LEHRSTUHL A FÜR MATHEMATIK
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Calculus and Linear Algebra, Worksheet 4

to be discussed on 6 November 2008

Exercise 1.

Investigate the position of the plane

$$\mathbb{P} = \left\{ \mathbf{x} = (1,2,2)^T + \lambda(1,-2,0)^T + \mu(4,0,2)^T \ : \ \lambda, \mu \in \mathbb{R} \right\}$$

relative to the line

$$\mathbb{L} = \left\{ \mathbf{x} = (0,1,1)^T + \alpha(5,-2,2)^T \ : \ \alpha \in \mathbb{R} \right\}$$

in $\mathbb{R}^3$. Depending on the result calculate the interception set or the distance $\operatorname{dist}(\mathbb{L}, \mathbb{P})$.

Exercise 2.

Consider the points $\mathbf{q} = (2,2,4)$ and $\mathbf{p}_\lambda = (\lambda, 2+\lambda, 1)$, where $\lambda \in \mathbb{R}$, and the line

$$\mathbb{L} = \left\{ \mathbf{x} = (2,1,0)^T + \mu(-2,0,4)^T \ : \ \mu \in \mathbb{R} \right\}$$

in $\mathbb{R}^3$. Let $\mathbb{P}_\lambda$ be the plane such that $\mathbf{q}$ is the interception point between $\mathbb{P}_\lambda$ and the line that goes through $\mathbf{p}_\lambda$ and is perpendicular to $\mathbb{P}_\lambda$.

a) Find an equation for $\mathbb{P}_\lambda$. Does $\mathbb{P}_\lambda$ contain the origin?

b) For which $\lambda \in \mathbb{R}$ is the line $\mathbb{L}$ not parallel to $\mathbb{P}_\lambda$? For this λ calculate the interception point $\mathbf{s}_\lambda$ of $\mathbb{L}$ and $\mathbb{P}_\lambda$. For all the other values of λ calculate $\operatorname{dist}(\mathbb{P}_\lambda, \mathbb{L})$.

Exercise 3.

Find all solutions of the following systems $Ax = b$ of linear equations.

a) $A = \begin{pmatrix} 2 & -1 & 4 & 4 \\ 1 & 0 & 4 & 2 \\ -1 & 3 & -9 & -2 \\ 4 & -4 & 7 & 8 \\ 1 & -4 & 4 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 4 \\ 0 \\ -5 \\ -8 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 5 & -1 & -3 \\ 3 & 1 & 0 & -1 \\ 2 & -1 & 3 & 5 \\ 5 & 0 & -1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 9 \\ 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 5 & -1 & -3 \\ 1 & 6 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 1 & 4 & 2 & 1 \\ 4 & 2 & 8 & 0 & 2 \\ -2 & -1 & 2 & 2 & 1 \\ 2 & 1 & 3 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ -1 \\ 4 \\ 1 \end{pmatrix}$.

Exercise 4.

Find the sets $\mathbb{S}(A, b^{(i)})$ of solutions of the systems of linear equations $Ax = b^{(i)}$, where $i = 1, 2, 3$.

a) $A = \begin{pmatrix} 27 & 1 & 11 \\ 28 & 1 & 12 \\ 29 & 1 & 13 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad b^{(2)} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad b^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

b) $A = \begin{pmatrix} 15 & 2 & 17 \\ 7 & 1 & 8 \\ 20 & 3 & 23 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad b^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}, \quad b^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

Exercise 5.

Find the solution of the following systems of equations using the Gauss algorithm.

a) $\begin{aligned} 2x_1+4x_2+2x_3-2x_4&=5 \\ x_1+2x_2+4x_3 &= 4 \\ -2x_1-4x_2-10x_3+3x_4&=-9 \\ 4x_1+8x_2-10x_3-8x_4&=3 \\ 2x_1+4x_2+4x_3-4x_4&=6 \end{aligned}$

b) $\begin{aligned} 2x_1+4x_2+2x_3-2x_4&=0 \\ x_1+2x_2+4x_3 &= 0 \\ -2x_1-4x_2-10x_3+3x_4&=1 \\ 4x_1+8x_2-10x_3-8x_4&=0 \\ 2x_1+4x_2+4x_3-4x_4&=0 \end{aligned}$

c) $\begin{aligned} x_1+3x_2-4x_3+3x_4&=9 \\ 3x_1+9x_2-2x_3-11x_4&=-3 \\ 4x_1+12x_2-6x_3-8x_4&=6 \\ 2x_1+6x_2+2x_3-14x_4&=-12 \end{aligned}$

Exercise 6.

Find the values of $\mu \in \mathbb{R}$ for which following systems of equations have a solution.

a) $\begin{aligned} x_1+x_2+x_3&=1 \\ -x_1+2x_3&=2 \\ 3x_1+2x_2&=\mu \end{aligned}$

b) $\begin{aligned} x_2-x_3+2x_4&=6 \\ 2x_1-x_2+3x_3&=-4 \\ \mu x_1+x_3&=-1 \\ 3x_1-x_2+4x_3&=-5 \end{aligned}$

c) $\begin{aligned} x_1+x_2+3x_3&=2 \\ 2x_1+2x_2+7x_3&=5 \\ \mu^2x_1+\mu x_2+4x_3&=4 \end{aligned}$

Exercise 7.

Let $\mu, \nu \in \mathbb{R}$. Investigate the solution of the following systems of equations and interpret the result geometrically.

a) $\begin{aligned} 2x-2y&=1 \\ x+y&=2 \end{aligned}$

b) $\begin{aligned} x+y&=1 \\ -x-y&=-2 \end{aligned}$

c) $\begin{aligned} x+2y&=2 \\ 3x+4y&=8 \\ x-y&=1 \end{aligned}$

d) $\begin{aligned} x+\mu y+2z&=1 \\ 4x+2y+2\mu z&=-2 \\ x+2\nu y+2z&=3 \end{aligned}$

Exercise 8.

For which $\alpha \in \mathbb{R}$ does the homogeneous linear system $A_\alpha x = 0$ have solutions different from $\mathbf{0}$? Find the set $\mathbb{S}(A_\alpha, 0)$ of solutions.

a) $A_\alpha = \begin{pmatrix} -2 & 4 & 3 \\ 1 & 3 & 5 \\ -4 & -2 & \alpha \end{pmatrix}$ b) $A_\alpha = \begin{pmatrix} 2 & 3 & -1 \\ 2 & 2 & \alpha \\ 1 & 2 & 4 \end{pmatrix}$

Exercise 9.

Let M_1, M_2, M_3 be metal alloys that contain copper, silver and gold in the following percentages. M_1 consists of 20% copper, 60% silver and 20% gold, M_2 consists of 70% copper, 10% silver and 20% gold, and M_3 consists of 50% copper and 50% silver.

a) Is it possible to mix M_1, M_2, M_3 to get a new alloy that contains 40% copper, 50% silver and 10% gold?

b) How high is the percentage of silver in the alloy that contains four times as much M_1 as M_2, and five times as much M_3 as M_2?

c) What is the percentage of M_3 in an alloy that consists of 5% gold?