Calculus and Linear Algebra, Worksheet 1

Exercise 1.

Let A, B, C, D be the following subsets of $\mathbb{N}$:

$A :=$ set of all even natural numbers, $B :=$ set of all odd natural numbers,
$C :=$ set of all prime numbers, $D :=$ set of all squares.

Determine:

a) $A \cap B$ b) $A \cup B$ c) $A \cap C$ d) $A \cap D$ e) $(A \cap D) \cup B$
f) $A \setminus B$ g) $B \setminus D$ h) $A \cap (C \cup D)$ i) $B \cup D$.

Exercise 2.

Find the error:

$$x = y \neq 0 \Rightarrow xy = y^2 \Rightarrow xy - x^2 = y^2 - x^2$$
$$\Rightarrow x(y - x) = (y - x)(y + x) \Rightarrow x = y + x \Rightarrow 1 = 2.$$

Exercise 3.

Simplify $(a, b, c > 0)$:

a) $a^{b-4}\, a^2\, a^{2b}$ b) $\sqrt[3]{\sqrt{a^{12}}}$ c) $\dfrac{9a^2\,(2b)^2\,c^5}{6a^3\,b\,c^3}$ d) $\left(\dfrac{48a^2\,a^4\,(b^4a)^{-3}}{3b^{-2}\,a^3} \right)^{1/2}$

e) $\left(a^{b+3}\,a^{8b+1} \right)^{9b-4}$ f) $\sqrt[3]{a}\,\sqrt[5]{a}$ g) $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}}$ h) $\dfrac{\sqrt{a^2\,b}}{b\,\sqrt{a}}$.

Exercise 4.

Let $a, b, c, d \neq 0$ and suppose that $\frac{a}{b} = \frac{c}{d}$ and $a \pm b \neq 0 \neq c \pm d$. Show that the following equalities hold:

a) $\dfrac{a \pm b}{b} = \dfrac{c \pm d}{d}$ b) $\dfrac{a}{a \pm b} = \dfrac{c}{c \pm d}$ c) $\dfrac{a + b}{a - b} = \dfrac{c + d}{c - d}$.

Exercise 5.

Find all real numbers x which satisfy the equality:

a) $2x - 0{,}5\,(x+7) = 2{,}5x - 3$

b) $\frac{2x+3}{4x-2} = \frac{4x+6}{8x+4}$

c) $7x^2 + 19 = 82$

d) $\sqrt{100x^7} - \sqrt{400x^5} = \sqrt{6400x^3}$

e) $12x^6 + 6x^5 = 0$

f) $56x^6 = 7x^7$

g) $\sqrt{x^2+6} = 3x - 4$

h) $135 - 5x^2 = 30x$

i) $2^{7x^2-21x} = 1^{-2{,}5}$

j) $-3x^4 + 3x^2 = -6$

k) $20x^2 - 64 = x^4$

l) $2x^2 + 4x = 0$

m) $4x^6 + 8x^5 = 96x^4$

n) $10^{2x+5} = (2y)^x$.

Exercise 6.

Determine the sets $S \subseteq \mathbb{R}$ of real numbers that satisfy the following inequalities:

a) $5x + 5 < 0$

b) $\sqrt{x-4} < 3$

c) $\frac{2-x}{4+x} - 5 < 0$

d) $\frac{1}{x+1} \le 2$

e) $|2x+5| \le 7$

f) $15 + \frac{2-2x}{3} < 7x - \frac{2x+3}{4}$

g) $4x + 24 \ge 15 - 5x$

h) $\frac{2-175x}{25-50x} \le x + 4$

i) $\frac{16x}{x^2+15/4} > 4$

j) $x^3 > 2x^2 - x$

k) $x^4 - 4x^3 > 0$

l) $x^7 + 24x^5 < 10x^6$.

Exercise 7.

Find all $x \in \mathbb{R}$ which satisfy the inequality:

a) $|x+1| > 3$

b) $x - 1 > \dfrac{5}{x+3}$

c) $3 - x \le x + 1$

d) $x^2 - 3x - 2 < 10 - 2x$

e) $x^2 + x < 0$

f) $x^2 - |x| \ge 0$

g) $x|x-4| < 5$

h) $\dfrac{1}{|x-1|} \le \dfrac{1}{|x-2|}$

i) $3 - x^2 + 2x > 0$

j) $x < \dfrac{9}{6-x}$

k) $x^2 + 6x > 7$

l) $x + 3 > \dfrac{x+18}{3x-2}$

m) $\dfrac{x}{x-2} > \dfrac{x-3}{3x-1}$

n) $|x-1| + x \le 5$

o) $|x-2| - 2x \ge 11$

p) $|x-2| + |x+3| \ge 5$

q) $\dfrac{(x+2)|x^2-1|}{x+1} > 4$

r) $\dfrac{8(x-1)}{x^2} < |3x-3| + x - 1$.

Exercise 8.

Determine the sets $S \subseteq \mathbb{R}$ of real numbers that satisfy the following (in-)equalities:

a) $-x - |x| + \sqrt{x^2-1} \le 0$

b) $\sqrt{x^4} = 2|x| - 1$

c) $\sqrt{x^2-9} \le \frac{x+3}{2}$

d) $\sqrt{x+5} \ge |x+1| - 2$

e) $\sqrt{x+1} \le \sqrt{3-x}$

f) $\sqrt{x+4} > x + 2$.