Modular Forms for the Orthogonal Group O(2, 5)

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Outline

History

- O(2, 1): Elliptic modular forms.
  It is well-known that the graded ring $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z}))$ of elliptic modular forms is a polynomial ring in the elliptic Eisenstein series g_2 and g_3 (of weight 4 and 6).
- O(2, 2): Hilbert modular forms.
  Cf. S. Mayer's talk.
- O(2, 3): Siegel modular forms of degree 2.
  J.-I. Igusa (1962): The graded ring $\mathcal{A}(\mathrm{Sp}_2(\mathbb{Z}))$ is a polynomial ring in the Siegel Eisenstein series E_4, E_6, E_{10} and E_{12}.

History (continued)

- $O(2,4)$: Hermitian modular forms of degree 2.
  E. Freitag (1967): The graded ring for $\mathbb{Q}(\sqrt{-1})$,
  T. Dern (2001): The graded ring for $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-2})$ (with A. Krieg).
- $O(2,5)$: This is the case we will consider.
- $O(2,6)$: Quaternionic modular forms of degree 2.
  A. Krieg (2005)

Symmetric Matrices and Quadratic Forms

- S: a symmetric, positive definite, even $\ell \times \ell$ matrix
- $S_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $S_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, signature of S_1 is $(2, \ell+2)$
- $(x,y)_T = {}^t x T y$ and $q_T(x) = \frac{1}{2}(x,x)_T = \frac{1}{2}\,{}^t x T x = \frac{1}{2} T[x]$
  Abbreviations:
  - $(\cdot,\cdot) = (\cdot,\cdot)_S$, $q = q_S$,
  - $(\cdot,\cdot)_0 = (\cdot,\cdot)_{S_0}$, $q_0 = q_{S_0}$,
  - $(\cdot,\cdot)_1 = (\cdot,\cdot)_{S_1}$, $q_1 = q_{S_1}$.
- Mostly $S = A_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$,
  $q_S(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_2^2 + x_2 x_3 + x_3^2$.

Lattices in Quadratic Spaces

- $\Lambda = \mathbb{Z}^\ell$, $\Lambda_0 = \mathbb{Z}^{\ell+2}$, $\Lambda_1 = \mathbb{Z}^{\ell+4}$ (lattices in quadratic spaces $(\Lambda \otimes \mathbb{R}, (\cdot,\cdot))$, ...)
- Dual lattices: $\Lambda_T^\sharp = \{\mu \in \Lambda \otimes \mathbb{R};\ (\lambda,\mu)_T \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\} = T^{-1}\Lambda$
- We have:
  - $\Lambda^\sharp = S^{-1}\mathbb{Z}^\ell$, $\Lambda_0^\sharp = \mathbb{Z} \times \Lambda^\sharp \times \mathbb{Z}$, $\Lambda_1^\sharp = \mathbb{Z} \times \Lambda_0^\sharp \times \mathbb{Z}$,
  - $\Lambda^\sharp/\Lambda \cong \Lambda_0^\sharp/\Lambda_0 \cong \Lambda_1^\sharp/\Lambda_1$,
  - $|\Lambda^\sharp/\Lambda| = \det S$.
- $\overline{q}_T : \Lambda_T^\sharp/\Lambda_T \to \mathbb{Q}/\mathbb{Z},\ \mu + \Lambda_T \mapsto q_T(\mu) + \mathbb{Z}$
- $S = A_3$: $\Lambda^\sharp/\Lambda = A_3^{-1}\mathbb{Z}^3/\mathbb{Z}^3$ is represented by $(0,0,0)$, $(\frac{1}{4},\frac{1}{2},-\frac{1}{4})$, $(\frac{1}{2},0,\frac{1}{2})$, $(-\frac{1}{4},\frac{1}{2},\frac{1}{4})$ with norm 0, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{8}$, respectively.

Orthogonal Groups and the Half-space

- $\mathrm{O}(T;\mathbb{R}) = \{M \in \mathrm{Mat}(\ell;\mathbb{R});\ T[M] := {}^tMTM = T\}$
  $= \{M \in \mathrm{Mat}(\ell;\mathbb{R});\ q_T(Mx) = q_T(x) \text{ for all } x \in \mathbb{R}^\ell\}.$
- $\mathrm{O}(\Lambda) = \{M \in \mathrm{O}(T;\mathbb{R});\ M\Lambda = \Lambda\}$
- $\mathcal{P}_S = \{v \in \mathbb{R}^{\ell+2};\ q_0(v) > 0,\ {}^tvS_0\,\mathrm{e} > 0\}$ where $\mathrm{e} = (1, 0, \ldots, 0, 1)$
- Half space: $\mathcal{H}_S = \{w = u + iv \in \mathbb{C}^{\ell+2};\ v = \mathrm{Im}(w) \in \mathcal{P}_S\}$
- $\mathrm{O}(S_1;\mathbb{R})$ acts on $\mathcal{H}_S \cup (-\mathcal{H}_S)$:

$$\begin{aligned} M\langle w\rangle &= j(M,w)^{-1}\cdot(-q_0(w)b + Aw + c) \\ j(M,w) &= -\gamma q_0(w) + {}^tdw + \delta \end{aligned} \qquad M = \begin{pmatrix} \alpha & {}^ta & \beta \\ b & A & c \\ \gamma & {}^td & \delta \end{pmatrix}$$

- $\mathrm{O}^+(S_1;\mathbb{R}) = \{M \in \mathrm{O}(S_1;\mathbb{R});\ M\langle\mathcal{H}_S\rangle = \mathcal{H}_S\}$
- $\Gamma_S = \mathrm{O}(\Lambda_1) \cap \mathrm{O}^+(S_1;\mathbb{R})$

Properties of the Orthogonal Modular Group

- Γ_S is (in our case) generated by J, T_λ, $\lambda \in \Lambda_0$, and R_A, $A \in \mathrm{O}(\Lambda)$, where

$$\begin{aligned} J\langle w\rangle &= -q_0(w)^{-1}\cdot(\tau_2, -z, \tau_1) && \text{(inversion)},\\ T_\lambda\langle w\rangle &= w+\lambda && \text{(translation)},\\ R_A\langle w\rangle &= (\tau_1, Az, \tau_2) && \text{(rotation)}.\end{aligned}$$

- Γ_S acts on $\Lambda_1^\sharp/\Lambda_1$ by multiplication.
  It permutes elements of $\Lambda_1^\sharp/\Lambda_1$ with the same norm (modulo $\mathbb{Z}$).
  The signs of those permutations are abelian characters of Γ_S.

- $S = A_3$: Abelian characters of Γ_{A_3}:

$$\Gamma_{A_3}^{\mathrm{ab}} = \langle \nu_\pi,\ \det\rangle\,,$$

  where ν_π is the sign of the permutation of the two elements of $\Lambda^\sharp/\Lambda$ of norm $\frac{3}{8}$.

What is an Orthogonal Modular Form?

Definition

An **(orthogonal) modular form** of weight $k \in \mathbb{Z}$ with respect to a subgroup Γ of Γ_S of finite index and an abelian character $\nu : \Gamma \to \mathbb{C}^\times$ of finite order is a holomorphic function $f : \mathcal{H}_S \to \mathbb{C}$ satifying

$$f(M\langle w\rangle) = \nu(M)\, j(M,w)^k\, f(w) \qquad \text{for all } w \in \mathcal{H}_S \text{ and } M \in \Gamma.$$

We denote the vector space of all such functions by $[\Gamma, k, \nu]$.

- If $-I \in \Gamma$ and $\nu(-I) \neq (-1)^k$ then $[\Gamma, k, \nu] = \{0\}$.
- If $k < 0$ then $[\Gamma, k, \nu] = \{0\}$.

What is Our Goal?

Products of modular forms are again modular forms. Thus

$$\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma_S, k, 1] \quad \text{and} \quad \mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma'_S, k, 1] = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{\nu\in\Gamma_S^{\mathrm{ab}}} [\Gamma_S, k, \nu]$$

form graded rings.

Goal: Determine generators and algebraic structure of $\mathcal{A}(\Gamma_{A_3})$ and $\mathcal{A}(\Gamma'_{A_3})$.

Due to $-I \in \Gamma_{A_3}$ and $\nu_\pi(-I) = \det(-I) = -1$ we get a first result:

- If k is even then $[\Gamma_{A_3}, k, \nu_\pi] = [\Gamma_{A_3}, k, \det] = \{0\}$.
- If k is odd then $[\Gamma_{A_3}, k, 1] = [\Gamma_{A_3}, k, \nu_\pi \det] = \{0\}$.

The Metaplectic Group

The metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is given by

$$\left\{(M,\varphi)\,;\ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),\ \varphi : \mathcal{H} \to \mathbb{C} \text{ holom.},\ \varphi^2(\tau) = c\tau + d\right\}.$$

It operates on the upper half-plane $\mathcal{H}$ via

$$(M,\varphi)\tau = M\tau = \frac{a\tau + b}{c\tau + d}$$

and is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{and} \quad J = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

The Weil Representation

Let

- $S \in \mathrm{Sym}(\ell; \mathbb{R})$ be a symmetric, even matrix of signature (b^+, b^-),
- $\Lambda = \mathbb{Z}^\ell$,
- $(\cdot, \cdot) = (\cdot, \cdot)_S$,
- $(e_\mu)_{\mu \in \Lambda^\sharp/\Lambda}$ be the standard basis of the group ring $\mathbb{C}[\Lambda^\sharp/\Lambda]$.

The Weil representation ρ_S of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[\Lambda^\sharp/\Lambda]$ is defined by

$$\rho_S(T)\, e_\mu = e^{\pi i (\mu,\mu)}\, e_\mu,$$

$$\rho_S(J)\, e_\mu = \frac{\sqrt{i}^{\,b^- - b^+}}{\sqrt{|\det S|}} \sum_{\nu \in \Lambda^\sharp/\Lambda} e^{-2\pi i (\mu,\nu)}\, e_\nu.$$

Vector-valued Modular Forms

Definition

A holomorphic function $f : \mathcal{H} \to \mathbb{C}[\Lambda^\sharp/\Lambda]$ is a **vector-valued modular form** of weight $k \in \frac{1}{2}\mathbb{Z}$ with respect to ρ_S if

$$f(M\tau) = \varphi(\tau)^{2k}\, \rho_S(M,\varphi)\, f(\tau), \qquad \text{for all } (M,\varphi) \in \mathrm{Mp}_2(\mathbb{Z})$$

and if f has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\substack{n \in q_S(\mu)+\mathbb{Z} \\ n \geq n_0}} c_\mu(n) q^n\, e_\mu .$$

- $n_0 \geq 0$: Holomorphic modular forms, $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_S]$,
- $n_0 < 0$: Nearly holomorphic modular forms, $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_S]_\infty$.

What to Know About Vector-valued Modular Forms

- Nearly holomorphic modular forms are uniquely determined by their principal part

$$\sum_{\mu\in\Lambda^\sharp/\Lambda}\ \sum_{\substack{n\in q_S(\mu)+\mathbb{Z}\\ n\leq 0}} c_\mu(n)q^n\, e_\mu.$$

- Skoruppa: Formula for dimension of $[\mathrm{Mp}_2(\mathbb{Z}), k, \rho_S]$ for $k \geq 2$.

- Examples:
  - Eisenstein series ($k \in \frac{1}{2}\mathbb{Z}$, $k > 2$)

$$E_k(\tau) = \frac{1}{2}\sum_{(M,\varphi)\in\langle T\rangle\backslash \mathrm{Mp}_2(\mathbb{Z})} \varphi(\tau)^{-2k}\ \rho_S(M,\varphi)^{-1}e_0.$$

    Bruinier, Kuss: Formula for Fourier coefficients of E_k.
  - Theta series.

Borcherds Theorem

Theorem

Let $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\ell/2, \rho_S^\sharp]_\infty$ *with Fourier coefficients* $c_\mu(n)$*, such that* $c_0(0) \in 2\mathbb{Z}$ *and* $c_\mu(n) \in \mathbb{Z}$ *whenever* $n < 0$*. Then there exists a Borcherds product* $\psi_k : \mathcal{H}_S \to \mathbb{C}$ *with the following properties:*

- ψ_k *is a meromorphic modular form of weight* $k = c_0(0)/2$ *with respect to* Γ_S *and some abelian character* χ*.*
- *The zeros and poles of* ψ_k *are explicitely known and depend only on the principal part of* f*.*
- ψ_k *is given by the normally convergent product expansion*

$$\psi_k(w) = e^{2\pi i(\varrho_f, w)} \prod_{\substack{\lambda_0 \in \Lambda_0^\sharp \\ \lambda_0 > 0}} \left(1 - e^{2\pi i(\lambda_0, w)}\right)^{c_{(0,\lambda_0,0)}(q_0(\lambda_0))}.$$

Borcherds' Obstruction Condition

A necessary and sufficient condition for the existence of nearly holomorphic modular forms

Theorem

There exists a nearly holomorphic modular form $f \in [\mathrm{Mp}_2(\mathbb{Z}), -\ell/2, \rho_S^\sharp]_\infty$ *with prescribed principal part*

$$\sum_{\mu \in \Lambda_1^\sharp/\Lambda_1} \sum_{\substack{n \in -q_S(\mu)+\mathbb{Z} \\ n \leq 0}} c_\mu(n) q^n \, e_\mu,$$

if and only if

$$\sum_{\mu \in \Lambda_1^\sharp/\Lambda_1} \sum_{\substack{n \in -q_S(\mu)+\mathbb{Z} \\ n \leq 0}} c_\mu(n) \, a_\mu(-n) = 0$$

for all holomorphic modular forms $g \in [\mathrm{Mp}_2(\mathbb{Z}), 2+\ell/2, \rho_S]$ *with Fourier expansion* $g(\tau) = \sum_{\mu \in \Lambda_1^\sharp/\Lambda_1} \sum_{n \in q_S(\mu)+\mathbb{Z},\ n \geq 0} a_\mu(n) q^n \, e_\mu$.

Input for Borcherds Theorem in the Case $S = A_3$

The obstruction space $[\mathrm{Mp}_2(\mathbb{Z}), 7/2, \rho_{A_3}]$ is of dimension 1 and spanned by

$$E_{7/2}(\tau) = 1\ e_0 - 8\ q^{3/8}\left(e_{\frac{1}{4}} + e_{-\frac{1}{4}}\right) - 18\ q^{1/2}\ e_{\frac{1}{2}} - 108\ q\ e_0 + O(q^{11/8}).$$

Thus the condition for the principal part of $f \in [\mathrm{Mp}_2(\mathbb{Z}), -3/2, \rho^{\sharp}_{A_3}]_\infty$ is

$$c_0(0) = 8\ \left(c_{\frac{1}{4}}(-\tfrac{3}{8}) + c_{-\frac{1}{4}}(-\tfrac{3}{8})\right) + 18\ c_{\frac{1}{2}}(-\tfrac{1}{2}) + 108\ c_0(-1) + \cdots .$$

Possible principal parts are given by

$$\begin{aligned} & & q^{-3/8}\ \left(e_{\frac{1}{4}} + e_{-\frac{1}{4}}\right) &+ 16\ e_0, \\ & q^{-1/2}\ e_{\frac{1}{2}} & &+ 18\ e_0, \\ q^{-1}\ e_0 & & &+ 108\ e_0. \end{aligned}$$

Borcherds Products for Γ_{A_3}

Theorem

There exist Borcherds products

$$\psi_8 \in [\Gamma_{A_3}, 8, 1], \quad \psi_9 \in [\Gamma_{A_3}, 9, \nu_\pi] \quad \textit{and} \quad \psi_{54} \in [\Gamma_{A_3}, 54, \nu_\pi \det].$$

The zeros of the products are all of first order and are given by

$$\bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{A_2} \rangle, \quad \bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{A_1^2} \rangle \quad \textit{and} \quad \bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{S_2} \rangle,$$

respectively, where

$$\begin{aligned}
\mathcal{H}_{A_2} &= \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3};\ z_3 = 0\},\\
\mathcal{H}_{A_1^2} &= \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3};\ z_2 = 0\},\\
\mathcal{H}_{S_2} &= \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3};\ z_1 + z_3 = 0\}.
\end{aligned}$$

Tools for Proving the Main Result

Corollary

Let $k \in \mathbb{Z}$ and $n \in \{0, 1\}$.

1. *If k is odd and $f \in [\Gamma_{A_3}, k, \nu_\pi^{n+1} \det^n]$, then f vanishes on $\mathcal{H}_{A_1^2}$ and $f/\psi_9 \in [\Gamma_{A_3}, k-9, \nu_\pi^n \det^n]$.*
2. *If $f \in [\Gamma_{A_3}, k, \nu_\pi^{k+1} \det]$, then f vanishes on $\mathcal{H}_{S_2}$ and $f/\psi_{54} \in [\Gamma_{A_3}, k-54, \nu_\pi^k]$.*

Theorem

(Dern 2001) The graded ring $\mathcal{A}(\Gamma_{A_2}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_{A_2}, 2k, 1]$ is a polynomial ring in

$$E_4|_{\mathcal{H}_{A_2}}, \quad E_6|_{\mathcal{H}_{A_2}}, \quad E_{10}|_{\mathcal{H}_{A_2}}, \quad E_{12}|_{\mathcal{H}_{A_2}} \quad \text{and} \quad \psi_9^2|_{\mathcal{H}_{A_2}}.$$

The Graded Ring of Modular Forms for Γ_{A_3}

Theorem

The graded ring $\mathcal{A}(\Gamma_{A_3}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ *is a polynomial ring in*

$$E_4, \quad E_6, \quad \psi_8, \quad E_{10}, \quad E_{12} \quad \text{and} \quad \psi_9^2.$$

Proof.

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern $f|_{\mathcal{H}_{A_2}}$ is equal to a polynomial p in $E_4|_{\mathcal{H}_{A_2}}$, $E_6|_{\mathcal{H}_{A_2}}$, $E_{10}|_{\mathcal{H}_{A_2}}$, $E_{12}|_{\mathcal{H}_{A_2}}$, $\psi_9^2|_{\mathcal{H}_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$ vanishes on $\mathcal{H}_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)/\psi_8 \in [\Gamma_{A_3}, 2k-8, 1]$.
- The assertion follows by induction. □

The Graded Ring of Modular Forms for Γ'_{A_3}

Theorem

The graded ring $\mathcal{A}(\Gamma'_{A_3}) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_{A_3}, k, 1]$ *is generated by*

$$E_4, \quad E_6, \quad \psi_8, \quad \psi_9, \quad E_{10}, \quad E_{12} \quad \text{and} \quad \psi_{54}$$

and is isomorphic to

$$\mathbb{C}[X_1, \dots, X_7]/(X_7^2 - p(X_1, \dots, X_6))$$

where $p \in \mathbb{C}[X_1, \dots, X_6]$ *is the uniquely determined polynomial with*

$$\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}).$$

Proof of the Second Main Result

Proof.

- Let $f \in [\Gamma'_{A_3}, k, 1]$.
- If k is odd, then f vanishes on $\mathcal{H}_{A_1^2}$ and we have $f/\psi_9 \in [\Gamma'_{A_3}, k-9, 1]$. So we can assume that k is even.
- We know that $[\Gamma'_{A_3}, 2k, 1] = [\Gamma_{A_3}, 2k, 1] \oplus [\Gamma_{A_3}, 2k, \nu_\pi \det]$. Thus $f = f_1 + f_{\nu_\pi \det}$ with $f_\nu \in [\Gamma_{A_3}, 2k, \nu]$.
- $f_{\nu_\pi \det}$ vanishes on $\mathcal{H}_{S_2}$ and we have $f_{\nu_\pi \det}/\psi_{54} \in [\Gamma_{A_3}, 2k-54, 1]$.
- Now f_1 and $f_{\nu_\pi \det}/\psi_{54}$ are polynomials in E_4, E_6, ψ_8, E_{10}, E_{12}, ψ_9^2. This completes the proof of the first result.

Proof of the Second Main Result (cont'd)

Proof.

- We have $\psi_{54}^2 \in [\Gamma_{A_3}, 108, 1]$. Thus $\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})$.
- We want to show that $\mathcal{A}(\Gamma'_S) \cong \mathbb{C}[X_1, \dots, X_7]/(X_7^2 - p(X_1, \dots, X_6))$. So let $Q \in \mathbb{C}[X_1, \dots, X_7]$ such that $Q(E_4, \dots, E_{12}, \psi_{54}) = 0$.
- There exist $Q_0, Q_1 \in \mathbb{C}[X_1, \dots, X_6]$ such that $Q \in Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \dots, X_6))$.
- Thus $Q_0(E_4, \dots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \dots, E_{12}) = 0$.
- There exists a modular substitution mapping ψ_{54} to $-\psi_{54}$ and leaving $E_4, \dots, E_{12}$ invariant; hence $Q_0(E_4, \dots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \dots, E_{12}) = 0$.
- We conclude $Q_0(E_4, \dots, E_{12}) = 0$ and $Q_1(E_4, \dots, E_{12}) = 0$.
- $E_4, \dots, E_{12}$ are algebraically independent. Therefore $Q_0 = Q_1 = 0$, which completes the proof. $\square$

Fields of Orthogonal Modular Functions

Corollary

1. *The field $\mathcal{K}(\Gamma_{A_3})$ of orthogonal modular functions with respect to Γ_{A_3} and the trivial character is a rational function field in the generators*

$$\frac{E_6^2}{E_4^3}, \quad \frac{\psi_8}{E_4^2}, \quad \frac{E_{10}}{E_4\,E_6}, \quad \frac{E_{12}}{E_4^3} \quad \textit{and} \quad \frac{\psi_9^2}{E_6^3}.$$

2. *The field $\mathcal{K}(\Gamma'_{A_3})$ of all orthogonal modular functions with respect to Γ'_{A_3} is an extension of degree 2 over $\mathcal{K}(\Gamma_{A_3})$ generated by ψ_{54}/ψ_9^6.*