Lehrstuhl A für Mathematik
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Calculus and Linear Algebra, Worksheet 2

Exercise 1.

Calculate the real part, the imaginary part and the modulus of the complex numbers z, z^2 and $|z|^2$.

a) $z = \dfrac{(1-i)(3-2i)}{3+i}$ b) $z = (2+i)^2 + 7 - 3i$ c) $z = \dfrac{1+3i}{-2+4i}$

d) $z = \dfrac{i+3}{2i-4}$ e) $z = \dfrac{(1+2i)[(4+3i)^2+1-22i]}{(2-i)^2-2+5i}$ f) $\dfrac{1-i}{1-2i}z = \dfrac{2+2i}{1+3i}$

g) $\left[\dfrac{1-i}{2+3i} - \dfrac{6+2i}{1+i}\right] z = \dfrac{3-i}{3+i}$ h) $\left[\dfrac{1-2i}{3+i} + \dfrac{6-i}{1+5i}\right] \bar{z} = \dfrac{3-6i}{1+i}$

Exercise 2.

Calculate z.

a) $2z + 3i\bar{z} - 5\bar{z} = 5(i-1)$ b) $\frac{5}{2}z - 3iz + \frac{1}{2}\bar{z} + 2i\bar{z} = 7 + 5i$ c) $-3z + \bar{z} - 2i\bar{z} = 2i$

Exercise 3.

Express z in polar coordinates.

a) $z = \sqrt{3} - i$ b) $z = \dfrac{1}{\sqrt{3}} + i$ c) $z = 4i$

d) $z = -2 + 2i$ e) $z = 2 + 2i$ f) $z = -8 - 8\sqrt{3}i$

Exercise 4.

Find the indicated roots and illustrate graphically:

a) $z^4 = 8(1 - i\sqrt{3})$ b) $z^2 = 4i$ c) $z^3 = \dfrac{1}{\sqrt{2}}(1-i)$

d) $z^4 = 1 + i\,\tan\alpha$ with $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ e) $z^6 = 1$ f) $z^4 = -16$

g) $(z-3i)^3 = 29\dfrac{3-5i}{2-5i} - 31 + 22i$ h) $z^3 + 2 = 2i$ i) $z^2 = \frac{1}{2}(1 - i\sqrt{3})$

Exercise 5.

Calculate $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

a) $z = \left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^{10}$ b) $z = (\frac{1-i}{1+i})^{10}$ c) $z = \left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^{20}$

d) $z = \left(\frac{\sqrt{3}+i\,3\sqrt{3}}{2+i}\right)^{15}$ e) $z = \left(\frac{-1+i\sqrt{3}}{6i}\right)^{1991}$ f) $z = \left(\frac{\sqrt{2}+i\sqrt{6}}{2(1+i)}\right)^{60}$

Exercise 6.

Find the imaginary part of all complex numbers z such that:

$$(1+\sqrt{3}i)(z+\sqrt{5}+i)^4 = -32.$$

Exercise 7.

Determine and sketch the regions of $\mathbb{C}$ defined by the following conditions.

a) $|z+1| - \mathrm{Re}(z) \geq 2$

b) $(4\mathrm{Re}(z))^2 - |z|^4 \geq 0$

c) $\mathrm{Re}(z) \leq |z|^2$

d) $\mathrm{Im}(\frac{z-1}{z+i}) \leq 0$

e) $\mathrm{Re}(\frac{z-1}{z+i}) < 0$

f) $|\frac{1}{z} - 1| = 2$

g) $|\frac{1}{z} - 1| = 1$

h) $|z-1| \leq \mathrm{Im}(z) + 1$

i) $(2+i)z + (2-i)\bar{z} - 2 = 0$

j) $z\bar{z} + 2\mathrm{Re}((1+i)z) - 2 = 0$

k) $3z^2 - 10z\bar{z} + 3\bar{z}^2 + 16 = 0$

l) $z^2 - 5z\bar{z} + 2\bar{z}^2 + 8 = 0$

m) $i(\frac{1}{z} - \frac{1}{\bar{z}}) < \frac{1}{\mathrm{Im}(z)}$ $(\mathrm{Im}(z) \neq 0)$

n) $|z-i| \leq |z-2+i|$ and $|\arg(z+2)| < \frac{\pi}{4}$

o) $|\arg z| < \frac{\pi}{4}$ and $2\mathrm{Re}(z) + \mathrm{Im}(z) < 1$

p) $|\arg(1+z)| < \frac{\pi}{3}$ and $|\arg(1-z)| < \frac{\pi}{3}$

q) $-\frac{3\pi}{4} \leq \arg(z-i) \leq -\frac{\pi}{4}$ and $2|\mathrm{Im}(z)| \leqslant 1$

Hint: The boundaries of the sets to be determined in Exercise 7 are often curves of the following types. Let $a, b, c, r \in \mathbb{R}$ fixed such that $r \geq 0$. The points (x, y) that satisfy the following conditions describe known curves in the plane.

$(x-a)^2 + (y-b)^2 = r^2$	circle with center (a, b) and radius r
$(\frac{x}{a})^2 + \left(\frac{y}{b}\right)^2 = 1$	ellipse with center $(0,0)$
$(\frac{x}{a})^2 - \left(\frac{y}{b}\right)^2 = 1$	hyperbola with center $(0,0)$
$y = ax^2 + bx + c$	parabola