Modular Forms for the Orthogonal Group $O(2,5)$

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Outline

History

- $O(2,1)$: Elliptic modular forms.
  It is well-known that the graded ring $\mathcal{A}(SL_2(\mathbb{Z}))$ of elliptic modular forms is a polynomial ring in the elliptic Eisenstein series g_2 and g_3 (of weight 4 and 6).
- $O(2,2)$: Hilbert modular forms.
- $O(2,3)$: Siegel modular forms of degree 2.
  J.-I. Igusa (1962): The graded ring $\mathcal{A}(Sp_2(\mathbb{Z}))$ is a polynomial ring in the Siegel Eisenstein series E_4, E_6, E_{10} and E_{12}.

History (continued)

- $O(2,4)$: Hermitian modular forms of degree 2.
  E. Freitag (1967): The graded ring for $\mathbb{Q}(\sqrt{-1})$,
  T. Dern (2001): The graded ring for $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-2})$ (with A. Krieg).
- $O(2,5)$: This is the case we will consider.
- $O(2,6)$: Quaternionic modular forms of degree 2.
  A. Krieg (2005)

Symmetric Matrices and Quadratic Forms

- S: a symmetric, positive definite, even $\ell \times \ell$ matrix
- $S_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
- $(x, y)_T = {}^t x T y \quad$ and $\quad q_T(x) = \frac{1}{2}(x, x)_T = \frac{1}{2}\, {}^t x T x = \frac{1}{2} T[x]$
  Abbreviations:
  - $(\cdot, \cdot) = (\cdot, \cdot)_S$, $q = q_S$,
  - $(\cdot, \cdot)_0 = (\cdot, \cdot)_{S_0}$, $q_0 = q_{S_0}$,
  - $(\cdot, \cdot)_1 = (\cdot, \cdot)_{S_1}$, $q_1 = q_{S_1}$.
- Mostly, $S = A_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ or $S = A_1^{(3)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Lattices in Quadratic Spaces

- $\Lambda = \mathbb{Z}^\ell$, $\Lambda_0 = \mathbb{Z}^{\ell+2}$, $\Lambda_1 = \mathbb{Z}^{\ell+4}$ (lattices in $(\Lambda \otimes \mathbb{R}, q)$, ...)
- Dual lattices:
  $\Lambda_T^\sharp = \{\mu \in \Lambda_T \otimes \mathbb{R};\ (\lambda, \mu)_T \in \mathbb{Z}$ for all $\lambda \in \Lambda_T\} = T^{-1}\Lambda_T$
- We have:
  - $\Lambda^\sharp = S^{-1}\mathbb{Z}^\ell$, $\Lambda_0^\sharp = \mathbb{Z} \times \Lambda^\sharp \times \mathbb{Z}$, $\Lambda_1^\sharp = \mathbb{Z} \times \Lambda_0^\sharp \times \mathbb{Z}$,
  - $\Lambda^\sharp/\Lambda \cong \Lambda_0^\sharp/\Lambda_0 \cong \Lambda_1^\sharp/\Lambda_1$,
  - $|\Lambda^\sharp/\Lambda| = \det S$.
- $\overline{q}_T : \Lambda_T^\sharp/\Lambda_T \to \mathbb{Q}/\mathbb{Z},\ \mu + \Lambda_T \mapsto q_T(\mu) + \mathbb{Z}$
- $S = A_3$: $\Lambda^\sharp/\Lambda$ is represented by 4 vectors of norm 0, $2 \times \frac{3}{8}$, $\frac{1}{2}$.
- $S = A_1^{(3)}$: $\Lambda^\sharp/\Lambda$ is represented by 8 vectors of norm 0, $3 \times \frac{1}{4}$, $3 \times \frac{1}{2}$, $\frac{3}{4}$.

Orthogonal Groups and the Half-Space

- $$\begin{aligned} \mathrm{O}(T;\mathbb{R}) &= \{M \in \mathrm{Mat}(\ell;\mathbb{R});\ T[M] := {}^tMTM = T\} \\ &= \{M \in \mathrm{Mat}(\ell;\mathbb{R});\ q_T(Mx) = q_T(x) \text{ for all } x \in \mathbb{R}^\ell\}. \end{aligned}$$
- $\mathrm{O}(\Lambda) = \{M \in \mathrm{O}(T;\mathbb{R});\ M\Lambda = \Lambda\}$
- $\mathcal{P}_S = \{v \in \mathbb{R}^{\ell+2};\ q_0(v) > 0,\ {}^tvS_0\,\mathrm{e} > 0\}$ where $\mathrm{e} = (1, 0, \dots, 0, 1)$
- Half space: $\mathcal{H}_S = \{w = u + iv \in \mathbb{C}^{\ell+2};\ v = \mathrm{Im}(w) \in \mathcal{P}_S\}$
- $\mathrm{O}(S_1;\mathbb{R})$ acts on $\mathcal{H}_S \cup (-\mathcal{H}_S)$:

$$\begin{aligned} M\langle w\rangle &= j(M,w)^{-1} \cdot (-q_0(w)b + Aw + c) \\ j(M,w) &= -\gamma q_0(w) + {}^tdw + \delta \end{aligned} \qquad M = \begin{pmatrix} \alpha & {}^ta & \beta \\ b & A & c \\ \gamma & {}^td & \delta \end{pmatrix}$$

- $\mathrm{O}^+(S_1;\mathbb{R}) = \{M \in \mathrm{O}(S_1;\mathbb{R});\ M\langle\mathcal{H}_S\rangle = \mathcal{H}_S\}$
- $\Gamma_S = \mathrm{O}(\Lambda_1) \cap \mathrm{O}^+(S_1;\mathbb{R})$

Properties of the Orthogonal Modular Group

- Γ_S is (in our case) generated by J, T_λ, $\lambda \in \Lambda_0$, and R_A, $A \in \mathrm{O}(\Lambda)$, where

$$J\langle w\rangle = -q_0(w)^{-1} \cdot (\tau_2, -z, \tau_1) \qquad \text{(inversion)},$$
$$T_\lambda\langle w\rangle = w + \lambda \qquad \text{(translation)},$$
$$R_A\langle w\rangle = (\tau_1, Az, \tau_2) \qquad \text{(rotation)}.$$

- Γ_S acts on $\Lambda_1^\sharp/\Lambda_1$ by multiplication. It permutes elements of $\Lambda_1^\sharp/\Lambda_1$ of the same norm (modulo $\mathbb{Z}$).

- Abelian characters of Γ_{A_3} and $\Gamma_{A_1^{(3)}}$:

$$\Gamma_{A_3}^{\mathrm{ab}} = \langle \nu_\pi,\ \det\rangle \quad \text{and} \quad \Gamma_{A_1^{(3)}}^{\mathrm{ab}} = \langle \nu_2,\ \nu_\pi,\ \det\rangle\,,$$

where ν_π is the sign of the permutation of the elements of $\Lambda_1^\sharp/\Lambda_1$ of same norm and ν_2 is an extension of the Siegel character.

What is an Orthogonal Modular Form?

Definition

An **(orthogonal) modular form** of weight $k \in \mathbb{Z}$ with respect to a subgroup Γ of Γ_S of finite index and an abelian character $\nu : \Gamma \to \mathbb{C}^\times$ of finite order is a holomorphic function $f : \mathcal{H}_S \to \mathbb{C}$ satifying

$$f(M\langle w \rangle) = \nu(M)\, j(M, w)^k\, f(w) \qquad \text{for all } w \in \mathcal{H}_S \text{ and } M \in \Gamma.$$

We denote the vector space of all such functions by $[\Gamma, k, \nu]$.

First results:

- If $-I \in \Gamma$ and $\nu(-I) \neq (-1)^k$ then $[\Gamma, k, \nu] = \{0\}$.
- If $k < 0$ then $[\Gamma, k, \nu] = \{0\}$.

Fourier Expansion

All modular forms $f \in [\Gamma, k, \nu]$ have a Fourier expansion of the form

$$f(w) = \sum_{\mu \in \Lambda_0^\sharp \cap \overline{\mathcal{P}_S}} \alpha_f(\mu)\, e^{2\pi i (\mu, w)_0 / h},$$

where h depends on $\Gamma \le \Gamma_S$ and the character ν.

If $\alpha_f(\mu) = 0$ unless $\mu \in \mathcal{P}_S$ then f is a **cusp form**, $f \in [\Gamma, k, \nu]_0$.

What is Our Goal?

Products of modular forms are again modular forms. Thus

$$\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma_S, k, 1] \quad \text{and} \quad \mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma'_S, k, 1] = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{\nu\in\Gamma_S^{\mathrm{ab}}} [\Gamma_S, k, \nu]$$

form graded rings.

Goal: Determine generators and algebraic structure of $\mathcal{A}(\Gamma_S)$ and $\mathcal{A}(\Gamma'_S)$.

Due to $-I \in \Gamma_{A_3}$ and $\nu_\pi(-I) = \det(-I) = -1$ we get a first result:

- If k is even then $[\Gamma_{A_3}, k, \nu_\pi] = [\Gamma_{A_3}, k, \det] = \{0\}$.
- If k is odd then $[\Gamma_{A_3}, k, 1] = [\Gamma_{A_3}, k, \nu_\pi \det] = \{0\}$.

We get similar results for $S = A_1^{(3)}$.

Vector-valued Modular Forms

Definition

A holomorphic function $f : \mathcal{H} \to \mathbb{C}[\Lambda^\sharp/\Lambda]$ is a **vector-valued modular form** of weight $k \in \frac{1}{2}\mathbb{Z}$ with respect to ρ_S if

$$f(M\tau) = \varphi(\tau)^{2k}\, \rho_S(M,\varphi)\, f(\tau), \qquad \text{for all } (M,\varphi) \in \mathrm{Mp}_2(\mathbb{Z})$$

and if f has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\substack{n \in q_S(\mu)+\mathbb{Z} \\ n \geq n_0}} c_\mu(n) q^n\, e_\mu.$$

- $n_0 \geq 0$: Holomorphic modular forms, $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_S]$,
- $n_0 < 0$: Nearly holomorphic modular forms, $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_S]_\infty$.

What to Know About Vector-valued Modular Forms

- Nearly holomorphic modular forms are uniquely determined by their principal part

$$\sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\substack{n \in q_S(\mu)+\mathbb{Z} \\ n \le 0}} c_\mu(n) q^n \, e_\mu .$$

- Skoruppa: Formula for dimension of $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_S]$ for $k \ge 2$.

- Examples:
  - Eisenstein series ($k \in \frac{1}{2}\mathbb{Z}$, $k > 2$)

$$E_k(\tau) = \frac{1}{2} \sum_{(M,\varphi) \in \langle T \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} \varphi(\tau)^{-2k} \, \rho_S(M,\varphi)^{-1} e_0 .$$

    Bruinier, Kuss: Formula for Fourier coefficients of E_k.
  - Theta series.

Borcherds Theorem

Theorem

Let $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\ell/2, \rho_S^\sharp]_\infty$ with Fourier coefficients $c_\mu(n)$, such that $c_0(0) \in 2\mathbb{Z}$ and $c_\mu(n) \in \mathbb{Z}$ whenever $n < 0$. Then there exists a Borcherds product $\psi_k : \mathcal{H}_S \to \mathbb{C}$ with the following properties:

- *ψ_k is a meromorphic modular form of weight $k = c_0(0)/2$ with respect to Γ_S and some abelian character χ.*
- *The zeros and poles of ψ_k are explicitely known and depend only on the principal part of f.*
- *ψ_k is given by the normally convergent product expansion*

$$\psi_k(w) = e^{2\pi i(\varrho_f, w)_0} \prod_{\substack{\lambda_0 \in \Lambda_0^\sharp \\ \lambda_0 > 0}} \left(1 - e^{2\pi i(\lambda_0, w)_0}\right)^{c_{(0,\lambda_0,0)}(q_0(\lambda_0))}.$$

Borcherds' Obstruction Condition

A necessary and sufficient condition for the existence of nearly holomorphic modular forms

Theorem

There exists a nearly holomorphic modular form $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\ell/2, \rho_S^\sharp]_\infty$ *with prescribed principal part*

$$\sum_{\mu \in \Lambda_1^\sharp/\Lambda_1} \sum_{\substack{n \in -q_S(\mu)+\mathbb{Z} \\ n \leq 0}} c_\mu(n) q^n \, e_\mu,$$

if and only if

$$\sum_{\mu \in \Lambda_1^\sharp/\Lambda_1} \sum_{\substack{n \in -q_S(\mu)+\mathbb{Z} \\ n \leq 0}} c_\mu(n) \, a_\mu(-n) = 0$$

for all holomorphic modular forms $g \in [\mathrm{Mp}_2(\mathbb{Z}), 2+\ell/2, \rho_S]$ *with Fourier expansion* $g(\tau) = \sum_{\mu \in \Lambda_1^\sharp/\Lambda_1} \sum_{n \in q_S(\mu)+\mathbb{Z},\, n \geq 0} a_\mu(n) q^n \, e_\mu$.

Input for Borcherds Theorem in the Case $S = A_3$

The obstruction space $[\mathrm{Mp}_2(\mathbb{Z}), 7/2, \rho_{A_3}]$ is of dimension 1 and spanned by

$$E_{7/2}(\tau) = 1\ e_0 - 8\ q^{3/8} \left(e_{\frac{1}{4}} + e_{-\frac{1}{4}}\right) - 18\ q^{1/2}\ e_{\frac{1}{2}} - 108\ q\ e_0 + O(q^{11/8}).$$

Thus the condition for the principal part of $f \in [\mathrm{Mp}_2(\mathbb{Z}), -3/2, \rho^\sharp_{A_3}]_\infty$ is

$$c_0(0) = 8\ \left(c_{\frac{1}{4}}(-\tfrac{3}{8}) + c_{-\frac{1}{4}}(-\tfrac{3}{8})\right) + 18\ c_{\frac{1}{2}}(-\tfrac{1}{2}) + 108\ c_0(-1) + \cdots .$$

Possible principal parts are given by

$$\begin{aligned} &q^{-3/8}\ \left(e_{\frac{1}{4}} + e_{-\frac{1}{4}}\right) + 16\ e_0,\\ &q^{-1/2}\ e_{\frac{1}{2}} + 18\ e_0,\\ &q^{-1}\ e_0 + 108\ e_0. \end{aligned}$$

Borcherds Products for Γ_{A_3}

Theorem

There exist Borcherds products

$$\psi_8 \in [\Gamma_{A_3}, 8, 1]_0, \quad \psi_9 \in [\Gamma_{A_3}, 9, \nu_\pi]_0 \quad \textit{and} \quad \psi_{54} \in [\Gamma_{A_3}, 54, \nu_\pi \det]_0.$$

The zeros of the products are all of first order and are given by

$$\bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{A_2} \rangle, \quad \bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{A_1^{(2)}} \rangle \quad \textit{and} \quad \bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{S_2} \rangle,$$

respectively, where

$$\begin{aligned}
\mathcal{H}_{A_2} &= \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3};\ z_3 = 0\} \cong H(2; \mathbb{Q}(\sqrt{-3})), \\
\mathcal{H}_{A_1^{(2)}} &= \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3};\ z_2 = 0\} \cong H(2; \mathbb{Q}(\sqrt{-1})), \\
\mathcal{H}_{S_2} &= \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3};\ z_1 + z_3 = 0\} \cong H(2; \mathbb{Q}(\sqrt{-2})).
\end{aligned}$$

Tools for Proving the Main Result

Corollary

Let $k \in \mathbb{Z}$ and $n \in \{0, 1\}$.

1. *If k is odd and $f \in [\Gamma_{A_3}, k, \nu_\pi^{n+1} \det^n]$, then f vanishes on $\mathcal{H}_{A_1^{(2)}}$ and $f/\psi_9 \in [\Gamma_{A_3}, k-9, \nu_\pi^n \det^n]$.*
2. *If $f \in [\Gamma_{A_3}, k, \nu_\pi^{k+1} \det]$, then f vanishes on $\mathcal{H}_{S_2}$ and $f/\psi_{54} \in [\Gamma_{A_3}, k-54, \nu_\pi^k]$.*

Theorem

(Dern 2001) The graded ring $\mathcal{A}(\Gamma_{A_2}) = \bigoplus_{k \in \mathbb{Z}} [\Gamma_{A_2}, 2k, 1]$ is a polynomial ring in

$$E_4|_{\mathcal{H}_{A_2}}, \quad E_6|_{\mathcal{H}_{A_2}}, \quad E_{10}|_{\mathcal{H}_{A_2}}, \quad E_{12}|_{\mathcal{H}_{A_2}} \quad \textit{and} \quad \psi_9^2|_{\mathcal{H}_{A_2}}.$$

The Graded Ring of Modular Forms for Γ_{A_3}

Theorem

a) The graded ring $\mathcal{A}(\Gamma_{A_3}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ *is a polynomial ring in*

$$E_4, \quad E_6, \quad \psi_8, \quad E_{10}, \quad E_{12} \quad \text{and} \quad \psi_9^2.$$

b) The graded ring $\mathcal{A}(\Gamma'_{A_3}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_{A_3}, k, 1]$ *is generated by*

$$E_4, \quad E_6, \quad \psi_8, \quad \psi_9, \quad E_{10}, \quad E_{12} \quad \text{and} \quad \psi_{54}$$

and is isomorphic to

$$\mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6))$$

where $p \in \mathbb{C}[X_1, \ldots, X_6]$ *such that* $\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})$.

Fields of Orthogonal Modular Functions for Γ_{A_3}

Corollary

1. *The field $\mathcal{K}(\Gamma_{A_3})$ of orthogonal modular functions with respect to Γ_{A_3} and the trivial character is a rational function field in the generators*

$$\frac{E_6^2}{E_4^3}, \quad \frac{\psi_8}{E_4^2}, \quad \frac{E_{10}}{E_4\,E_6}, \quad \frac{E_{12}}{E_4^3} \quad \text{and} \quad \frac{\psi_9^2}{E_6^3}.$$

2. *The field $\mathcal{K}(\Gamma'_{A_3})$ of all orthogonal modular functions with respect to Γ'_{A_3} is an extension of degree 2 over $\mathcal{K}(\Gamma_{A_3})$ generated by ψ_{54}/ψ_9^6.*

Borcherds Products for $\Gamma_{A_1^{(3)}}$

Theorem

There exist Borcherds products

$$\psi_3 \in [\Gamma_{A_1^{(3)}}, 3, \nu_2\nu_\pi \det]_0, \quad \psi_{18} \in [\Gamma_{A_1^{(3)}}, 18, \nu_\pi]_0, \quad \psi_{30} \in [\Gamma_{A_1^{(3)}}, 30, \nu_2]_0.$$

The zeros of the products are all of first order and are given by

$$\bigcup_{M \in \Gamma_{A_1^{(3)}}} M\langle \mathcal{H}_{A_1^{(2)}} \rangle, \quad \bigcup_{M \in \Gamma_{A_1^{(3)}}} M\langle \mathcal{H}_{S_2} \rangle \quad \text{and} \quad \bigcup_{M \in \Gamma_{A_1^{(3)}}} M\langle \mathcal{H}_8 \rangle,$$

respectively, where

$$\mathcal{H}_{A_1^{(2)}} = \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_1^{(3)}};\ z_3 = 0\} \cong H(2; \mathbb{Q}(\sqrt{-1})),$$
$$\mathcal{H}_{S_2} = \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_1^{(3)}};\ z_2 = z_3\} \cong H(2; \mathbb{Q}(\sqrt{-2})),$$
$$\mathcal{H}_8 = \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_1^{(3)}};\ z_3 = \tfrac{1}{2}\}.$$

Tools for Proving the Main Result

Corollary

Let $S = A_1^{(3)}$, $k \in \mathbb{Z}$ and $n \in \{0, 1\}$.

1. *If k is odd and $f \in [\Gamma'_S, k, 1]$, then f vanishes on $\mathcal{H}_{A_1^{(2)}}$ and $f/\psi_3 \in [\Gamma'_S, k-3, 1]$.*
2. *If $f \in [\Gamma_S, k, \nu_2^n \nu_\pi^{k+1} \det^k]$, then f vanishes on $\mathcal{H}_{S_2}$ and $f/\psi_{18} \in [\Gamma_S, k-18, \nu_2^n \nu_\pi^k \det^k]$.*
3. *If $f \in [\Gamma_S, k, \nu_2^{k+1} \nu_\pi^n \det^k]$, then f vanishes on $\mathcal{H}_8$ and $f/\psi_{30} \in [\Gamma_S, k-30, \nu_2^k \nu_\pi^n \det^k]$.*

Theorem

The graded ring $\mathcal{A}(\Gamma_{A_1^{(2)}}) \cong \mathcal{A}(\Gamma(2, \mathbb{Q}(\sqrt{-1})))$ is a polynomial ring in the restrictions of functions h_4, h_6, h_8, h_{10}, $h_{12} \in [\Gamma_{A_1^{(3)}}, k, 1]$ which are invariant polynomials in the restrictions of five quaternionic theta series.

The Graded Ring of Modular Forms for $\Gamma_{A_1^{(3)}}$

Theorem

a) The graded ring $\mathcal{A}(\Gamma_{A_1^{(3)}}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_{A_1^{(3)}}, 2k, 1]$ *is a polynomial ring in*

$$E_4, \quad E_6, \quad \psi_3^2, \quad h_8, \quad E_{10} \quad \text{and} \quad E_{12}.$$

b) The graded ring $\mathcal{A}(\Gamma'_{A_1^{(3)}}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_{A_1^{(3)}}, k, 1]$ *is generated by*

$$\psi_3, \quad E_4, \quad E_6, \quad h_8, \quad E_{10}, \quad E_{12}, \quad \psi_{18} \quad \text{and} \quad \psi_{30}$$

and is isomorphic to

$$\mathbb{C}[X_1, \ldots, X_8]/(X_7^2 - p(X_1, \ldots, X_6),\ X_8^2 - q(X_1, \ldots, X_6))$$

where $p, q \in \mathbb{C}[X_1, \ldots, X_6]$ *such that* $\psi_{18}^2 = p(\psi_3, E_4, E_6, h_8, E_{10}, E_{12})$ *and* $\psi_{30}^2 = q(\psi_3, E_4, E_6, h_8, E_{10}, E_{12})$.

Fields of Orthogonal Modular Functions for $\Gamma_{A_1^{(3)}}$

Corollary

1. *The field $\mathcal{K}(\Gamma_{A_1^{(3)}})$ of orthogonal modular functions with respect to $\Gamma_{A_1^{(3)}}$ and the trivial character is a rational function field in five generators.*
2. *The field $\mathcal{K}(\Gamma'_{A_1^{(3)}})$ of all orthogonal modular functions with respect to $\Gamma'_{A_1^{(3)}}$ is an extension of degree 4 over $\mathcal{K}(\Gamma_{A_1^{(3)}})$.*