Lehrstuhl A für Mathematik
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Calculus and Linear Algebra, Worksheet 3

to be discussed on 30 October 2008

Exercise 1.

a) Consider the points $\mathbf{a} = (3, 1, 4)$, $\mathbf{b} = (0, 2, -3)$ and $\mathbf{c} = (3, -1, 3)$ in $\mathbb{R}^3$. Prove that the triangle $\mathbf{abc}$ is a right triangle and calculate the length of the hypotenuse.

b) Consider the points $\mathbf{a} = (2, 2, 0)$, $\mathbf{b} = (4, -2, 0)$ and $\mathbf{c}_\alpha = (3, 0, \alpha)$ $(\alpha \in \mathbb{R}, \alpha \neq 0)$. Show that the three points built an isosceles triangle for any $\alpha \in \mathbb{R}, \alpha \neq 0$. Which $\alpha \in \mathbb{R}$ makes this triangle equilateral?

Exercise 2.

Find the components $a_1 (\leqslant 0), a_2$ of the vector $\mathbf{a}^T = (a_1, a_2, -6, 4, -2)$ with $|\mathbf{a}| = 9$ and the components b_3, b_4, b_5 of the vector $\mathbf{b}^T = (20, -15, b_3, b_4, b_5)$ which is a multiple of $\mathbf{a}$.

Exercise 3.

a) Consider the vectors $\mathbf{a} = (1, 3, -1)^T$, $\mathbf{b} = (3, 2, 1)^T$ and $\mathbf{c} = (1, 2, 3)^T$. Find:

   i) all possible vector products with $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$;

   ii) the area A of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$;

   iii) a vector $\mathbf{d} \in \mathbb{R}^3$ orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

b) Do i) - iii) for $\mathbf{a} = (1, 1, -2)^T$, $\mathbf{b} = (3, 2, -3)^T$ and $\mathbf{c} = (-2, 1, 1)^T$.

Notation: Let $\mathbf{e}^{(i)} \in \mathbb{R}^n$ denote the vector with coordinates $\mathbf{e}^{(i)}_i = 1$ and $\mathbf{e}^{(i)}_j = 0$ for $j \neq i$ (e.g. $\mathbf{e}^1 = (1, 0, 0)^T$, $\mathbf{e}^{(2)} = (0, 1, 0)^T$, $\mathbf{e}^{(3)} = (0, 0, 1)^T$ in $\mathbb{R}^3$).

Exercise 4.

Consider $\mathbf{b} = (\sqrt{2}, 0, -1)^T$.

a) Find $\mathbf{c} = (c_1, c_2, c_3)^T \in \mathbb{R}^3$ of length 2 that is perpendicular to $\mathbf{b}$ and forms the angle $\frac{\pi}{3}$ with $\mathbf{e}^{(2)}$.

b) Find $\mathbf{a} = (a_1, a_2, a_3)^T$ of length 8 that is perpendicular to $\mathbf{b}$ and to $\mathbf{e}^{(1)} + \mathbf{e}^{(2)}$.

Exercise 5.

Consider the vectors $\mathbf{a} = (2,1,1)^T$ and $\mathbf{b} = (1,-1,-1)^T$. Find all vectors $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ with $(\mathbf{a} \times \mathbf{x}) \cdot \mathbf{b} = -9$ and $\mathbf{x} \cdot \mathbf{b} = 0$ such that the angle between $\mathbf{a}$ and $\mathbf{x}$ is $\frac{2\pi}{3}$.

Exercise 6.

a) Consider the vectors $\mathbf{v}, \mathbf{a} \in \mathbb{R}^n$ with $\mathbf{a} \neq 0$. Calculate $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}$ is parallel to $\mathbf{a}$, $\mathbf{x} + \mathbf{y} = \mathbf{v}$ and $\mathbf{y} \perp \mathbf{a}$.

b) Solve a) for $\mathbf{v} = (2,3,1)^T$ and $\mathbf{a} = (1,1,1)^T$.

Exercise 7.

Consider the straight line $\{(x_1, x_2)^T : x_2 = 2x_1 + 5\}$ in the plane. Find a parametric equation of the line and its Hesse normal form.

Exercise 8.

a) Consider the points $\mathbf{p}_1 = (1,-2)$ and $\mathbf{p}_2 = (4,2)$ in $\mathbb{R}^2$.
   i) Find an equation in parametric form of the straight line $\mathbb{L} \subset \mathbb{R}^2$ that passes through $\mathbf{p}_1$ and $\mathbf{p}_2$.
   ii) Find the Hesse normal form of $\mathbb{L}$.

b) Do i) and ii) for $\mathbf{p}_1 = (1,1)$ and $\mathbf{p}_2 = (3,4)$.

Exercise 9.

Find the straight line $\mathbb{L} \subset \mathbb{R}^2$ that intercepts the straight line $\mathbb{L}_1 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (\sqrt{3}, -1)^T + \lambda(1,3)^T\}$ at the point $\mathbf{p}_1 = (\sqrt{3}, 1)$ with angle $\frac{\pi}{4}$.

Exercise 10.

Let $\mathbb{L}_1 \subset \mathbb{R}^3$ be the straight line that goes through the points $\mathbf{p}_1 = (-2,-1,6)$ and $\mathbf{p}_2 = (4,-4,12)$.

a) Write $\mathbb{L}_1$ in parametric form.

b) Let $\mathbb{L}_2$ be another straight line that passes through the point $(0,0,\alpha)$ in the direction of $(1,0,1)^T$.
   i) For which value of α do $\mathbb{L}_1$ and $\mathbb{L}_2$ intercept each other?
   ii) Calculate the interception point $\mathbf{s}$ of $\mathbb{L}_1$ and $\mathbb{L}_2$.

Exercise 11.

Let $\mathbb{P}_1$ be the plane that contains the points $\mathbf{p}_1 = (1,0,1)$, $\mathbf{p}_2 = (1,1,0)$ and $\mathbf{p}_3 = (2,2,1)$. Find the plane $\mathbb{P}$ that is parallel to $\mathbb{P}_1$ and contains the point $\mathbf{p}_0 = (1,0,2)$. Write $\mathbb{P}$ in parametric form and in Hesse's normal form. Is the point $\mathbf{0} = (0,0,0)$ between $\mathbb{P}$ and $\mathbb{P}_1$?

Exercise 12.

Let $\mathbf{p} = (1, 0, 5)$ and $\mathbb{P} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot (1, 2, -1)^T = -3\}$ in $\mathbb{R}^3$. Calculate the distance $\text{dist}(\mathbf{p}, \mathbb{P})$ between the point and the plane.

Exercise 13.

a) Consider the equations of two parallel planes $\mathbb{P}_1$ and $\mathbb{P}_2$ in $\mathbb{R}^3$ written in HNF. Give a simple formula for the distance of the planes, using the Hesse normal forms of both planes.

b) Calculate the distance between $\mathbb{P}_1 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot (1, 1, -1)^T = 3\}$ and $\mathbb{P}_2 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot (2, 2, -2)^T = -4\}$.

Exercise 14.

Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} \lambda \\ 2 \\ \lambda \end{pmatrix} \quad (\lambda \in \mathbb{R}).$$

Determine whether the following subsets are linearly dependent or independent:

a) $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, b) $\{\mathbf{a}, \mathbf{d}\}$, c) $\{\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$, d) $\{\mathbf{b}, \mathbf{e}\}$,
e) $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$, f) $\{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$, g) $\{\mathbf{b}, \mathbf{c}, \mathbf{d}\}$, h) $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$.