ is a global minimum point in $[-3, 10]$.
- $x_{\text{max}} = -1/3$ is a global maximum point in $[-3, 10]$.



Back to the initial example

We study the function $f(x) = 2x^2 - \sqrt{x}$ on the interval $[0,1]$. f is continuous on $[0,1]$ and differentiable on $(0,1)$.

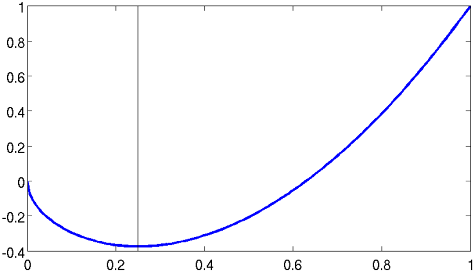
- $f'(x) = 4x - \frac{1}{2}x^{-1/2}$. Hence

$$f'(x) = 0 \Leftrightarrow 4x = \frac{1}{2}x^{-1/2} \Leftrightarrow x^{3/2} = \frac{1}{8} = \left(\frac{1}{2}\right)^3 ,$$

  hence $f'(x_0) = 0$ only for $x_0 = \frac{1}{4}$.
- $f''(x) = 4 + \frac{1}{4}x^{-3/2} > 0$, for all $x \in (0,1)$. In particular, $f''(1/4) > 0$, which makes $1/4$ a local minimum point.
- For $x < 1/4$, we have $4x < 1$, and $x^{-1/2} > 1/2$. Hence $f'(x) < 0$, and f is strictly decreasing on $[0, 1/4]$.
- Similarly, $f'(x) > 0$ for $x > 1/4$, and f is strictly increasing.
- In particular, 0 is a local maximum point, and 1 is a local minimum point.

Back to the initial example

Conclusion: f strictly decreases on $[0, 1/4]$, and strictly increases on $[1/4, 1]$. The boundaries are local maximum point, $1/4$ is the unique local minimum point, which is therefore global.
$f(1) = 1$ is the maximum of f on $[0, 1]$, and $f(1/4) = -0.375$ the minimum.



Application: Deriving inequalities

We want to prove the inequality

$$\sin(x) \geq \frac{2x}{\pi}$$

for all $x \in [0, \pi/4]$. For this purpose we let $f(x) = \sin(x) - \frac{2x}{\pi}$. We then need to show that $f(x) \geq 0$ for all $x \in [0, \pi/4]$. We make the following observations:

- $f(0) = 0$.
- $f'(x) = \cos(x) - \frac{2}{\pi}$.
  cos is decreasing on $[0, \pi/4]$, hence

$$f'(x) \geq f'(\pi/4) = \cos(\pi/4) - \frac{2}{\pi} = \approx 0.0705 > 0$$

- Hence f increases on $[0, \pi/4]$, in particular $f(x) \geq f(0) = 0$.

Convexity

Definition. A function $f : [a, b] \to \mathbb{R}$ is called convex if for all $x, y \in [a, b]$ and all $0 < \lambda < 1$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \ .$$

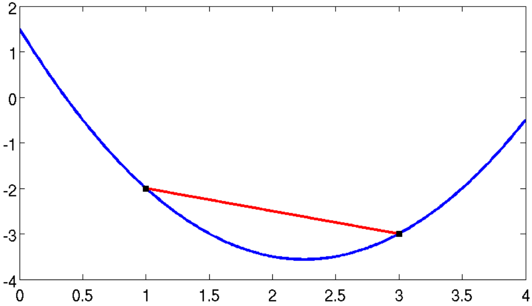
f is called concave if $-f$ is convex.

Note: As λ runs through $(0, 1)$, the points $(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y))$ run through all points on the secant between x and y.
Thus, convexity means that the secant between two points on the graph is above (or on) the graph.

Graphically, convexity means that the graph of f curves upwards.

Illustration: Convexity

Graphically: The function $f : [a, b] \to \mathbb{R}$ is convex iff for all x, y, the secant between x and y is above the graph of f:



Convexity and the second derivative

Theorem 5. Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then f is convex iff f' is increasing.
In particular, if f is twice differentiable, f is convex iff for all $x \in (a, b)$, $f''(x) \geq 0$.

Example: The function $f(x) = x^k$ (with $k \in \mathbb{N}$) is convex on $\mathbb{R}$ iff $k \leq 1$, or if k is even:

- If $k \leq 1$, then $f''(x) = 0 \geq 0$.
- If k is even, then $f''(x) = k(k-1)x^{k-2} \geq 0$, because $k - 2$ is even.
- If k is odd, then $f''(x) \leq 0$ for $x \leq 0$, hence f is concave on $(-\infty, 0]$, and $f''(x) \geq 0$ for $x \geq 0$ implies that f is convex on $[0, \infty)$.

Inflection points

Definition. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. $x_0 \in (a, b)$ is called inflection point if it is a local extremum of f'.

Remarks:

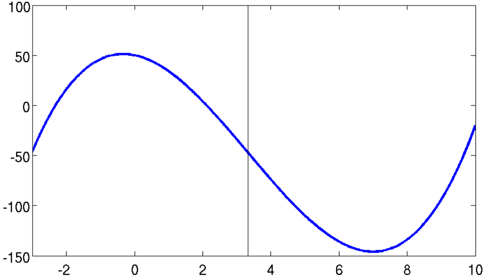
- At inflection points, f changes between convexity and concavity.
- Inflection points are determined from higher derivatives of f by applying Theorems 3 and 4 to f'. In particular, all inflection points are roots of the second derivative.

Example: Determining an inflection point

Consider $f(x) = x^3 - 10x^2 - 7x + 50$. Candidates for inflection points are the roots of f''. Here we have

$$f'(x) = 3x^2 - 20x - 7\ ,\ \ f''(x) = 6x - 20\ ,\ \ f'''(x) = 6 > 0\ .$$

Hence $x_0 = \frac{10}{3}$ is an inflection point.
f is concave on $(-\infty, 10/3]$, and convex on $[10/3, \infty)$.



Application: Inequalities from convexity

We want to prove the inequality

$$\sin(x) \geq \frac{2x}{\pi}$$

for all $x \in [0, \pi/2]$. For this purpose we let $f(x) = \sin(x) - \frac{2x}{\pi}$. We want to show $f \geq 0$ on $[0, \pi/2]$.

Noting that $f''(x) = -\sin(x) \leq 0$, for all $x \in [0, \pi/2]$, we conclude that f is concave on $[0, \pi/2]$.

In particular, the secant between $0, \pi/2$ is below the graph of f. But

$$f(0) = 0 = f\left(\frac{\pi}{2}\right)$$

shows that the secant through $0, \pi/2$ is on the x-axis, hence $f(x) \geq 0$ for all $x \in [0, \pi/2]$.

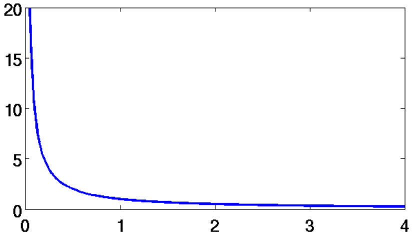
Convergence to ∞

Definition. Let $f : (a, b) \to \mathbb{R}$, and $x_0 \in \mathbb{R}$. Then $\lim_{x \to x_0} f(x) = \infty$ if

- There exists a sequence $(x_n)_{n \in \mathbb{N}} \subset (a, b)$ with $x_0 = \lim_{n \to \infty} x_n$
- For every sequence $(x_n)_{n \in \mathbb{N}} \subset (a, b)$ with $x_0 = \lim_{n \to \infty} x_n$,

$$\lim_{n \to \infty} f(x_n) = \infty \ .$$

Example: $f(x) = \frac{1}{x}$, defined on $(0, 1)$, fulfills $\lim_{x \to 0} f(x) = \infty$



L'Hospital's theorem

Theorem 6. Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be differentiable functions, and $x_0 \in [a, b]$. Assume that either

$$\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = 0 \text{ or } \lim_{x \longrightarrow x_0} |f(x)| = \lim_{x \longrightarrow x_0} |g(x)| = \infty \ .$$

If there exists $y \in \mathbb{R}$ such that

$$y = \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)}$$

then

$$y = \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)}$$

Sample applications of L'Hospital's theorem

- Consider $f(x) = \frac{\sin(x)}{x}$, for $x \neq 0$.
  Both denominator and enumerator converge to 0 as $x \longrightarrow 0$.
  Hence, taking derivatives of both,

$$\lim_{x \longrightarrow x_0} \frac{\sin(x)}{x} = \lim_{x \longrightarrow 0} \frac{\cos(x)}{1} = 1 \ .$$

- Consider $g(x) = \frac{\cos(x)-1}{x^2}$, for $x \neq 0$.
  Both denominator and enumerator converge to 0 as $x \longrightarrow 0$.
  Taking derivatives of both gives $\frac{\sin(x)}{x}$, which we know to converge. Hence

$$\lim_{x \longrightarrow 0} \frac{\cos(x) - 1}{x^2} = \lim_{x \longrightarrow 0} \frac{-\sin(x)}{2x} = -\frac{1}{2} \ .$$

  Note that we obtained this result by a repeated application of L'Hospital's theorem.

Summary

- Properties of curves: Monotonicity, local and global extrema, convexity
- Criteria based on derivatives
- A systematic analysis of functions is based on
  - Computation of derivatives.
  - Computation of roots, signs of derivatives on
  - Interpretation of signs and roots: Roots of f' correspond to extrema, roots of f'' to inflection points. The sign of f' corresponds to monotonicity, the sign of f'' to convexity.
- L'Hopital's theorem for the computation of limits.