Lehrstuhl A für Mathematik
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# Calculus and Linear Algebra, Worksheet 12

to be discussed on Thursday, 15 January 2009

**Exercise 1.**

Let $f : [a, b] \to \mathbb{R}$ be a function and let $\mathcal{P}$ be a partition of $[a, b]$ as given below.

a) $a = 0$, $b > 0$, $\mathcal{P} = \{0, \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{(n-1)b}{n}, b\}$;

   i) $f(x) = x$;

   ii) $f(x) = x^2$;

   iii) $f(x) = x^\alpha$, where $\alpha > 0$;

   iv) $f(x) = e^x$.

b) $0 < a < b$, $\mathcal{P} = \{a = a \cdot (\frac{b}{a})^{0/n}, a \cdot (\frac{b}{a})^{1/n}, a \cdot (\frac{b}{a})^{2/n}, \ldots, a \cdot (\frac{b}{a})^{(n-1)/n}, a \cdot (\frac{b}{a})^{n/n} = b\}$; $f(x) = \frac{1}{x}$.

Compute $\underline{S}(\mathcal{P})$ and $\overline{S}(\mathcal{P})$ and show that

$$\lim_{n\to\infty} \underline{S}(\mathcal{P}) = \lim_{n\to\infty} \overline{S}(\mathcal{P}).$$

**Exercise 2.**

Let $f$ be continuous on $\mathbb{R}$, let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and let $a \in \mathbb{R}$. Let the function $H$ be defined by

$$H(x) := \int_a^{g(x)} f(t)\, dt.$$

Show that $H'(x) = f(g(x)) \cdot g'(x)$.

**Exercise 3.**

Let $a, b \in \mathbb{R}$ and $a < b$. Differentiate the following functions with respect to $x$ using Exercise 2.

a) $\displaystyle\int_a^{x^3} \frac{1}{1+\sin^2(t)}\,dt$ b) $\displaystyle\int_{x^3}^{a} \frac{1}{1+\sin^2(t)}\,dt$ c) $\displaystyle\int_a^{b} \frac{x}{1+\sin^2(t)}\,dt$

d) $\displaystyle\int_a^{x^3} \sin^3(t)\,dt$ e) $\displaystyle\int_x^{b} \frac{1}{1+t^2+\sin^2(t)}\,dt$ f) $\displaystyle\int_a^{b} \frac{1}{1+t^2+\sin^2(t)}\,dt$

**Exercise 4.**

Compute the length $\ell(\mathcal{C})$ of the curve $\mathcal{C}$ parameterised by $f$ in the given interval.

a) $f : [0, T] \to \mathbb{R}^3, \quad f(t) = (2\cos^2 2t + 1, 1 + 2\sin 2t \cos 2t, 3t), \quad T > 0;$

b) $f : [0, \frac{\pi}{2}] \to \mathbb{R}^2, \quad f(t) = (\sin^3(t), \cos^3(t));$

c) $f : [0, T] \to \mathbb{R}^2, \quad f(t) = (e^{-t}\cos(t), e^{-t}\sin(t)), \quad T > 0;$

d) $f : [0, 2\pi] \to \mathbb{R}^2, \quad f(t) = (t - \sin(t), 1 - \cos(t)).$

**Hint for d):** Use an addition theorem.

**Exercise 5.**

Find the length of the curve parameterised by $f$ in the given interval.

a) $f(t) = (t^2, t - \frac{1}{3}t^3),\ t \in [0, 1];$

b) $f(t) = (t\sin t + \cos t, \sin t - t\cos t),\ t \in [0, \pi];$

c) $f(t) = (\cos t, \sin t, t),\ t \in [0, 2\pi];$

d) $f(t) = (\arcsin t, t, \sqrt{1 - t^2}),\ t \in [-1, 1].$