Lehrstuhl A für Mathematik
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Calculus and Linear Algebra, Worksheet 11

to be discussed on Thursday, 8 January 2009

Exercise 1.

Compute the radius of convergence of the following power series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{(4n+5)5^n}} x^n$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{3}\left(x+\frac{3}{2}\right)^n$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{2n-1}(x-13)^n$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-e)^n$ e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} x^n$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+2}}{\sqrt{n+1}} x^n$ h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^{n^2}} x^n$

Exercise 2.

Compute the limit $\lim\limits_{x\to 0} f(x)$ for $f(x) =$

a) $\displaystyle\frac{(e^x-1)\sin x}{1-\cos x}$ b) $\displaystyle\frac{1}{x}\ln(1+x)$ c) $\displaystyle\frac{1}{x}(\ln(a+x)-\ln a)$ $(a>0)$

d) $\displaystyle\frac{2-3x}{e^x-1}-\frac{2}{x}$ e) $\displaystyle\frac{x^3\sin x}{(1-\cos x)^2}$ f) $\displaystyle\frac{\cos x+\frac{x^2}{2}-1}{[\sin x]^4}$

Hint: Use the power series of f.

Exercise 3.

Show for $|x|<1$:

$$\sqrt{1+x} = 1+\frac{1}{2}x+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot \ldots \cdot (2n-3)}{2\cdot 4\cdot \ldots \cdot 2n}x^n$$

Exercise 4.

Determine the Taylor series

$$T_{\infty,x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

of f. For which $x \in \mathbb{R}$ does the Taylor series converge? For which $x \in \mathbb{R}$ is $f(x) = T_{\infty,x_0}(x)$?

a) $f(x) = \frac{1-x}{1+x}, \quad x \in \mathbb{R} \setminus \{-1\}, \quad x_0 = 0$ b) $f(x) = x^{\frac{3}{2}}, \quad x \geq 0, \quad x_0 = 1$

Hint for a): To compute the interval of convergence, consider the Taylor series for $x \geqslant 0$, $0 > x \geqslant -\frac{1}{2}$ and $x < -\frac{1}{2}$.

Exercise 5.

Show that:

a) $(\cos x e^{-x})^{(4)} = -4\cos x e^{-x}$

b) $(\ln(\ln x))' = \frac{1}{x \ln x}$

c) $((1+x^2)^{\sin x})' = (1+x^2)^{\sin x}(\frac{2x \sin x}{1+x^2} + \ln(1+x^2)\cos x)$

d) $((\frac{1+x}{1-x})^{x^2})' = (\frac{1+x}{1-x})^{x^2} 2x(\ln\frac{1+x}{1-x} + \frac{x}{1-x^2})$

e) $(\sqrt{e^{x^2+x+1}})' = (x+\frac{1}{2})\sqrt{e^{x^2+x+1}}$

f) $(\sqrt{e^{\sin\sqrt{x}}})' = \frac{\cos\sqrt{x}}{4\sqrt{x}}\sqrt{e^{\sin\sqrt{x}}}$

Exercise 6.

Find the domain of the following functions and calculate their derivatives. Consider $f(x) =$

a) $\frac{1}{2}\tan^2 x + \ln(\cos x)$ b) $\ln|x^2 - x|$ c) $(\cos x)^{\cos x}$ d) e^{x^2+x+1}