Lehrstuhl A für Mathematik
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Calculus and Linear Algebra

Test

Exercise 1 (3 points).

Let $z = \frac{3}{5} - \frac{4}{5}i \in \mathbb{C}$.

- Compute $|z|$.
  Answer: $|z| = 1$.
- Which of the following numbers is z^{-1}?

  a) $w = 1 + i$ b) $w = 1$ c) $w = \frac{3}{5} + \frac{4}{5}i$ d) $w = \dfrac{5+14i}{13}$

  Answer: c) $z^{-1} = \overline{z}/|z| = \overline{z} = \frac{3}{5} + \frac{4}{5}i$.

Exercise 2 (4 points)(We subtract a point per false answer)

Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 7 & 2 \\ 2 & -2 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 3 & -1 \\ 4 & 7 \\ 3 & 2 \end{pmatrix}, \; C = \begin{pmatrix} 4 & 8 & 7 \\ 1 & -2 & 1 \end{pmatrix}$$

For the following matrix products, determine whether they are well-defined:
a) AC b) BCA c) $B^T A$ d) CC^T
Answers: a) is not well-defined, all others are.

Exercise 3 (2 points).

Let $z \in \mathbb{C} \setminus \{0\}$ have polar coordinates (r, α). What are the polar coordinates of $z^{-1}\overline{z}$?

a) $(r, 2\alpha)$ b) (r^2, α) c) $(1, -2\alpha)$ d) $(2\alpha, 1)$.
Answer: c): $|z^{-1}\overline{z}| = |z|^{-1}|\overline{z}| = 1$, hence $z^{-1}\overline{z}$ has length 1. This leaves c) as only answer.

Exercise 4 (4 points)(We subtract 2 points per false answer).

For each of the following systems of linear equations, decide whether it has a unique solution.

a) $\begin{array}{rrrcl} 2x & +3y & -z & = & 5 \\ -4x & +8y & +2z & = & 2 \\ 4x & +6y & -2z & = & 10 \end{array}$ b) $\begin{array}{rrrcl} 2x & +3y & -z & = & 5 \\ & y & +2z & = & 1 \\ & & z & = & 2 \end{array}$

Answer: a) has no unique solution: The third line is twice the first line. Thus, if there exists

a solution, it is not unique.
b) is in simple form, with nonzero left-hand side, thus uniquely solvable.

Exercise 5 (3 points).

Compute the determinant of the matrix

$$\begin{pmatrix} a & a^2 & 1+a^3 \\ 0 & a-1 & a^2 \\ 0 & 0 & a-2 \end{pmatrix} .$$

For which values of a is the matrix invertible?
Answer: The matrix has determinant $a(a-1)(a-2)$. It is invertible precisely if the determinant is nonzero, i.e., whenever $a \notin \{0,1,2\}$.

Exercise 6 (4 points).

Consider the line

$$\mathbb{L} = \{\mathbf{c} \in \mathbb{R}^2 : \mathbf{c} \cdot (1,3)^T = 10\} \text{ ?}$$

(a) Which of the following points belong to $\mathbb{L}$?
i) $(3,-1)^T$ ii) $(1,3)^T$ iii) $(7,-1)^T$ iv) $(-2,4)^T$
Answer: ii) and iv).

(b) Compute the distance of the point $(2,5)^T$ to $\mathbb{L}$.
Answer: Dividing the equation by the norm of $(1,3)$ results in the the Hesse normal form:

$$\mathbf{c} \cdot \frac{1}{\sqrt{10}}(1,3)^T = \sqrt{10} .$$

Thus $(2,5)$ has the distance

$$|(2,5) \cdot \frac{1}{\sqrt{10}}(1,3)^T - \sqrt{10}| = \frac{17}{\sqrt{10}} - \sqrt{10} = \frac{17-10}{\sqrt{10}} = \frac{7}{\sqrt{10}} .$$

Some statistics:

Score histogram:

0-9	10-15	16-20
2	3	1

Average score: 11.13
Lowest score: 8
Highest score: 17

Percentage scored per exercise:

1	2	3	4	5	6
78	71	33	67	56	25