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Calculus and Linear Algebra, Worksheet 7

to be discussed on Thursday, 27 December 2008

Exercise 1.

Let $f : A \to B$ and $g : B \to C$ be two mappings as given below. Determine the composition $h = g \circ f : A \to C$.

a) $A = \{1,2,3\}$, $B = \{1,2\}$, $C = \{2,3\}$;
$f(1) = f(2) = 1$, $f(3) = 2$, $g(1) = 2$, $g(2) = 3$

b) $A = \mathbb{Z}$, $B = C = \mathbb{N}$;
$f(z) = z^2 + 1$, $z \in A$; $g(n) = 2n + 1$, $n \in B$

Exercise 2.

Which of the following mappings $f : A \to B$ are injective, surjective or bijective? If f is bijective, determine the inverse mapping $f^{-1} : B \to A$.

a) $A = B = \{1,2,3\}$; $f(1) = 3$, $f(2) = 1$, $f(3) = 2$.

b) $A = \{1,2\}$, $B = \{1,2,3\}$; $f(1) = 2$, $f(2) = 3$.

c) $A = \{1,2,3\}$, $B = \{2\}$; $f(1) = f(2) = f(3) = 2$.

d) $A = B = \mathbb{Q} \setminus \{0\}$; $f(q) = \frac{1}{q}$ for $q \in A$.

e) $A = B = \mathbb{R}$; $f(x) = 2x + 3$ for $x \in A$.

f) $A = \mathbb{R}$, $B = \{x \in \mathbb{R} \mid x \geqslant 0\}$; $f(x) = x^2$ for $x \in A$.

Exercise 3.

Find the domain $D(f)$.

a) $f(x) = \dfrac{1-x}{1+x}$ b) $f(x) = \dfrac{5\,|x^2-9|}{6(x+3)}$ c) $f(x) = \dfrac{x-1}{(2x+1)^2}$

d) $f(x) = \dfrac{x^3-1}{x^2-1}$ e) $f(x) = \dfrac{x^n-1}{x-1}$ f) $f(x) = \dfrac{x^2+3x-10}{(x-2)\sqrt{2x^2-4}}$

g) $f(x) = \dfrac{(x+|x|)\sqrt{x^2-4x+3}}{1-x^2}$ h) $f(x) = \dfrac{4-x^2}{x(1-|x-1|)}$

Exercise 4.

Consider the following subsets $G \subseteq \mathbb{R}^2$. For which sets G exists a mapping f such that $G = G_f$? If possible, determine the domain and range of f.

a) $\{(x,y) \in \mathbb{R}^2 : x + y = 1\}$ b) $\{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1, xy > 0\}$

c) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ d) $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$

Exercise 5.

Find the domain of the following functions:

a) $f(x) = \tan \dfrac{\pi}{1+x}$ b) $g(x) = \sqrt{\sqrt{(x^3 + 7x^2 - 73x + 66)} - 1}$

c) $h(x) = \dfrac{x}{2\sin(x) - 1}$ d) $k(x) = \sqrt{\cos(-x)}$

Exercise 6.

Let the function f be defined as below.

a) $f(x) = \begin{cases} \frac{1}{x-1} & ; \ x > 2 \\ x^2 - \frac{7}{2} & ; \ -1 \leqslant x \leqslant 2 \\ \frac{5}{2x} & ; \ x < -1 \end{cases}$ b) $f(x) = \begin{cases} \frac{-x}{1-x}, & x < 0 \\ x, & 0 < x < 1 \\ x^2, & x > 1 \end{cases}$

c) $f(x) = \begin{cases} 2 - x^2 & ; \ |x| \leqslant 2 \\ -\frac{4}{|x|} & ; \ |x| > 2. \end{cases}$ d) $f(x) = \begin{cases} |x| & ; \ x \leqslant 0 \text{ or } x \geqslant 1 \\ \frac{1}{x} & ; \ 0 < x < 1 \end{cases}$

i) Calculate the domain of f;

ii) Determine whether f is bounded from below or from above;

iii) Determine the monotony intervals of f;

iv) Sketch f.

Exercise 7.

Are the following functions bounded from below or from above? Determine the monotony intervals and sketch f.

a) $f(x) = \dfrac{1-x}{1+x}$ b) $f(x) = \dfrac{x^2-1}{x-1}$ c) $f(x) = \dfrac{x}{1+x}$ d) $f(x) = -|x-1|$

Exercise 8.

Find the domain $D = D(f) \subseteq \mathbb{R}^n$.

a) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}, \ n = 3$ b) $f(x,y) = x^3 - 2x^2y^2 + 4xy^3 + y^4 + 10, \ n = 2$

c) $f(x,y) = \dfrac{x-y}{x+2y}, \ n = 2$ d) $f(x,y) = \dfrac{x}{y}\sqrt[3]{y^2 - x}, \ n = 2$