Lehrstuhl A für Mathematik
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Calculus and Linear Algebra, Worksheet 10

to be discussed on Thursday, 18 December 2008

Exercise 1.

Compute all local and global extrema and the corresponding extremal values for the function $f : A \to \mathbb{R}$.

a) $A = [-1, 1],\ f(x) = x^4$

b) $A = [0, 5],\ f(x) = x^3 - 5x^2 + 3x + 1$

c) $A = [-2, 2],\ f(x) = x(x^2 - 1)$

d) $A = \mathbb{R},\ f(x) = x - 2\sqrt{x^2 + 1}$

e) $A = [-2, 3],\ f(x) = \dfrac{x^3}{3} - x^2$

f) $A = \mathbb{R},\ f(x) = \dfrac{x^2 - 2x + 3}{x^2 + 2x + 3}$

g) $A = [-1, \frac{1}{2}],\ f(x) = \dfrac{x + 1}{x^2 + 1}$

h) $A = (1, \infty),\ f(x) = \sqrt{\dfrac{x^3}{x - 1}}$

j) $A = [0, \pi),\ f(x) = \sin^3 x + \cos^3 x$

Exercise 2.

Let the function $f : A \to \mathbb{R}$ be defined by

a) $A = [0, 2\pi),\ f(x) = 5|\cos(x)|$

b) $A = \mathbb{R},\ f(x) = |x^3 - 2x^2 - x + 2|$

c) $A = [0, 2],\ f(x) = \sin(x^2 - 2x + 1)$

Compute the global minima of f and the corresponding minimal values.

Exercise 3.

Determine the inflection points and the intervals of convexity and concavity for the function f. Let $f(x) =$

a) x^3 b) $\dfrac{x^2}{x - 1}$ c) $\frac{3}{2}x^4 - x^2 + 1$ d) $x^r,\ r \in \mathbb{Z},\ x > 0$

Exercise 4.

Let the function f be defined by

a) $f(x) = \dfrac{2x + 5}{\sqrt{4x + 1}}$

b) $f(x) = |x|\dfrac{\sqrt{x + 2}}{\sqrt{x + 1}}$

c) $f(x) = x - 2\sqrt{x^2 + 1}$

d) $f(x) = \dfrac{x^2 + 2x - 7}{x + 4}$

e) $f(x) = \dfrac{\sqrt{(x + 1)^3 + 16}}{\sqrt{x + 1}}$

f) $f(x) = 2\sqrt{x^2 - 24x + 80}$

g) $f(x) = \dfrac{8x^3}{(3x-2)^2}$ h) $f(x) = 2x + \dfrac{2}{x}$

Answer the following questions:

i) compute the domain $D(f)$ and the domain of continuity $C(f)$;

ii) compute the limits of f at the boundaries of $D(f)$; is there a continuous extension of f?;

iii) where is f differentiable?; where is f monotonic; what are the local and global extrema of f?;

iv) (for c) and d):) what are the inflection points of f?; where is f convex or concave?;

v) sketch f.

Exercise 5.

Find the mistake in the following application of L'Hospital's rule

$$\lim_{x\to 1} \frac{x^3+x^2-x-1}{x^2-1} = \lim_{x\to 1} \frac{3x^2+2x-1}{2x} = \lim_{x\to 1} \frac{6x+2}{2} = 4$$

and show that 2 is the correct value.

Exercise 6.

Using L'Hospital's theorem, show that

a) $\displaystyle\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = -\frac{1}{3}$ b) $\displaystyle\lim_{x\to \frac{\pi}{2}} \frac{\sin(\pi - 2x)}{\pi - 2x + \cos x} = \frac{2}{3}$

c) $\displaystyle\lim_{x\to\infty} \sqrt{1+x^2}\sin\tfrac{1}{x} = 1$ d) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = 1$

e) $\displaystyle\lim_{x\to 5} \frac{x^2-3x-10}{x^2-4x-5} = \frac{7}{6}$ f) $\displaystyle\lim_{x\to\infty} \left(\frac{1+x^2}{1+x} - \sqrt{x^2-1}\right) = -1.$

Exercise 7.

Compute the limits.

a) $\displaystyle\lim_{x\to 0} \frac{x-\tan x}{x-\sin x}$ b) $\displaystyle\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$ c) $\displaystyle\lim_{x\to 0} (\cos x)^{x^{-2}}$

d) $\displaystyle\lim_{x\to 0} \left(x\left(\ln x + \sin\frac{1}{x}\right)\right)$ e) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$ f) $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$