Lehrstuhl A für Mathematik
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Aachen, 1 December 2008

Calculus and Linear Algebra, Worksheet 8

to be discussed on Thursday, 4 December 2008

Exercise 1.

Let f by the function given by

a) $f(x) = 1 + \frac{2x}{|x|} - (1-|x|)^2$; b) $f(x) = \frac{2}{3}\frac{|x^2-4|}{x-2}$; c) $f(x) = \frac{x^2-3x+2}{x-2}$;

d) $f(x) = \frac{x^3-7x^2+16x-12}{|x^2+x-6|}$; e) $f(x) = \frac{\sqrt{4-x}-\sqrt{4-x^2}}{x|x-1|}$; f) $f(x) = \frac{x}{1+x}$;

g) $f(x) = \frac{x^2+3x+2}{x-2}$; h) $f(x) = -|x-1|$.

Calculate the domain of continuity $C(f)$. Are there any points where the discontinuity of f is removable?

Exercise 2.

Investigate the continuity of the following functions at the point $(0,0)$. Let $f(0,0) = 0$ and for $(x,y) \neq (0,0)$ let $f(x,y) =$

a) $\frac{3x^2+2y^2}{x^2+y^2}$ b) $\frac{x^3y^2}{(x^2+y^2)^{5/2}}$ c) $\frac{x^4+y^4}{(x^2+y^2)^{3/2}}$ d) $\frac{x^2y}{x^2+y^2}$

e) $\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$ f) $\frac{x^2y}{y^2+x^4}$ g) $\frac{x^2-y^2}{(x^2+y^2)^{\alpha/2}}$ with $\alpha \in \mathbb{R}$

Exercise 3.

Let f be defined as below. Show that f is continuous and stricly monotonic on the given domain. Determine the inverse mapping f^{-1} and the domain of f^{-1}.

a) $f(x) = \frac{ax+b}{cx+d}$ $(ad-bc \neq 0,\ c \neq 0,\ a,b,c,d \in \mathbb{R})$, $x \in (-\frac{d}{c}, \infty)$;

b) $f(x) = x^3 + 1$, $x \in \mathbb{R}$;

c) $f(x) = \sqrt{1-x^2}$, $x \in [0,1]$;

d) $f(x) = \frac{x}{1-x^2}$, $x \in (-1,1)$.

Exercise 4.

Let f be the piecewise defined function as given below. Calculate the domain $D(f)$, the domain of continuity $C(f)$, the roots of f and the limits of f at the boundaries of $D(f)$. Does f have a maximum or a minimum?

a) $f(x) = \begin{cases} \frac{5}{2x}, & x < -1 \\ x^2 - \frac{7}{2}, & -1 \leqslant x \leqslant 2 \\ \frac{1}{x-1}, & x > 2 \end{cases}$ b) $f(x) = \begin{cases} \frac{-x}{1-x}, & x < 0 \\ x, & 0 \leqslant x \leqslant 1 \\ x^2, & x > 1 \end{cases}$

c) $f(x) = \begin{cases} 2 - x^2, & |x| < 2 \\ -\frac{4}{|x|}, & |x| \geqslant 2 \end{cases}$ d) $f(x) = \begin{cases} \frac{1}{3}x - \frac{2}{3}, & x < -1 \\ 2x + 1, & x \geqslant -1 \end{cases}$

e) $f(x) = \begin{cases} |x|, & x \leqslant 0 \text{ or } x \geqslant 1 \\ \frac{1}{x}, & 0 < x < 1 \end{cases}$

Exercise 5.

Let f be given by

a) $f(x) = x^4 - x - 10$; b) $f(x) = x^2 - 3$; c) $f(x) = 3x^3 - 17x^2 + 23x - 5$.

Determine the sign of $f(x)$ for $x = 0, 1, 2, \ldots$ to determine the number of positive roots of f. These roots can be approximated via the interval subdivision method. Calculate the positive roots with an error less than 10^{-2}.

Exercise 6.

Solve the inequalities of Worksheet 1 Exercise 7 in the following way. Find an equivalent inequality $\begin{cases} f(x) < 0 \\ f(x) \leqslant 0 \end{cases}$, where f is a continuous function. Then compute the roots $-\infty < x_1 < x_2 < \ldots < x_n < \infty$ of f and determine the sign of f on the intervals (x_i, x_{i+1}) by evaluating $f(y_i)$ for some arbitrary $y_i \in (x_i, x_{i+1})$ (for $i = 0, 1, \ldots, n$ with $x_0 = -\infty$ and $x_n = \infty$).