Modular Forms for the Orthogonal Group $O(2, 5)$

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Outline

Introduction

Orthogonal Modular Forms

Vector-valued Modular Forms

Borcherds Products

The Graded Ring of Modular Forms
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Vector-valued Modular Forms

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The Graded Ring of Modular Forms
History

- **O(2, 1):** Elliptic modular forms.
  It is well-known that the graded ring $A(\text{SL}_2(\mathbb{Z}))$ of elliptic modular forms is a polynomial ring in the elliptic Eisenstein series $g_2$ and $g_3$ (of weight 4 and 6).

- **O(2, 2):** Hilbert modular forms.
  Cf. S. Mayer’s talk.

- **O(2, 3):** Siegel modular forms of degree 2.
  J.-I. Igusa (1962): The graded ring $A(\text{Sp}_2(\mathbb{Z}))$ is a polynomial ring in the Siegel Eisenstein series $E_4$, $E_6$, $E_{10}$ and $E_{12}$. 
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History (continued)

- **O(2, 4):** Hermitian modular forms of degree 2.
  - E. Freitag (1967): The graded ring for \( \mathbb{Q}(\sqrt{-1}) \).
  - T. Dern (2001): The graded ring for \( \mathbb{Q}(\sqrt{-3}) \) and \( \mathbb{Q}(\sqrt{-2}) \) (with A. Krieg).

- **O(2, 5):** This is the case we will consider.

- **O(2, 6):** Quaternionic modular forms of degree 2.
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The Graded Ring of Modular Forms
Symmetric Matrices and Quadratic Forms

- $S$: a symmetric, positive definite, even $\ell \times \ell$ matrix

$$
S_0 := \begin{pmatrix}
0 & 0 & 1 \\
0 & -S & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad S_1 := \begin{pmatrix}
0 & 0 & 1 \\
0 & S_0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \text{signature of } S_1 \text{ is } (2, \ell + 2)
$$

- $(x, y)_T = txTy$ and $q_T(x) = \frac{1}{2} (x, x)_T = \frac{1}{2} txTx = \frac{1}{2} T[x]$

Abbreviations:

- $(\cdot, \cdot)_S, q = q_S$
- $(\cdot, \cdot)_0 = (\cdot, \cdot)_{S_0}, q_0 = q_{S_0}$
- $(\cdot, \cdot)_1 = (\cdot, \cdot)_{S_1}, q_1 = q_{S_1}$

- Mostly $S = A_3 = \begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{pmatrix}$,

$$
q_S(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_2^2 + x_2 x_3 + x_3^2.
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Symmetric Matrices and Quadratic Forms

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Lattices in Quadratic Spaces

- $\Lambda = \mathbb{Z}^\ell$, $\Lambda_0 = \mathbb{Z}^{\ell+2}$, $\Lambda_1 = \mathbb{Z}^{\ell+4}$ (lattices in quadratic spaces $(\Lambda \otimes \mathbb{R}, (\cdot, \cdot))$, ...)

- Dual lattices: $\Lambda^\#_T = \{ \mu \in \Lambda \otimes \mathbb{R}; (\lambda, \mu)_T \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \} = T^{-1}\Lambda$

- We have:
  - $\Lambda^\# = S^{-1}\mathbb{Z}^\ell$, $\Lambda^\#_0 = \mathbb{Z} \times \Lambda^\# \times \mathbb{Z}$, $\Lambda^\#_1 = \mathbb{Z} \times \Lambda^\#_0 \times \mathbb{Z}$,
  - $\Lambda^\#/\Lambda \cong \Lambda^\#_0/\Lambda_0 \cong \Lambda^\#_1/\Lambda_1$,
  - $|\Lambda^\#/\Lambda| = \det S$.

- $q_T: \Lambda^\#_T/\Lambda_T \to \mathbb{Q}/\mathbb{Z}$, $\mu + \Lambda_T \mapsto q_T(\mu) + \mathbb{Z}$

- $S = A_3$: $\Lambda^\#/\Lambda = A_3^{-1}\mathbb{Z}^3/\mathbb{Z}^3$ is represented by $(0, 0, 0)$, $(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4})$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(-\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ with norm $0$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{8}$, respectively.
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- \( \Lambda = \mathbb{Z}^l, \Lambda_0 = \mathbb{Z}^{l+2}, \Lambda_1 = \mathbb{Z}^{l+4} \) (lattices in quadratic spaces \((\Lambda \otimes \mathbb{R}, (\cdot, \cdot)), \ldots\))

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Orthogonal Groups and the Half-space

- $O(T; \mathbb{R}) = \{ M \in \text{Mat}(\ell; \mathbb{R}); \ T[M] := tMTM = T \} = \{ M \in \text{Mat}(\ell; \mathbb{R}); \ q_T(Mx) = q_T(x) \text{ for all } x \in \mathbb{R}^\ell \}$.

- $O(\Lambda) = \{ M \in O(T; \mathbb{R}); \ MA_\Lambda = \Lambda \}$

- $P_S = \{ v \in \mathbb{R}^{\ell+2}; \ q_0(v) > 0, \ ^tvS_0e > 0 \}$ where $e = (1, 0, \ldots, 0, 1)$

- Half space: $H_S = \{ w = u + iv \in \mathbb{C}^{\ell+2}; \ v = \text{Im}(w) \in P_S \}$

- $O(S_1; \mathbb{R})$ acts on $H_S \cup (-H_S)$:

  $$M\langle w \rangle = j(M, w)^{-1} \cdot (-q_0(w)b + Aw + c) \quad \quad M = \begin{pmatrix} \alpha & t_a & \beta \\ b & A & c \\ \gamma & t_d & \delta \end{pmatrix}$$

  $$j(M, w) = -\gamma q_0(w) + \ ^tdw + \delta$$

- $O^+(S_1; \mathbb{R}) = \{ M \in O(S_1; \mathbb{R}); \ M\langle H_S \rangle = H_S \}$

- $\Gamma_S = O(\Lambda_1) \cap O^+(S_1; \mathbb{R})$
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Orthogonal Groups and the Half-space

- \( \Omega(T; \mathbb{R}) = \{ M \in \text{Mat}(\ell; \mathbb{R}); \ T[M] := tM^T M = T \} = \{ M \in \text{Mat}(\ell; \mathbb{R}); \ q_T(Mx) = q_T(x) \text{ for all } x \in \mathbb{R}^\ell \} \).

- \( \Omega(\Lambda) = \{ M \in \Omega(T; \mathbb{R}); \ M\Lambda = \Lambda \} \)

- \( \mathcal{P}_S = \{ v \in \mathbb{R}^{\ell+2}; \ q_0(v) > 0, \ t_v S_0 e > 0 \} \) where \( e = (1, 0, \ldots, 0, 1) \)

- Half space: \( \mathcal{H}_S = \{ w = u + iv \in \mathbb{C}^{\ell+2}; \ v = \text{Im}(w) \in \mathcal{P}_S \} \)

- \( \Omega(S_1; \mathbb{R}) \) acts on \( \mathcal{H}_S \cup (-\mathcal{H}_S): \)

\[
M\langle w \rangle = j(M, w)^{-1} \cdot (-q_0(w)b + Aw + c) \\
j(M, w) = -\gamma q_0(w) + tdw + \delta
\]

\[
M = \begin{pmatrix} \alpha & t a & \beta \\ b & A & c \\ \gamma & t d & \delta \end{pmatrix}
\]

- \( \Omega^+(S_1; \mathbb{R}) = \{ M \in \Omega(S_1; \mathbb{R}); \ M\langle \mathcal{H}_S \rangle = \mathcal{H}_S \} \)

- \( \Gamma_S = \Omega(\Lambda_1) \cap \Omega^+(S_1; \mathbb{R}) \)
Orthogonal Groups and the Half-space

$\mathbf{O}(T; \mathbb{R}) = \{ M \in \text{Mat}(\ell; \mathbb{R}); \ T[M] := tMTM = T \} = \{ M \in \text{Mat}(\ell; \mathbb{R}); \ q_T(Mx) = q_T(x) \text{ for all } x \in \mathbb{R}^\ell \}$. 

$\mathbf{O}(\Lambda) = \{ M \in \mathbf{O}(T; \mathbb{R}); \ M\Lambda = \Lambda \}$

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$\Gamma_S = \mathbf{O}(\Lambda_1) \cap \mathbf{O}^+(S_1; \mathbb{R})$
Properties of the Orthogonal Modular Group

- $\Gamma_S$ is (in our case) generated by $J$, $T_\lambda$, $\lambda \in \Lambda_0$, and $R_A$, $A \in O(\Lambda)$, where

\[
J\langle w \rangle = -q_0(w)^{-1} \cdot (\tau_2, -z, \tau_1) \quad \text{(inversion)},
\]
\[
T_\lambda\langle w \rangle = w + \lambda \quad \text{(translation)},
\]
\[
R_A\langle w \rangle = (\tau_1, Az, \tau_2) \quad \text{(rotation)}.
\]

- $\Gamma_S$ acts on $\Lambda_1^\# / \Lambda_1$ by multiplication.
  It permutes elements of $\Lambda_1^\# / \Lambda_1$ with the same norm (modulo $\mathbb{Z}$).
  The signs of those permutations are abelian characters of $\Gamma_S$.

- $S = A_3$: Abelian characters of $\Gamma_{A_3}$:

\[
\Gamma_{A_3}^{ab} = \langle \nu_\pi, \det \rangle,
\]

where $\nu_\pi$ is the sign of the permutation of the two elements of $\Lambda_1^\# / \Lambda$ of norm $\frac{3}{8}$. 

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What is an Orthogonal Modular Form?

Definition

An \textit{(orthogonal) modular form} of weight \( k \in \mathbb{Z} \) with respect to a subgroup \( \Gamma \) of \( \Gamma_S \) of finite index and an abelian character \( \nu : \Gamma \to \mathbb{C}^\times \) of finite order is a holomorphic function \( f : \mathcal{H}_S \to \mathbb{C} \) satisfying

\[
f(M\langle w \rangle) = \nu(M) j(M, w)^k f(w) \quad \text{for all } w \in \mathcal{H}_S \text{ and } M \in \Gamma.
\]

We denote the vector space of all such functions by \([\Gamma, k, \nu]\).

\begin{itemize}
\item If \(-I \in \Gamma\) and \(\nu(-I) \neq (-1)^k\) then \([\Gamma, k, \nu] = \{0\}\).
\item If \(k < 0\) then \([\Gamma, k, \nu] = \{0\}\).
\end{itemize}
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Ingo Klöcker (RWTH Aachen)
What is Our Goal?

Products of modular forms are again modular forms. Thus

\[ A(\Gamma_S) = \bigoplus_{k \in \mathbb{Z}} [\Gamma_S, k, 1] \quad \text{and} \quad A(\Gamma'_S) = \bigoplus_{k \in \mathbb{Z}} [\Gamma'_S, k, 1] = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{\nu \in \Gamma_{ab}^S} [\Gamma_S, k, \nu] \]

form graded rings.

Goal: Determine generators and algebraic structure of \( A(\Gamma_{A_3}) \) and \( A(\Gamma'_{A_3}) \).

Due to \(-I \in \Gamma_{A_3}\) and \( \nu_\pi(-I) = \det(-I) = -1 \) we get a first result:

- If \( k \) is even then \( [\Gamma_{A_3}, k, \nu_\pi] = [\Gamma_{A_3}, k, \det] = \{0\} \).
- If \( k \) is odd then \( [\Gamma_{A_3}, k, 1] = [\Gamma_{A_3}, k, \nu_\pi \det] = \{0\} \).
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The Graded Ring of Modular Forms
The Metaplectic Group

The metaplectic group $\text{Mp}_2(\mathbb{Z})$ is given by

$$\{(M, \varphi); \ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \ \varphi: \mathcal{H} \to \mathbb{C} \text{ holom.}, \ \varphi^2(\tau) = c\tau + d\}.$$

It operates on the upper half-plane $\mathcal{H}$ via

$$(M, \varphi)\tau = M\tau = \frac{a\tau + b}{c\tau + d}$$

and is generated by

$$T = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \text{ and } J = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$
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The Weil Representation

Let

- \( S \in \text{Sym}(\ell; \mathbb{R}) \) be a symmetric, even matrix of signature \((b^+, b^-)\),
- \( \Lambda = \mathbb{Z}^\ell \),
- \((\cdot, \cdot) = (\cdot, \cdot)_S\),
- \((e_\mu)_{\mu \in \Lambda^\# / \Lambda}\) be the standard basis of the group ring \(\mathbb{C}[^\# / \Lambda]\).

The Weil representation \(\rho_S\) of \(\text{Mp}_2(\mathbb{Z})\) on \(\mathbb{C}[\Lambda^\# / \Lambda]\) is defined by

\[
\rho_S(T) e_\mu = e^{\pi i (\mu, \mu)} e_\mu,
\]

\[
\rho_S(J) e_\mu = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|\det S|}} \sum_{\nu \in \Lambda^\# / \Lambda} e^{-2\pi i (\mu, \nu)} e_\nu.
\]
Vector-valued Modular Forms

Definition
A holomorphic function $f : \mathcal{H} \to \mathbb{C}[\Lambda^\# / \Lambda]$ is a **vector-valued modular form** of weight $k \in \frac{1}{2} \mathbb{Z}$ with respect to $\rho_S$ if

$$f(M\tau) = \varphi(\tau)^{2k} \rho_S(M, \varphi) f(\tau), \quad \text{for all } (M, \varphi) \in \text{Mp}_2(\mathbb{Z})$$

and if $f$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in \Lambda^\# / \Lambda} \sum_{n \in q_S(\mu) + \mathbb{Z}} c_\mu(n) q^n e_\mu.$$

- $n_0 \geq 0$: Holomorphic modular forms, $[\text{Mp}_2(\mathbb{Z}), k, \rho_S]$, 
- $n_0 < 0$: Nearly holomorphic modular forms, $[\text{Mp}_2(\mathbb{Z}), k, \rho_S]_\infty$. 
What to Know About Vector-valued Modular Forms

- Nearly holomorphic modular forms are uniquely determined by their principal part

\[ \sum_{\mu \in \Lambda^\# / \Lambda} \sum_{\substack{n \in q_S(\mu) + \mathbb{Z} \\ n \leq 0}} c_\mu(n) q^n e_\mu. \]

- Skoruppa: Formula for dimension of \([Mp_2(\mathbb{Z}), k, \rho_S]\) for \(k \geq 2\).

- Examples:
  - Eisenstein series \((k \in \frac{1}{2} \mathbb{Z}, k > 2)\)

\[ E_k(\tau) = \frac{1}{2} \sum_{(M, \varphi) \in \langle T \rangle \setminus Mp_2(\mathbb{Z})} \varphi(\tau)^{-2k} \rho_S(M, \varphi)^{-1} e_0. \]

  Bruinier, Kuss: Formula for Fourier coefficients of \(E_k\).

  - Theta series.
What to Know About Vector-valued Modular Forms

- Nearly holomorphic modular forms are uniquely determined by their principal part
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The Graded Ring of Modular Forms
Borcherds Theorem

Theorem

Let \( f \in [\text{Mp}_2(\mathbb{Z}), -\ell/2, \rho_S^\#]_\infty \) with Fourier coefficients \( c_\mu(n) \), such that \( c_0(0) \in 2\mathbb{Z} \) and \( c_\mu(n) \in \mathbb{Z} \) whenever \( n < 0 \). Then there exists a Borcherds product \( \psi_k : \mathcal{H}_S \to \mathbb{C} \) with the following properties:

- \( \psi_k \) is a meromorphic modular form of weight \( k = c_0(0)/2 \) with respect to \( \Gamma_S \) and some abelian character \( \chi \).

- The zeros and poles of \( \psi_k \) are explicitly known and depend only on the principal part of \( f \).

- \( \psi_k \) is given by the normally convergent product expansion

\[
\psi_k(w) = e^{2\pi i (\ell, w)} \prod_{\lambda_0 \in \Lambda^\#_0} \left(1 - e^{2\pi i (\lambda_0, w)}\right)^{c(0, \lambda_0, 0)(q_0(\lambda_0))}.
\]
Borcherds Theorem

Theorem

Let $f \in \text{Mp}_2(\mathbb{Z}), -\ell/2, \rho_S^{\#}]_\infty$ with Fourier coefficients $c_\mu(n)$, such that $c_0(0) \in 2\mathbb{Z}$ and $c_\mu(n) \in \mathbb{Z}$ whenever $n < 0$. Then there exists a Borcherds product $\psi_k : \mathcal{H}_S \to \mathbb{C}$ with the following properties:

- $\psi_k$ is a meromorphic modular form of weight $k = c_0(0)/2$ with respect to $\Gamma_S$ and some abelian character $\chi$.

- The zeros and poles of $\psi_k$ are explicitly known and depend only on the principal part of $f$.

- $\psi_k$ is given by the normally convergent product expansion

$$\psi_k(w) = e^{2\pi i (\varrho_f, w)} \prod_{\lambda_0 \in \Lambda_0^{\#}, \lambda_0 > 0} \left(1 - e^{2\pi i (\lambda_0, w)}\right)^{c(0, \lambda_0, 0)(q_0(\lambda_0))}.$$
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\]
Borcherds’ Obstruction Condition
A necessary and sufficient condition for the existence of nearly holomorphic modular forms

**Theorem**

*There exists a nearly holomorphic modular form* \( f \in [Mp_2(\mathbb{Z}), -\ell/2, \rho^\#_S]_\infty \)

*with prescribed principal part*

\[
\sum_{\mu \in \Lambda_1^\# / \Lambda_1} \sum_{\substack{n \in -qs(\mu) + \mathbb{Z} \atop n \leq 0}} c_\mu(n) q^n e_\mu,
\]

*if and only if*

\[
\sum_{\mu \in \Lambda_1^\# / \Lambda_1} \sum_{\substack{n \in -qs(\mu) + \mathbb{Z} \atop n \leq 0}} c_\mu(n) \ a_\mu(-n) = 0
\]

*for all holomorphic modular forms* \( g \in [Mp_2(\mathbb{Z}), 2 + \ell/2, \rho_S] \) *with Fourier expansion* \( g(\tau) = \sum_{\mu \in \Lambda_1^\# / \Lambda_1} \sum_{n \in qs(\mu) + \mathbb{Z}, \ n \geq 0} a_\mu(n) q^n e_\mu.\)
Input for Borcherds Theorem in the Case $S = A_3$

The obstruction space $[\text{Mp}_2(\mathbb{Z}), 7/2, \rho_{A_3}]$ is of dimension 1 and spanned by

$$E_{7/2}(\tau) = 1 e_0 - 8 q^{3/8} \left( e_{1/4} + e_{-1/4} \right) - 18 q^{1/2} e_{1/2} - 108 q e_0 + O(q^{11/8}).$$

Thus the condition for the principal part of $f \in [\text{Mp}_2(\mathbb{Z}), -3/2, \rho_{A_3}^\#]_\infty$ is

$$c_0(0) = 8 \left( c_{1/4} \left( -\frac{3}{8} \right) + c_{-1/4} \left( -\frac{3}{8} \right) \right) + 18 c_{1/2} \left( -\frac{1}{2} \right) + 108 c_0(-1) + \cdots.$$ 

Possible principal parts are given by

- $q^{-3/8} \left( e_{1/4} + e_{-1/4} \right) + 16 e_0,$
- $q^{-1/2} e_{1/2} + 18 e_0,$
- $q^{-1} e_0 + 108 e_0.$
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The obstruction space $[Mp_2(\mathbb{Z}), 7/2, \rho_{A_3}]$ is of dimension 1 and spanned by

$$E_{7/2}(\tau) = 1 \cdot e_0 - 8 \cdot q^{3/8} \left( e_{1/4} + e_{-1/4} \right) - 18 q^{1/2} \cdot e_{1/2} - 108 q \cdot e_0 + O(q^{11/8}).$$

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$$q^{-1} \cdot e_0 + 108 e_0.$$
**Borcherds Products for $\Gamma_{A_3}$**

**Theorem**

*There exist Borcherds products*

$$\psi_8 \in [\Gamma_{A_3}, 8, 1], \quad \psi_9 \in [\Gamma_{A_3}, 9, \nu_\pi] \quad \text{and} \quad \psi_{54} \in [\Gamma_{A_3}, 54, \nu_\pi \det].$$

*The zeros of the products are all of first order and are given by*

$$\bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{A_2} \rangle, \quad \bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{A_1^2} \rangle \quad \text{and} \quad \bigcup_{M \in \Gamma_{A_3}} M\langle \mathcal{H}_{S_2} \rangle,$$

*respectively, where*

$$\mathcal{H}_{A_2} = \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3}; \ z_3 = 0\},$$

$$\mathcal{H}_{A_1^2} = \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3}; \ z_2 = 0\},$$

$$\mathcal{H}_{S_2} = \{(\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_{A_3}; \ z_1 + z_3 = 0\}.$$
Outline

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**Corollary**

Let \( k \in \mathbb{Z} \) and \( n \in \{0, 1\} \).

1. If \( k \) is odd and \( f \in [\Gamma_{A_3}, k, \nu^{n+1}_\pi \det^n] \), then \( f \) vanishes on \( \mathcal{H}_{A_2} \) and \( f/\psi_9 \in [\Gamma_{A_3}, k - 9, \nu^n_\pi \det^n] \).

2. If \( f \in [\Gamma_{A_3}, k, \nu^{k+1}_\pi \det] \), then \( f \) vanishes on \( \mathcal{H}_{S_2} \) and \( f/\psi_{54} \in [\Gamma_{A_3}, k - 54, \nu^k_\pi] \).

**Theorem**

*(Dern 2001)* The graded ring \( A(\Gamma_{A_2}) = \bigoplus_{k \in \mathbb{Z}} [\Gamma_{A_2}, 2k, 1] \) is a polynomial ring in

\[
E_4|\mathcal{H}_{A_2}, \quad E_6|\mathcal{H}_{A_2}, \quad E_{10}|\mathcal{H}_{A_2}, \quad E_{12}|\mathcal{H}_{A_2} \quad \text{and} \quad \psi_9^2|\mathcal{H}_{A_2}.
\]
Corollary

Let $k \in \mathbb{Z}$ and $n \in \{0, 1\}$.

1. If $k$ is odd and $f \in [\Gamma_{A_3}, k, \nu^{n+1}_\pi \det^n]$, then $f$ vanishes on $\mathcal{H}_{A_2}$ and $f / \psi_9 \in [\Gamma_{A_3}, k - 9, \nu^n_\pi \det^n]$.

2. If $f \in [\Gamma_{A_3}, k, \nu^{k+1}_\pi \det]$, then $f$ vanishes on $\mathcal{H}_{S_2}$ and $f / \psi_{54} \in [\Gamma_{A_3}, k - 54, \nu^k_\pi]$.

Theorem

(Dern 2001) The graded ring $\mathcal{A}(\Gamma_{A_2}) = \bigoplus_{k \in \mathbb{Z}} [\Gamma_{A_2}, 2k, 1]$ is a polynomial ring in

$$E_4|_{\mathcal{H}_{A_2}}, \ E_6|_{\mathcal{H}_{A_2}}, \ E_{10}|_{\mathcal{H}_{A_2}}, \ E_{12}|_{\mathcal{H}_{A_2}} \ \text{and} \ \psi_9^2|_{\mathcal{H}_{A_2}}.$$
The Graded Ring of Modular Forms for $\Gamma_{A_3}$

**Theorem**

The graded ring $A(\Gamma_{A_3}) = \bigoplus_{k \in \mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ is a polynomial ring in $E_4, E_6, \psi_8, E_{10}, E_{12}$ and $\psi_9^2$.

**Proof.**

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern, $f|_{\mathcal{H}_{A_2}}$ is equal to a polynomial $p$ in $E_4|_{\mathcal{H}_{A_2}}, E_6|_{\mathcal{H}_{A_2}}, E_{10}|_{\mathcal{H}_{A_2}}, E_{12}|_{\mathcal{H}_{A_2}}, \psi_9^2|_{\mathcal{H}_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$ vanishes on $\mathcal{H}_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2))/\psi_8 \in [\Gamma_{A_3}, 2k - 8, 1]$.
- The assertion follows by induction.
The Graded Ring of Modular Forms for $\Gamma_{A_3}$

**Theorem**

The graded ring $\mathcal{A}(\Gamma_{A_3}) = \bigoplus_{k \in \mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ is a polynomial ring in $E_4, E_6, \psi_8, E_{10}, E_{12}$ and $\psi_9^2$.

**Proof.**

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern $f|_{\mathcal{H}_{A_2}}$ is equal to a polynomial $p$ in $E_4|_{\mathcal{H}_{A_2}}, E_6|_{\mathcal{H}_{A_2}}, E_{10}|_{\mathcal{H}_{A_2}}, E_{12}|_{\mathcal{H}_{A_2}}, \psi_9^2|_{\mathcal{H}_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$ vanishes on $\mathcal{H}_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2))/\psi_8 \in [\Gamma_{A_3}, 2k - 8, 1]$.
- The assertion follows by induction.
The Graded Ring of Modular Forms for $\Gamma_{A_3}$

Theorem

The graded ring $A(\Gamma_{A_3}) = \bigoplus_{k \in \mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ is a polynomial ring in $E_4, E_6, \psi_8, E_{10}, E_{12}$ and $\psi^2_9$.

Proof.

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern $f|_{H_{A_2}}$ is equal to a polynomial $p$ in $E_4|_{H_{A_2}}, E_6|_{H_{A_2}}, E_{10}|_{H_{A_2}}, E_{12}|_{H_{A_2}}, \psi^2_9|_{H_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi^2_9)$ vanishes on $H_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi^2_9)/\psi_8 \in [\Gamma_{A_3}, 2k - 8, 1]$.
- The assertion follows by induction.
The Graded Ring of Modular Forms for $\Gamma_{A_3}$

Theorem

The graded ring $A(\Gamma_{A_3}) = \bigoplus_{k \in \mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ is a polynomial ring in

$$E_4, \ E_6, \ \psi_8, \ E_{10}, \ E_{12} \ \text{and} \ \psi_9^2.$$

Proof.

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern $f|_{\mathcal{H}_{A_2}}$ is equal to a polynomial $p$ in $E_4|_{\mathcal{H}_{A_2}}, \ E_6|_{\mathcal{H}_{A_2}}, \ E_{10}|_{\mathcal{H}_{A_2}}, \ E_{12}|_{\mathcal{H}_{A_2}}, \ \psi_9^2|_{\mathcal{H}_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$ vanishes on $\mathcal{H}_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)/\psi_8 \in [\Gamma_{A_3}, 2k - 8, 1]$.
- The assertion follows by induction.
The Graded Ring of Modular Forms for $\Gamma_{A_3}$

**Theorem**

The graded ring $A(\Gamma_{A_3}) = \bigoplus_{k \in \mathbb{Z}}[\Gamma_{A_3}, 2k, 1]$ is a polynomial ring in $E_4$, $E_6$, $\psi_8$, $E_{10}$, $E_{12}$ and $\psi_9^2$.

**Proof.**

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern $f|_{\mathcal{H}_{A_2}}$ is equal to a polynomial $p$ in $E_4|_{\mathcal{H}_{A_2}}$, $E_6|_{\mathcal{H}_{A_2}}$, $E_{10}|_{\mathcal{H}_{A_2}}$, $E_{12}|_{\mathcal{H}_{A_2}}$, $\psi_9^2|_{\mathcal{H}_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$ vanishes on $\mathcal{H}_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)/\psi_8 \in [\Gamma_{A_3}, 2k - 8, 1]$.
- The assertion follows by induction.
The Graded Ring of Modular Forms for $\Gamma_{A_3}$

**Theorem**

The graded ring $A(\Gamma_{A_3}) = \bigoplus_{k \in \mathbb{Z}} [\Gamma_{A_3}, 2k, 1]$ is a polynomial ring in $E_4, E_6, \psi_8, E_{10}, E_{12}$ and $\psi_9^2$.

**Proof.**

- Let $f \in [\Gamma_{A_3}, 2k, 1]$.
- According to Dern $f|_{\mathcal{H}_{A_2}}$ is equal to a polynomial $p$ in $E_4|_{\mathcal{H}_{A_2}}, E_6|_{\mathcal{H}_{A_2}}, E_{10}|_{\mathcal{H}_{A_2}}, E_{12}|_{\mathcal{H}_{A_2}}, \psi_9^2|_{\mathcal{H}_{A_2}}$.
- Thus $f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$ vanishes on $\mathcal{H}_{A_2}$.
- Then $(f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)/\psi_8 \in [\Gamma_{A_3}, 2k - 8, 1]$.
- The assertion follows by induction.
The Graded Ring of Modular Forms for $\Gamma'_A_3$

Theorem
The graded ring $\mathcal{A}(\Gamma'_A_3) = \bigoplus_{k \in \mathbb{Z}} [\Gamma'_A_3, k, 1]$ is generated by

\[ E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12} \text{ and } \psi_{54} \]

and is isomorphic to

\[ \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6)) \]

where $p \in \mathbb{C}[X_1, \ldots, X_6]$ is the uniquely determined polynomial with

\[ \psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}). \]
Proof of the Second Main Result

Proof.

- Let $f \in [\Gamma'_{A_3}, k, 1]$.
  - If $k$ is odd, then $f$ vanishes on $\mathcal{H}_{A_1}$ and we have $f/\psi_9 \in [\Gamma'_{A_3}, k - 9, 1]$. So we can assume that $k$ is even.
  - We know that $[\Gamma'_{A_3}, 2k, 1] = [\Gamma_{A_3}, 2k, 1] \oplus [\Gamma_{A_3}, 2k, \nu \pi \det]$. Thus $f = f_1 + f_{\nu \pi \det}$ with $f_\nu \in [\Gamma_{A_3}, 2k, \nu]$.
  - $f_{\nu \pi \det}$ vanishes on $\mathcal{H}_{S_2}$ and we have $f_{\nu \pi \det}/\psi_{54} \in [\Gamma_{A_3}, 2k - 54, 1]$.
  - Now $f_1$ and $f_{\nu \pi \det}/\psi_{54}$ are polynomials in $E_4, E_6, \psi_8, E_{10}, E_{12}, \psi_9^2$. This completes the proof of the first result.
Proof of the Second Main Result

Proof.

- Let $f \in \Gamma_{A_3}' \ast k \ast 1$.
- If $k$ is odd, then $f$ vanishes on $\mathcal{H}_{A_1}^2$ and we have $f/\psi_9 \in \Gamma_{A_3}' \ast k - 9 \ast 1$. So we can assume that $k$ is even.
- We know that $[\Gamma_{A_3}' \ast 2k \ast 1] = [\Gamma_{A_3} \ast 2k \ast 1] \oplus [\Gamma_{A_3} \ast 2k \ast \nu \pi \det]$. Thus $f = f_1 + f_{\nu \pi \det}$ with $f_{\nu} \in \Gamma_{A_3} \ast 2k \ast \nu$.
- $f_{\nu \pi \det}$ vanishes on $\mathcal{H}_{S_2}$ and we have $f_{\nu \pi \det}/\psi_{54} \in \Gamma_{A_3} \ast 2k - 54 \ast 1$.
- Now $f_1$ and $f_{\nu \pi \det}/\psi_{54}$ are polynomials in $E_4$, $E_6$, $\psi_8$, $E_{10}$, $E_{12}$, $\psi_9^2$. This completes the proof of the first result.
Proof of the Second Main Result

Proof.

- Let $f \in \Gamma_{A_3}' \cdot k \cdot 1$.
- If $k$ is odd, then $f$ vanishes on $\mathcal{H}_{A_1^2}$ and we have $f/\psi_9 \in \Gamma_{A_3}' \cdot k - 9 \cdot 1$. So we can assume that $k$ is even.
- We know that $\Gamma_{A_3}' \cdot 2k \cdot 1 = \Gamma_{A_3} \cdot 2k \cdot 1 \oplus \Gamma_{A_3} \cdot 2k \cdot \nu \cdot \det$. Thus $f = f_1 + f_{\nu \cdot \det}$ with $f_\nu \in \Gamma_{A_3} \cdot 2k \cdot \nu$.
- $f_{\nu \cdot \det}$ vanishes on $\mathcal{H}_{S_2}$ and we have $f_{\nu \cdot \det}/\psi_{54} \in \Gamma_{A_3} \cdot 2k - 54 \cdot 1$.
- Now $f_1$ and $f_{\nu \cdot \det}/\psi_{54}$ are polynomials in $E_4, E_6, \psi_8, E_{10}, E_{12}, \psi_9^2$.

This completes the proof of the first result.
Proof of the Second Main Result

Proof.

- Let \( f \in [\Gamma'_{A_3}, k, 1] \).
- If \( k \) is odd, then \( f \) vanishes on \( \mathcal{H}_{A_1}^2 \) and we have \( f / \psi_9 \in [\Gamma'_{A_3}, k - 9, 1] \). So we can assume that \( k \) is even.
- We know that \([\Gamma'_{A_3}, 2k, 1] = [\Gamma_{A_3}, 2k, 1] \oplus [\Gamma_{A_3}, 2k, \nu \pi \det] \). Thus \( f = f_1 + f_{\nu \pi \det} \) with \( f_{\nu} \in [\Gamma_{A_3}, 2k, \nu] \).
- \( f_{\nu \pi \det} \) vanishes on \( \mathcal{H}_{S_2} \) and we have \( f_{\nu \pi \det} / \psi_{54} \in [\Gamma_{A_3}, 2k - 54, 1] \).
- Now \( f_1 \) and \( f_{\nu \pi \det} / \psi_{54} \) are polynomials in \( E_4, E_6, \psi_8, E_{10}, E_{12}, \psi_9^2 \).
  This completes the proof of the first result.
Proof of the Second Main Result

Proof.

- Let $f \in [\Gamma'_{A_3}, k, 1]$.
- If $k$ is odd, then $f$ vanishes on $\mathcal{H}_{A_1}^2$ and we have $f/\psi_9 \in [\Gamma'_{A_3}, k - 9, 1]$. So we can assume that $k$ is even.
- We know that $[\Gamma'_{A_3}, 2k, 1] = [\Gamma_{A_3}, 2k, 1] \oplus [\Gamma_{A_3}, 2k, \nu \pi \det]$. Thus $f = f_1 + f_{\nu \pi \det}$ with $f_{\nu} \in [\Gamma_{A_3}, 2k, \nu]$.
- $f_{\nu \pi \det}$ vanishes on $\mathcal{H}_{S_2}$ and we have $f_{\nu \pi \det}/\psi_{54} \in [\Gamma_{A_3}, 2k - 54, 1]$.
- Now $f_1$ and $f_{\nu \pi \det}/\psi_{54}$ are polynomials in $E_4, E_6, \psi_8, E_{10}, E_{12}, \psi_9^2$.

This completes the proof of the first result.
Proof of the Second Main Result (cont’d)

Proof.

—we have $\psi_{54}^2 \in [\Gamma_{A_3}, 108, 1]$. Thus $\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})$.

—we want to show that $\mathcal{A}(\Gamma'_S) \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6))$.

So let $Q \in \mathbb{C}[X_1, \ldots, X_7]$ such that $Q(E_4, \ldots, E_{12}, \psi_{54}) = 0$.

—there exist $Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6]$ such that

$Q \in Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \ldots, X_6))$.

—thus $Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0$.

—there exists a modular substitution mapping $\psi_{54}$ to $-\psi_{54}$ and leaving $E_4, \ldots, E_{12}$ invariant; hence

$Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0$.

—we conclude $Q_0(E_4, \ldots, E_{12}) = 0$ and $Q_1(E_4, \ldots, E_{12}) = 0$.

$E_4, \ldots, E_{12}$ are algebraically independent. Therefore $Q_0 = Q_1 = 0$, which completes the proof.
Proof of the Second Main Result (cont’d)

Proof.

- We have \( \psi_{54}^2 \in [\Gamma_{A_3}, 108, 1] \). Thus \( \psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) \).
- We want to show that \( \mathcal{A}(\Gamma'_S) \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6)) \).
  
  So let \( Q \in \mathbb{C}[X_1, \ldots, X_7] \) such that \( Q(E_4, \ldots, E_{12}, \psi_{54}) = 0 \).
- There exist \( Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6] \) such that \( Q \in Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \ldots, X_6)) \).
- Thus \( Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- There exists a modular substitution mapping \( \psi_{54} \) to \( -\psi_{54} \) and leaving \( E_4, \ldots, E_{12} \) invariant; hence \( Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- We conclude \( Q_0(E_4, \ldots, E_{12}) = 0 \) and \( Q_1(E_4, \ldots, E_{12}) = 0 \).
- \( E_4, \ldots, E_{12} \) are algebraically independent. Therefore \( Q_0 = Q_1 = 0 \), which completes the proof.
Proof of the Second Main Result (cont’d)

Proof.

- We have $\psi_{54}^2 \in [\Gamma_{A_3}, 108, 1]$. Thus $\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})$.
- We want to show that $A(\Gamma'_S) \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6))$. So let $Q \in \mathbb{C}[X_1, \ldots, X_7]$ such that $Q(E_4, \ldots, E_{12}, \psi_{54}) = 0$.
- There exist $Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6]$ such that $Q = Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \ldots, X_6))$.
- Thus $Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0$.
- There exists a modular substitution mapping $\psi_{54}$ to $-\psi_{54}$ and leaving $E_4, \ldots, E_{12}$ invariant; hence $Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0$.
- We conclude $Q_0(E_4, \ldots, E_{12}) = 0$ and $Q_1(E_4, \ldots, E_{12}) = 0$.
- $E_4, \ldots, E_{12}$ are algebraically independent. Therefore $Q_0 = Q_1 = 0$, which completes the proof.
Proof of the Second Main Result (cont’d)

Proof.

- We have \( \psi_{54}^2 \in [\Gamma_{A_3}, 108, 1] \). Thus \( \psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) \).
- We want to show that \( \mathcal{A}(\Gamma'_S) \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6)) \). So let \( Q \in \mathbb{C}[X_1, \ldots, X_7] \) such that \( Q(E_4, \ldots, E_{12}, \psi_{54}) = 0 \).
- There exist \( Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6] \) such that \( Q \in Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \ldots, X_6)) \).
- Thus \( Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- There exists a modular substitution mapping \( \psi_{54} \) to \( -\psi_{54} \) and leaving \( E_4, \ldots, E_{12} \) invariant; hence \( Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- We conclude \( Q_0(E_4, \ldots, E_{12}) = 0 \) and \( Q_1(E_4, \ldots, E_{12}) = 0 \).
- \( E_4, \ldots, E_{12} \) are algebraically independent. Therefore \( Q_0 = Q_1 = 0 \), which completes the proof.
Proof of the Second Main Result (cont’d)

Proof.

- We have \( \psi_{54}^2 \in [\Gamma_{A_3}, 108, 1] \). Thus \( \psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) \).
- We want to show that \( \mathcal{A}(\Gamma'_5) \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6)) \).

So let \( Q \in \mathbb{C}[X_1, \ldots, X_7] \) such that \( Q(E_4, \ldots, E_{12}, \psi_{54}) = 0 \).

- There exist \( Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6] \) such that \( Q \in Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \ldots, X_6)) \).
- Thus \( Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- There exists a modular substitution mapping \( \psi_{54} \) to \( -\psi_{54} \) and leaving \( E_4, \ldots, E_{12} \) invariant; hence \( Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- We conclude \( Q_0(E_4, \ldots, E_{12}) = 0 \) and \( Q_1(E_4, \ldots, E_{12}) = 0 \).
- \( E_4, \ldots, E_{12} \) are algebraically independent. Therefore \( Q_0 = Q_1 = 0 \), which completes the proof.

▶
Proof of the Second Main Result (cont’d)

Proof.

- We have $\psi_{54}^2 \in [\Gamma_{A_3}, 108, 1]$. Thus $\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})$.
- We want to show that $A(\Gamma'_S) \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6))$. So let $Q \in \mathbb{C}[X_1, \ldots, X_7]$ such that $Q(E_4, \ldots, E_{12}, \psi_{54}) = 0$.
- There exist $Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6]$ such that $Q \in Q_0 + X_7 Q_1 + (X_7^2 - p(X_1, \ldots, X_6))$.
- Thus $Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0$.
- There exists a modular substitution mapping $\psi_{54}$ to $-\psi_{54}$ and leaving $E_4, \ldots, E_{12}$ invariant; hence $Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0$.
- We conclude $Q_0(E_4, \ldots, E_{12}) = 0$ and $Q_1(E_4, \ldots, E_{12}) = 0$.
- $E_4, \ldots, E_{12}$ are algebraically independent. Therefore $Q_0 = Q_1 = 0$, which completes the proof.
Proof of the Second Main Result (cont’d)

Proof.

- We have \( \psi_{54}^2 \in [\Gamma_{A_3}, 108, 1] \). Thus \( \psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) \).
- We want to show that \( \mathcal{A}(\Gamma_S') \cong \mathbb{C}[X_1, \ldots, X_7]/(X_7^2 - p(X_1, \ldots, X_6)) \).
  So let \( Q \in \mathbb{C}[X_1, \ldots, X_7] \) such that \( Q(E_4, \ldots, E_{12}, \psi_{54}) = 0 \).
- There exist \( Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6] \) such that
  \( Q \in Q_0 + X_7Q_1 + (X_7^2 - p(X_1, \ldots, X_6)) \).
- Thus \( Q_0(E_4, \ldots, E_{12}) + \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- There exists a modular substitution mapping \( \psi_{54} \) to \( -\psi_{54} \) and leaving \( E_4, \ldots, E_{12} \) invariant; hence
  \( Q_0(E_4, \ldots, E_{12}) - \psi_{54} \cdot Q_1(E_4, \ldots, E_{12}) = 0 \).
- We conclude \( Q_0(E_4, \ldots, E_{12}) = 0 \) and \( Q_1(E_4, \ldots, E_{12}) = 0 \).
- \( E_4, \ldots, E_{12} \) are algebraically independent. Therefore \( Q_0 = Q_1 = 0 \), which completes the proof.
Fields of Orthogonal Modular Functions

Corollary

1. The field $\mathcal{K}(\Gamma_{A_3})$ of orthogonal modular functions with respect to $\Gamma_{A_3}$ and the trivial character is a rational function field in the generators

$$\frac{E_6^2}{E_4^3}, \quad \frac{\psi_8}{E_4^2}, \quad \frac{E_{10}}{E_4 E_6}, \quad \frac{E_{12}}{E_4^3} \quad \text{and} \quad \frac{\psi_9^2}{E_6^3}.$$

2. The field $\mathcal{K}(\Gamma'_{A_3})$ of all orthogonal modular functions with respect to $\Gamma'_{A_3}$ is an extension of degree 2 over $\mathcal{K}(\Gamma_{A_3})$ generated by $\psi_5^4/\psi_9^6$. 
Fields of Orthogonal Modular Functions

Corollary

1. The field $\mathcal{K}(\Gamma_{A_3})$ of orthogonal modular functions with respect to $\Gamma_{A_3}$ and the trivial character is a rational function field in the generators $\frac{E_6^2}{E_4^3}$, $\frac{\psi_8}{E_4^2}$, $\frac{E_{10}}{E_4 E_6}$, $\frac{E_{12}}{E_4^3}$ and $\frac{\psi_9^2}{E_6^3}$.

2. The field $\mathcal{K}(\Gamma'_{A_3})$ of all orthogonal modular functions with respect to $\Gamma'_{A_3}$ is an extension of degree 2 over $\mathcal{K}(\Gamma_{A_3})$ generated by $\psi_{54}/\psi_9^6$. 