Lehrstuhl A für Mathematik
Prof. Dr. H. Führ
Dr. D. Meierling

Aachen, 17 November 2008

Calculus and Linear Algebra, Worksheet 6

to be discussed on 20 November 2008

Exercise 1.

Calculate the limit a of the sequence $(a_n)_{n\in\mathbb{N}}$, and find $N \in \mathbb{N}$ such that $|a_n - a| < 10^{-3}$ for $n > N$.

a) $1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}$; b) $\frac{1}{4^n}$; c) $\frac{6n-2}{3n+7}$.

Exercise 2.

Calculate, if possible, the limits of the following sequences:

a) $a_n = \dfrac{5n}{1-2n}$ b) $a_n = \sqrt{n^2+1} - n$ c) $\dfrac{2n^3+3n}{5n^3-1}$

d) $a_n = \dfrac{\sqrt{2}n + (-1)^n n}{n+1}$ e) $a_n = \dfrac{1}{\sqrt{n^2+5}}\dbinom{n}{2}$ f) $(-i)^n$

g) $a_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{(n-1)(n)}$ h) $\dfrac{2n^4+3n}{5n^3-1}$

Exercise 3.

Show that

$$\lim_{n\to\infty} \sqrt[n]{n^3(3^n+5)} = 3.$$

Hint: You can use $\lim_{n\to\infty} n^{1/n} = 1$ as well as the following fact. If $x = \lim_{n\to\infty} x_n$ for $x, x_n \geq 0$, then $x^c = \lim_{n\to\infty} x_n^c$, for all $c > 0$.

Exercise 4.

Knowing that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n-1}\right)^n = e$, show that $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n = \dfrac{1}{e}$.

Exercise 5.

Find the limits of the following sequences:

a) $\left(\dfrac{n+3}{n+2}\right)^{3n-5}$ b) $\left(1+\dfrac{1}{2n}\right)^{4n}$

c) $\left(\dfrac{n}{n+1}\right)^{2-5n}$ d) $\left(1-\dfrac{1}{1-3n}\right)^{9n}$

Hint: You can use the following fact. If $x = \lim_{n\to\infty} x_n$, and $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, then $x = \lim_{k\to\infty} x_{n_k}$.

Exercise 6.

Investigate whether the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent or divergent and calculate the sum in case of convergence.

a) $a_n = \dfrac{1}{4n^2-1}$ b) $a_n = (-1)^n \dfrac{4}{3^{n-2}}$ c) $a_n = \dfrac{1}{2^n}$

d) $a_n = 3^n$ e) $a_n = \dfrac{n-1}{n!}$ f) $a_n = \dfrac{1}{n^2+13n+42}$

Exercise 7.

Investigate the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$, where a_n is given by

a) $a_n = \dfrac{1}{\sqrt{n^2+1}}$ b) $a_n = (-1)^n \left(1+\dfrac{1}{\sqrt{n}}\right)$ c) $a_n = \dfrac{n!}{n^n}$

d) $a_n = \left(\dfrac{n+1}{2n+1}\right)^n$ e) $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ f) $a_n = \left(-\dfrac{1}{2}\right)^n$

g) $a_n = (-1)^n \left(\dfrac{1}{3} - \dfrac{n-1}{3n-2}\right)$ h) $a_n = \dfrac{1}{\sqrt{n}}$ i) $a_n = \dfrac{1}{n^2+\pi}$

j) $a_n = \dfrac{k^{2n}}{n!}$, $k \in \mathbb{N}$ fixed k) $a_n = \left(\dfrac{n}{n+1}\right)^n$ l) $a_n = n\left(\dfrac{n}{n+1}\right)^{n^2}$

m) $a_n = \dfrac{2}{n^2} + (-1)^n \dfrac{n^2+1}{n^3}$ n) $a_n = \dfrac{n}{2^{2n}}$ o) $a_n = (-1)^n \dfrac{n}{n+1}$

p) $a_n = n\left(1-\left(1-\dfrac{1}{2n}\right)^3\right)$ q) $a_n = \dfrac{\sqrt{n^2+1}-\sqrt{n^2}}{\sqrt{n^2+1}}$ r) $a_n = \dfrac{n!\left(1+i\sqrt{3}\right)^n}{n^n}$