Modular Forms for the Orthogonal Group O(2,5)

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Introduction

We consider modular forms for orthogonal groups $\mathrm{O}(2, l+2)$ with particular emphasis on the case $l = 3$. Modular forms for $\mathrm{O}(2,3)$ correspond to Siegel modular forms of degree 2. In the 1960's Igusa [Ig64] used theta constants in order to describe the graded ring of Siegel modular forms of degree 2. Using Igusa's method Freitag [Fr67] was able to determine the graded ring of symmetric Hermitian modular forms of degree 2 over the Gaussian number field $\mathbb{Q}(\sqrt{-1})$ which corresponds to the case of modular forms for $\mathrm{O}(2,4)$. Nagaoka [Na96], Ibukiyama [Ib99b] and Aoki [AI05] completed the description the graded ring in terms of generators and relations. Other cases corresponding to modular forms for $\mathrm{O}(2,4)$ where dealt with by Dern and Krieg. They determined the graded rings of Hermitian modular forms of degree 2 including the Abelian characters for the number fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$ (cf. [De01], [DK03], [DK04]). Instead of using estimations on theta series as in Igusa's approach they applied the theory of Borcherds products (cf. [Bo98]) in order to obtain Hermitian modular forms with known zeros. Then a similar reduction process as the one used by Igusa and Freitag yields their structure theorems. The general case of modular forms for $\mathrm{O}(2, l+2)$ was studied by Freitag and Hermann [FH00] from a geometrical point of view. They derived partial results on modular forms for $\mathrm{O}(2,5)$ by embedding suitable lattices into the Hurwitz quaternions.

Using similar methods as Dern and Krieg we will characterize the graded rings of orthogonal modular forms for two maximal discrete subgroups of $\mathrm{O}(2,5)$. Let S be an even positive definite symmetric matrix of rank l, and let

$$S_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ S_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The bilinear form associated to S_0 is given by $(a,b)_0 = {}^t a S_0 b$ for $a, b \in \mathbb{R}^{l+2}$ and the corresponding quadratic form is $q_0 = \frac{1}{2}(\cdot,\cdot)_0$. The attached half-space is

$$\mathcal{H}_S = \{w = u + iv \in \mathbb{C}^{l+2};\ v \in \mathcal{P}_S\},$$

where $\mathcal{P}_S = \{v \in \mathbb{R}^{l+2};\ (v,v)_0 > 0,\ (v,\mathrm{e}) > 0\}$, $\mathrm{e} = {}^t(1,0,\ldots,0,1)$. The orthogonal group

$$\mathrm{O}(S_1;\mathbb{R}) = \{M \in \mathrm{Mat}(l+4;\mathbb{R});\ {}^t M S_1 M = S_1\}$$

acts on $\mathcal{H}_S \cup (-\mathcal{H}_S)$ as group of biholomorphic rational transformations via

$$w \mapsto M\langle w\rangle = (-q_0(w)b + Aw + c)\, j(M,w)^{-1} \quad \text{for } M = \left(\begin{smallmatrix} \alpha & {}^t a & \beta \\ b & A & c \\ \gamma & {}^t d & \delta \end{smallmatrix}\right) \in \mathrm{O}(S_1;\mathbb{R}),$$

where $j(M,w) = -\gamma q_0(w) + {}^t dw + \delta$. The orthogonal modular group is given by

$$\Gamma_S = \{M \in \mathrm{O}(S_1;\mathbb{R});\ M\langle\mathcal{H}_S\rangle = \mathcal{H}_S,\ M\Lambda_1 = \Lambda_1\}.$$

An orthogonal modular form of weight $k \in \mathbb{Z}$ with respect to an Abelian character ν of Γ_S is a holomorphic function $f : \mathcal{H}_S \to \mathbb{C}$ satisfying

$$(f|_k M)(w) := j(M,w)^{-k}\, f(M\langle w\rangle) = \nu(M)\, f(w) \quad \text{for all } w \in \mathcal{H}_S,\ M \in \Gamma_S.$$

The vector space $[\Gamma_S, k, \nu]$ of those functions is finite dimensional. If $f_j \in [\Gamma_S, k_j, \nu_j]$, $j = 1, 2$, then $f_1 f_2 \in [\Gamma_S, k_1 + k_2, \nu_1\nu_2]$. Thus

$$\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma_S, k, 1] \quad \text{and} \quad \mathcal{A}(\Gamma_S') = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{\nu\in\Gamma_S^{\mathrm{ab}}} [\Gamma_S, k, \nu],$$

where Γ_S' is the commutator subgroup of Γ_S and Γ_S^{ab} is the group of Abelian characters of Γ_S, form graded rings. Our main goal is the explicit description of those graded rings in terms of generators for

$$S = A_3 = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right) \quad \text{and} \quad S = A_1^{(3)} = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right).$$

It turns out that in both cases the graded ring $\mathcal{A}(\Gamma_S)$ is a polynomial ring in six (algebraically independent) generators while the graded rings $\mathcal{A}(\Gamma_S')$ are free $\mathcal{R}$-modules of rank 2 and 4, respectively, where in both cases $\mathcal{R}$ is an extension of degree two of $\mathcal{A}(\Gamma_S)$. In the case of $S = A_3$ we can simply take certain Eisenstein series and Borcherds products as generators. In the other case we determine the invariant ring of a finite representation which is given by the action of a subgroup of the quaternionic symplectic group on quaternionic theta series. The restrictions of the primary invariants and some Borcherds products generate the graded rings for $S = A_1^{(3)}$. In both cases Borcherds products play an important role. In a first step the explicitly known zeros of the Borcherds products allow us to reduce the problem of determining the graded ring $\mathcal{A}(\Gamma_S')$ of modular forms with Abelian characters to the problem of determining the graded ring $\mathcal{A}(\Gamma_S)$ of modular forms of even weight with respect to the trivial character. In the next step we use the fact that we already know the generators of the graded rings of modular forms living on certain submanifolds of $\mathcal{H}_S$ on which suitable Borcherds products vanish of first order. In the case of $S = A_3$ we can derive our results from Dern's result for $\mathbb{Q}(\sqrt{-3})$, and in the case of $S = A_1^{(3)}$ we use the results for $\mathbb{Q}(\sqrt{-1})$. As an application of our results we describe the attached fields of orthogonal modular functions.

We now briefly describe the content of this thesis:

In the first chapter we collect the necessary facts and results about orthogonal groups. In particular, we explicitly determine generators and Abelian characters of certain orthogonal modular groups Γ_S, and we introduce the paramodular subgroup of Γ_S.

In the second chapter we define the main object of our studies, the orthogonal modular forms, and state some fundamental results. In particular, we show that, unlike elliptic modular forms, orthogonal modular forms automatically possess an absolutely and locally uniformly convergent Fourier series due to Koecher's principle. Moreover, we introduce the notion of cusp forms and show that, as usual, the subspace of cusp forms can be characterized by Siegel's Φ-operator. Then we consider a certain differential operator which allows us to construct non-trivial orthogonal modular forms from a number of algebraically independent orthogonal modular forms. The next two sections deal with Jacobi forms and the Maaß space. An explicit formula for the dimension of certain Maaß spaces is derived from a dimension formula for spaces of Jacobi forms. Next we take a look at restrictions of orthogonal modular forms to submanifolds and give a brief introduction into Hermitian and quaternionic modular forms of degree 2. We translate the results about graded rings of Hermitian modular forms of degree 2 from the symplectic point of view to our terminology, and we define orthogonal Eisenstein series for $S = A_3$ and $S = A_1^{(3)}$ as restrictions of quaternionic Eisenstein series. Finally, we consider a 5-dimensional finite representation of $\Gamma_{A_1^{(3)}}$, determine its invariant ring using the MAGMA and get five algebraically independent modular forms for $\Gamma_{A_1^{(3)}}$ whose restrictions to a certain submanifold generate the graded ring of orthogonal modular forms of even weight and trivial character corresponding to Hermitian modular forms over the Gaussian number field.

In the third chapter we recall fundamental facts about vector-valued elliptic modular forms for the metaplectic group $\mathrm{Mp}(2;\mathbb{Z})$. We focus on holomorphic vector-valued modular forms with respect to the Weil representation ρ_S attached to a certain quadratic module $(\Lambda^\sharp/\Lambda, \overline{q}_S)$ associated to S. A dimension formula for spaces of holomorphic vector-valued modular forms is given, and two classes of vector-valued modular forms whose Fourier expansions can be explicitly calculated are introduced: Eisenstein series and theta series. Moreover, so-called nearly holomorphic vector-valued modular forms, that is vector-valued modular forms with a pole in the cusp ∞, are defined.

In the fourth chapter we briefly review the theory of Borcherds products specializing Borcherds's results to our setting. Borcherds products are constructed from nearly holomorphic vector-valued modular forms of weight $-l/2$ with respect to the dual Weil representation $\rho_S^\sharp$. They are orthogonal modular forms, but in general they are not holomorphic. The most remarkable property of a Borcherds product is the fact that its zeros and poles are completely determined by the principal part of the nearly holomorphic modular form the Borcherds product is constructed from. The zeros and poles lie on so-called rational quadratic divisors which correspond to embedded orthogonal half-spaces of codimension 1. It is intuitively clear that it is desirable to find holomorphic Borcherds products with as few zeros of as low order as possible. The existence of nearly holomorphic modular forms with suitably nice principal part is controlled by the so-called obstruction space, the space of holomorphic vector-valued modular forms of weight $2 + l/2$ with respect to ρ_S.

In the fifth chapter we derive our main results. For $S = A_3$ and $S = A_1^{(3)}$ we start by determining nice Borcherds products. In the first case the obstruction space is 1-dimensional and spanned by an Eisenstein series while in the other case it is 3-dimensional and spanned by an Eisenstein series and two theta series. Nevertheless in both cases the existence of principal parts of nearly holomorphic modular forms mainly depends only on the Fourier coefficients of the Eisenstein series. This allows us to construct Borcherds products which vanish only on one rational quadratic divisor and only of first order. Orthogonal modular forms with non-trivial character have to vanish on certain rational quadratic divisors. Since the Borcherds products we constructed vanish of first order we can divide orthogonal modular forms with non-trivial character by suitable Borcherds products. This way we can reduce all orthogonal modular forms to orthogonal modular forms with respect to the trivial character. It turns out that in the two cases we consider all non-trivial orthogonal modular forms with respect to the trivial character are of even weight. Thus it remains to determine the graded rings of modular forms of even weight and with trivial character. In the case of $S = A_3$ we show that the ring of orthogonal modular forms of even weight and with trivial character corresponding to Hermitian modular forms for $\mathbb{Q}(\sqrt{-3})$ is generated by the restrictions of four orthogonal Eisenstein series and the restriction of the square of a Borcherds product. So by subtracting a suitable polynomial in those functions from an arbitrary modular form of even weight and with trivial character we get a function which vanishes on a submanifold corresponding to the Hermitian half-space for $\mathbb{Q}(\sqrt{-3})$. Again we can divide by a suitable Borcherds product and by induction we get our main result in the case of $S = A_3$. In the other case we use the five algebraically independent modular forms we determined in chapter two in order to derive a corresponding result. We conclude the chapter by a few corollaries including the determination of the algebraic structure of the fields of orthogonal modular functions.

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0. Basic Notation

We use the following notation (for a detailed list see the table of notation on pages 125 ff.): $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0$ is the set of non-negative integers, $\mathbb{Z}$ is the ring of the integers, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of rational, real and complex numbers, respectively, and $\mathbb{H}$ is the skew field of Hamilton quaternions with standard basis $1, \mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3 = \mathrm{i}_1\mathrm{i}_2$.

Let R be a suitable ring with unity, i.e., commutative whenever necessary. $\mathrm{Mat}(n,m;R)$ is the group of $n \times m$ matrices over R, $\mathrm{Mat}(n;R)$ is the ring of $n \times n$ matrices over R, $\mathrm{GL}(n;R)$ and $\mathrm{SL}(n;R)$ are the general linear group and the special linear group in $\mathrm{Mat}(n;R)$, respectively. $\mathrm{Sym}(n;R)$ denotes the set of symmetric matrices, $\mathrm{Her}(n;R)$ the set of Hermitian matrices, and $\mathrm{Pos}(n;R) \subset \mathrm{Her}(n;R)$ the ring of positive definite Hermitian matrices in $\mathrm{Mat}(n;R)$. For $H \in \mathrm{Her}(n;R)$ we write $H > 0$ if H is positive definite and we write $H \geq 0$ if H is positive semi-definite. I_n is the identity matrix in $\mathrm{Mat}(n;R)$. If the dimension is obvious then we also write simply I.

For $A \in \mathrm{Mat}(n;R)$ and $B \in \mathrm{Mat}(n,m;R)$ we denote the transpose of B by ${}^tB$, the conjugate transpose of B by ${}^t\overline{B}$, and we define $A[B] := {}^t\overline{B}AB$. For matrices $A_j \in \mathrm{Mat}(n_j;R)$, $1 \leq j \leq n$, we define

$$A_1 \times \ldots \times A_n := \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{pmatrix},$$

and for $a_1, \ldots, a_n \in R$ we denote the diagonal matrix with diagonal elements a_j by $[a_1, \ldots, a_n]$.

Let G be a group. For $g, h \in G$ we define the commutator of g and h by $[g,h] := ghg^{-1}h^{-1}$. We denote the commutator subgroup of G by G' and the commutator factor group of G by $G^{\mathrm{ab}} := G/G'$. The latter coincides with the group of Abelian characters $G \to \mathbb{C}^\times$ which we also denote by G^{ab}.

We will sometimes write column vectors as row vectors because row vectors take less vertical space. In this case we will omit the "transpose" symbol whenever it is clear from the context what we actually mean.

1. Orthogonal Groups

1.1. Lattices and orthogonal groups

Definition 1.1 *A* lattice *is a free $\mathbb{Z}$-module of finite rank equipped with a symmetric $\mathbb{Z}$-valued bilinear form $(\cdot,\cdot)$. We call a lattice Λ* even *if (λ,λ) is even for all $\lambda \in \Lambda$. The associated quadratic form q is defined by*

$$q(\lambda) = \frac{1}{2}(\lambda,\lambda) \quad \textit{for all } \lambda \in \Lambda.$$

Let Λ be a lattice. If Λ is even then q obviously takes its values in $\mathbb{Z}$.

Henceforth we always assume that Λ is non-degenerate. We set $V := \Lambda \otimes \mathbb{R}$. Since Λ contains a basis of the $\mathbb{R}$-vector space V the bilinear form $(\cdot,\cdot)$ on $\Lambda \times \Lambda$ induces a bilinear form on $V \times V$ which we again denote by $(\cdot,\cdot)$. The associated quadratic form is again denoted by q. Then the pair (V,q) is a quadratic space.

Definition 1.2 *For a lattice Λ with attached bilinear form $(\cdot,\cdot)$ the* dual lattice $\Lambda^\sharp$ *is defined by*

$$\Lambda^\sharp := \{\mu \in V;\ (\mu,\lambda) \in \mathbb{Z} \textit{ for all } \lambda \in \Lambda\}.$$

We obviously have $\Lambda \subset \Lambda^\sharp$. Therefore the following definitions make sense.

Definition 1.3 *Let Λ be a lattice.*

a) The finite Abelian group

$$\mathrm{Dis}(\Lambda) := \Lambda^\sharp/\Lambda$$

is called the discriminant group *of Λ.*

b) The level *of the lattice Λ is defined by*

$$\min\{n \in \mathbb{N};\ nq(\mu) \in \mathbb{Z} \textit{ for all } \mu \in \Lambda^\sharp\}.$$

c) $\mu \in \Lambda^\sharp$ is called primitive *if $\mathbb{Q}\mu \cap \Lambda^\sharp = \mathbb{Z}\mu$, i.e., $\max\{n \in \mathbb{N};\ \frac{1}{n}\mu \in \Lambda^\sharp\} = 1$.*

Proposition 1.4 *Let Λ be an even lattice. Then the map $\overline{q} : \mathrm{Dis}(\Lambda) \to \mathbb{Q}/\mathbb{Z}$ which is induced by q on $\mathrm{Dis}(\Lambda)$, i.e., which is given by*

$$\overline{q}(\mu + \Lambda) = q(\mu) + \mathbb{Z}$$

for all $\mu \in \Lambda^\sharp$, is well defined.

PROOF Let $\mu + \Lambda = \mu' + \Lambda \in \mathrm{Dis}(\Lambda)$. Then $\mu' = \mu + \lambda$ for some $\lambda \in \Lambda$ and

$$q(\mu') - q(\mu) = q(\mu + \lambda) - q(\mu) = (\mu, \lambda) + q(\lambda) \in \mathbb{Z}.$$

Thus $\overline{q}(\mu' + \Lambda) = \overline{q}(\mu + \Lambda)$. ∎

Now let $l \in \mathbb{N}$, $\Lambda = \mathbb{Z}^l$, $V = \Lambda \otimes \mathbb{R} \cong \mathbb{R}^l$, and let $S \in \mathrm{Sym}(l; \mathbb{R}) \cap \mathrm{GL}(l; \mathbb{R})$ be a nonsingular real symmetric matrix. We define the symmetric bilinear form $(\cdot, \cdot)_S$ on V associated to S by $(x, y)_S = {}^t xSy$ for $x, y \in V$, and we denote the corresponding quadratic form by q_S, i.e.,

$$q_S(x) = \frac{1}{2}(x, x)_S = \frac{1}{2}S[x]$$

for $x \in V$. If it is clear to which matrix S the bilinear form $(\cdot, \cdot)_S$ and the quadratic form q_S correspond to then we simply write $(\cdot, \cdot)$ and q respectively.

If $S \in \mathrm{Sym}(l; \mathbb{Z}) \cap \mathrm{GL}(l; \mathbb{R})$ is a nonsingular integral symmetric matrix then Λ together with $(\cdot, \cdot)_S$ is a lattice of rank l, the lattice associated to S. Obviously we have $\Lambda^\sharp = S^{-1}\Lambda$, and thus the discriminant group $\mathrm{Dis}(\Lambda)$ is of order $\det S$. We call S an *even matrix* if the associated lattice is an even lattice.

Definition 1.5 *Let $S \in \mathrm{Sym}(l; \mathbb{R}) \cap \mathrm{GL}(l; \mathbb{R})$ be a nonsingular real symmetric matrix. The* real orthogonal group $\mathrm{O}(S; \mathbb{R})$ *with respect to S is defined by*

$$\begin{aligned} \mathrm{O}(S; \mathbb{R}) &:= \{M \in \mathrm{Mat}(l; \mathbb{R});\ S[M] = S\} \\ &= \{M \in \mathrm{Mat}(l; \mathbb{R});\ q_S(Mx) = q_S(x) \text{ for all } x \in \mathbb{R}^l\}. \end{aligned}$$

Remark 1.6 *Up to isomorphism the real orthogonal group $\mathrm{O}(S; \mathbb{R})$ only depends on the signature (b^+, b^-) of S. Therefore one often writes $\mathrm{O}(b^+, b^-)$ for $\mathrm{O}(S; \mathbb{R})$. Moreover, note that* $\det(S[M]) = \det S$ *yields* $\det M = \pm 1$ *for all $M \in \mathrm{O}(S; \mathbb{R})$.*

Definition 1.7 *Suppose that $S \in \mathrm{Sym}(l; \mathbb{Z}) \cap \mathrm{GL}(l; \mathbb{R})$ is a nonsingular integral symmetric matrix. Let Λ be the lattice associated to S. The stabilizer of Λ in $\mathrm{O}(S; \mathbb{R})$ is denoted by* $\mathrm{O}(\Lambda)$*, i.e., we have*

$$\mathrm{O}(\Lambda) = \{M \in \mathrm{O}(S; \mathbb{R});\ M\Lambda = \Lambda\}.$$

Remark 1.8 *The condition $M\Lambda = \Lambda$ is equivalent to $M \in \mathrm{GL}(l; \mathbb{Z})$. Thus $\mathrm{O}(\Lambda)$ is a subgroup of* $\mathrm{GL}(l; \mathbb{Z})$*. In fact we have* $\mathrm{O}(\Lambda) = \mathrm{O}(S; \mathbb{R}) \cap \mathrm{GL}(l; \mathbb{Z})$.

One easily verifies that we have $M\Lambda^\sharp = \Lambda^\sharp$ for all $M \in \mathrm{O}(\Lambda)$. Thus $\mathrm{O}(\Lambda)$ acts on the discriminant group $\mathrm{Dis}(\Lambda)$ of Λ which leads to the following definition.

Definition 1.9 *Let Λ be the lattice associated to a nonsingular integral symmetric matrix S. The* discriminant kernel $\mathrm{O_d}(\Lambda)$ *of* $\mathrm{O}(\Lambda)$ *is the kernel of the action of* $\mathrm{O}(\Lambda)$ *on the discriminant group* $\mathrm{Dis}(\Lambda)$.

Finally we define a property of lattices which will be crucial for the existence of a nice system of generators of the corresponding modular group.

Definition 1.10 *Let* Λ *be the lattice associated to a positive definite symmetric matrix* S*. We call* Λ Euclidean *if for all* $x \in \mathbb{R}^l$ *there exists* $\lambda \in \Lambda$ *such that*

$$q_S(x+\lambda) < 1.$$

1.2. $\mathrm{O}(2,l+2)$ and the attached half-space

We are particularly interested in certain integral symmetric matrices of signature $(2,l+2)$, $l \in \mathbb{N}$. Let $S \in \mathrm{Pos}(l;\mathbb{R})$ be a positive definite even matrix. We set

$$S_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } S_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then S_0 is of signature $(1,l+1)$ and S_1 is of signature $(2,l+2)$. We use the following abbreviations for the associated bilinear forms and quadratic forms:

$$\begin{aligned} (\cdot,\cdot) &= (\cdot,\cdot)_S, & q &= q_S, \\ (\cdot,\cdot)_0 &= (\cdot,\cdot)_{S_0}, & q_0 &= q_{S_0}, \\ (\cdot,\cdot)_1 &= (\cdot,\cdot)_{S_1}, & q_1 &= q_{S_1}. \end{aligned}$$

Moreover, we set $\mathrm{e} := {}^t(1,0,\ldots,0,1) \in \mathbb{R}^{l+2}$ and define

$$\mathcal{H}_S := \{w = u + iv \in \mathbb{C}^{l+2};\ v \in \mathcal{P}_S\},$$

where

$$\begin{aligned} \mathcal{P}_S &:= \{v \in \mathbb{R}^{l+2};\ q_0(v) > 0,\ (v,\mathrm{e})_0 > 0\} \\ &= \{(v_0,\tilde{v},v_{l+1}) \in \mathbb{R} \times \mathbb{R}^l \times \mathbb{R};\ v_0 v_{l+1} > q_S(\tilde{v}),\ v_0 > 0\}. \end{aligned}$$

Then $\mathcal{P}_S$ is the domain of positivity of a certain Jordan algebra with unit element e, and $\mathcal{H}_S$ is a Hermitian symmetric space of type (IV) in Cartan's classification and a Siegel domain of genus 1 (cf. [PS69]) and corresponds to the group $\mathrm{O}(2,l+2)$ (cf. [Kr96]).

Note that we have

$$\mathcal{H}_S \subset \mathcal{H} \times \mathbb{C}^l \times \mathcal{H}$$

where $\mathcal{H} = \{\tau \in \mathbb{C};\ \mathrm{Im}(\tau) > 0\}$ denotes the complex upper half plane. Therefore we will usually write the elements of $\mathcal{H}_S$ in the form $w = (\tau_1, z, \tau_2)$, $\tau_1,\tau_2 \in \mathcal{H}$, $z \in \mathbb{C}^l$.

In the orthogonal context we write a matrix $M \in \mathrm{Mat}(l+4;\mathbb{R})$ always in the form

$$M = \begin{pmatrix} \alpha & {}^t a & \beta \\ b & A & c \\ \gamma & {}^t d & \delta \end{pmatrix}, \text{ where } A \in \mathrm{Mat}(l+2;\mathbb{R}).$$

Then we have $M \in \mathrm{O}(S_1;\mathbb{R})$ if and only if

$$\begin{pmatrix} 2\alpha\gamma + S_0[b] & \alpha\,{}^t\!d + {}^t\!bS_0A + \gamma\,{}^t\!a & \alpha\delta + {}^t\!bS_0c + \beta\gamma \\ \alpha d + {}^t\!AS_0b + \gamma a & a\,{}^t\!d + S_0[A] + d\,{}^t\!a & \beta d + {}^t\!AS_0c + \delta a \\ \alpha\delta + {}^t\!bS_0c + \beta\gamma & \beta\,{}^t\!d + {}^t\!cS_0A + \delta\,{}^t\!a & 2\beta\delta + S_0[c] \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

The real orthogonal group $\mathrm{O}(S_1;\mathbb{R})$ acts transitively on $\mathcal{H}^S := \mathcal{H}_S \cup (-\mathcal{H}_S)$ as a group of biholomorphic automorphisms via

$$w \mapsto M\langle w\rangle := (-q_0(w)b + Aw + c)(M\{w\})^{-1},$$

where

$$M\{w\} := -\gamma q_0(w) + {}^t\!dw + \delta$$

(cf. [Bü96],[Kr96]). In fact all biholomorphic automorphisms of $\mathcal{H}^S$ have this form, and they either induce an automorphism of $\mathcal{H}_S$ (and $-\mathcal{H}_S$) or they permute the two connected components $\mathcal{H}_S$ and $-\mathcal{H}_S$ of $\mathcal{H}^S$. A matrix $M \in \mathrm{O}(S_1;\mathbb{R})$ acts trivially on $\mathcal{H}^S$ if and only if M lies in the center $\mathrm{Cent}(\mathrm{O}(S_1;\mathbb{R})) = \{\pm I\}$ of $\mathrm{O}(S_1;\mathbb{R})$. Thus the group of biholomorphic automorphisms of $\mathcal{H}^S$, denoted by $\mathrm{Bihol}(\mathcal{H}^S)$, is isomorphic to $\mathrm{PO}(S_1;\mathbb{R}) := \mathrm{O}(S_1;\mathbb{R})/\{\pm I\}$.

Definition 1.11 *We define*

$$\mathrm{O}^+(S_1;\mathbb{R}) := \{M \in \mathrm{O}(S_1;\mathbb{R});\ M\langle\mathcal{H}_S\rangle = \mathcal{H}_S\}$$

as the subgroup of $\mathrm{O}(S_1;\mathbb{R})$ *stabilizing* $\mathcal{H}_S$.

Remark 1.12 $\mathrm{O}^+(S_1;\mathbb{R})$ *acts transitively on* $\mathcal{H}_S$ *as a group of biholomorphic automorphisms (cf. [Bü96, Satz 2.17]) and we have*

$$\mathrm{Bihol}(\mathcal{H}_S) \cong \mathrm{PO}^+(S_1;\mathbb{R}) := \mathrm{O}^+(S_1;\mathbb{R})/\{\pm I\}.$$

Proposition 1.13 *Let* $M = \left(\begin{smallmatrix} * & * & * \\ C & * & D \end{smallmatrix}\right) \in \mathrm{O}(S_1;\mathbb{R})$, $C, D \in \mathrm{Mat}(2;\mathbb{R})$. *Then*

$$M \in \mathrm{O}^+(S_1;\mathbb{R}) \text{ if and only if } \det\left(C\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + D\right) > 0.$$

PROOF For all $M \in \mathrm{O}(S_1;\mathbb{R})$ we have either $M\langle\mathcal{H}_S\rangle = \mathcal{H}_S$ or $M\langle\mathcal{H}_S\rangle = -\mathcal{H}_S$. Thus $M \in \mathrm{O}^+(S_1;\mathbb{R})$ holds if and only if $M\langle i\mathrm{e}\rangle \in \mathcal{H}_S$ where $i\mathrm{e} = {}^t(i, 0, \dots, 0, i) \in \mathcal{H}_S$. For details confer [Bü96, Satz 2.15]. ∎

1.3. The orthogonal modular group

Let $\Lambda = \mathbb{Z}^l$, $\Lambda_0 = \mathbb{Z} \times \Lambda \times \mathbb{Z}$ and $\Lambda_1 = \mathbb{Z} \times \Lambda_0 \times \mathbb{Z}$. Λ, Λ_0 and Λ_1 are even lattices with respect to S, S_0 and S_1, respectively. The corresponding dual lattices are $\Lambda^\sharp = S^{-1}\mathbb{Z}^l$,

$\Lambda_0^\sharp = \mathbb{Z} \times \Lambda^\sharp \times \mathbb{Z}$ and $\Lambda_1^\sharp = \mathbb{Z} \times \Lambda_0^\sharp \times \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \Lambda^\sharp \times \mathbb{Z} \times \mathbb{Z}$, respectively. Thus we obviously have $\mathrm{Dis}(\Lambda) \cong \mathrm{Dis}(\Lambda_0) \cong \mathrm{Dis}(\Lambda_1)$ where the isomorphisms are given by

$$\mathrm{Dis}(\Lambda) \to \mathrm{Dis}(\Lambda_0),\ \lambda + \Lambda \mapsto (0, \lambda, 0) + \Lambda_0,$$
$$\mathrm{Dis}(\Lambda_0) \to \mathrm{Dis}(\Lambda_1),\ \lambda_0 + \Lambda \mapsto (0, \lambda_0, 0) + \Lambda_1.$$

Moreover, note that

$$\overline{q}(\lambda + \Lambda) = \overline{q}_0((0, \lambda, 0) + \Lambda_0) = \overline{q}_1((0, 0, \lambda, 0, 0) + \Lambda_1) \quad \text{for all } \lambda \in \Lambda^\sharp.$$

Because of this we will often use the three discriminant groups interchangeably, and, by abuse of notation, we will often simply write λ instead of $(0, \lambda, 0)$ or $(0, 0, \lambda, 0, 0)$.

Definition 1.14 *The* orthogonal modular group Γ_S *with respect to S is defined by*

$$\Gamma_S := \mathrm{O}(\Lambda_1) \cap \mathrm{O}^+(S_1; \mathbb{R}).$$

In Section 1.1 we already saw that $\mathrm{O}(\Lambda_1)$ acts on the discriminant group $\mathrm{Dis}(\Lambda_1)$. Thus Γ_S also acts on $\mathrm{Dis}(\Lambda_1)$. We can say even more about this action.

Proposition 1.15 Γ_S *acts on the sets of elements of* $\mathrm{Dis}(\Lambda_1)$ *with the same value of* $\overline{q}_1$*. For*

$$M = \begin{pmatrix} * & * & * \\ * & A & * \\ * & * & * \end{pmatrix} \in \Gamma_S,$$

where $A \in \mathrm{Mat}(l; \mathbb{Z})$*, both, the action of* Γ_S *on* $\mathrm{Dis}(\Lambda_1)$ *and the action of* Γ_S *on the sets of elements of* $\mathrm{Dis}(\Lambda_1)$ *with the same value of* $\overline{q}_1$*, only depend on* A*.*

PROOF For all $M \in \Gamma_S$ we have

$$\overline{q}_1(M(\mu + \Lambda_1)) = \overline{q}_1(M\mu + \Lambda_1) = q_1(M\mu) + \mathbb{Z} = q_1(\mu) + \mathbb{Z} = \overline{q}_1(\mu + \Lambda_1).$$

Since $\mathrm{Dis}(\Lambda_1) = \{0 + \mathbb{Z}\} \times \{0 + \mathbb{Z}\} \times \mathrm{Dis}(\Lambda) \times \{0 + \mathbb{Z}\} \times \{0 + \mathbb{Z}\}$ it is clear that both actions only depend on A. ■

Proposition 1.16 *The following matrices belong to* Γ_S*:*

(1) $\pm I_{l+4}$,

(2) $J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \tilde{J} & 0 \\ -1 & 0 & 0 \end{pmatrix}$, *where* $\tilde{J} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_l & 0 \\ -1 & 0 & 0 \end{pmatrix}$,

(3) $T_g = \begin{pmatrix} 1 & -{}^t g S_0 & -q_0(g) \\ 0 & I_{l+2} & g \\ 0 & 0 & 1 \end{pmatrix}$, $g \in \Lambda_0$,

(4) $U_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widetilde{U}_\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *where* $\widetilde{U}_\lambda = \begin{pmatrix} 1 & {}^t\lambda S & q(\lambda) \\ 0 & I_l & \lambda \\ 0 & 0 & 1 \end{pmatrix}$, $\lambda \in \Lambda$,

(5) ${}_\lambda U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & {}_\lambda\widetilde{U} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *where* ${}_\lambda\widetilde{U} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & I_l & 0 \\ q(\lambda) & {}^t\lambda S & 1 \end{pmatrix}$, $\lambda \in \Lambda$,

(6) $R_g = \begin{pmatrix} \varepsilon_g & 0 & 0 \\ 0 & \widetilde{R}_g & 0 \\ 0 & 0 & \varepsilon_g \end{pmatrix}$, *where* $\widetilde{R}_g = (I_{l+2} - \varepsilon_g g\, {}^t g S_0)\tilde{J}$, *if* $g \in \Lambda_0$ *such that* $\varepsilon_g = q_0(g) = \pm 1$,

(7) $M_D = \begin{pmatrix} \begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix} & 0 & 0 \\ 0 & I_l & 0 \\ 0 & 0 & \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \end{pmatrix}$, $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})$,

(8) $M_D^* := \begin{pmatrix} \begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix} & 0 & \begin{smallmatrix} -\beta & 0 \\ 0 & \beta \end{smallmatrix} \\ 0 & I_l & 0 \\ \begin{smallmatrix} -\gamma & 0 \\ 0 & \gamma \end{smallmatrix} & 0 & \begin{smallmatrix} \delta & 0 \\ 0 & \delta \end{smallmatrix} \end{pmatrix}$, $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})$,

(9) $P = (1) \times \widetilde{P} \times (1)$, *where* $\widetilde{P} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_l & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

(10) $R_A = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_2 \end{pmatrix}$, $A \in \mathrm{O}(\Lambda)$.

PROOF In [Bü96, Prop. 2.27] Bühler proved that matrices of the forms (1)–(7) belong to Γ_S. For the remaining matrices one easily verifies that they belong to $\mathrm{O}^+(S_1;\mathbb{R})$ by using the definition of $\mathrm{O}(S_1;\mathbb{R})$ and the characterisation of $\mathrm{O}^+(S_1;\mathbb{R})$ (Proposition 1.13). It remains to be proved that $M\Lambda_1 = \Lambda_1$ for those matrices. According to the remark following Definition 1.14, this is equivalent to $M \in \mathrm{GL}(l+4;\mathbb{Z})$ which follows immediately from $\det M_D^* = \det\left(\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \times \left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix}\right) \times I_l\right) = 1$ for $D = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{Z})$, $\det P = -1$ and $\det R_A = \det A = \pm 1$ for $A \in \mathrm{O}(\Lambda) \subset \mathrm{GL}(l;\mathbb{Z})$. ∎

The above elements of Γ_S act as follows on $w = (\tau_1, z, \tau_2) \in \mathcal{H}_S$:

$$
\begin{aligned}
J\langle w\rangle &= -q_0(w)^{-1}(\tau_2, -z, \tau_1), && \\
T_g\langle w\rangle &= w + g, && \text{for } g \in \Lambda_0,\\
U_\lambda\langle w\rangle &= (\tau_1 + {}^t\lambda S z + q(\lambda)\tau_2, z + \lambda\tau_2, \tau_2), && \text{for } \lambda \in \Lambda,\\
R_g\langle w\rangle &= q_0(g)\widetilde{R}_g w, && \text{for } g \in \Lambda_0 \text{ with } q_0(g) = \pm 1,\\
M_D\langle w\rangle &= \left(\tau_1 - \frac{\gamma q(z)}{\gamma\tau_2+\delta}, \frac{z}{\gamma\tau_2+\delta}, \frac{\alpha\tau_2+\beta}{\gamma\tau_2+\delta}\right), && \text{for } D = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}),\\
M_D^*\langle w\rangle &= \left(\frac{\alpha\tau_1+\beta}{\gamma\tau_1+\delta}, \frac{z}{\gamma\tau_1+\delta}, \tau_2 - \frac{\gamma q(z)}{\gamma\tau_1+\delta}\right), && \text{for } D = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}),\\
P\langle w\rangle &= (\tau_2, z, \tau_1), && \\
R_A\langle w\rangle &= (\tau_1, Az, \tau_2), && \text{for } A \in \mathrm{O}(\Lambda).
\end{aligned}
$$

1.4. Generators of certain orthogonal modular groups

In this section we will show that for certain S the orthogonal modular group Γ_S is nicely generated. We will consider the following matrices:

$$
\begin{gathered}
D_4 = \begin{pmatrix}2&0&0&1\\0&2&0&1\\0&0&2&1\\1&1&1&2\end{pmatrix},\ A_1^{(3)} = \begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix},\ A_3 = \begin{pmatrix}2&1&0\\1&2&1\\0&1&2\end{pmatrix},\\
A_1^{(2)} = \begin{pmatrix}2&0\\0&2\end{pmatrix},\ A_2 = \begin{pmatrix}2&1\\1&2\end{pmatrix},\ S_2 = \begin{pmatrix}2&0\\0&4\end{pmatrix}.
\end{gathered}
\tag{1.2}
$$

The quadratic spaces associated to those matrices are isomorphic to subspaces of the Hamilton quaternions $\mathbb{H}$. Since we will later make use of this fact we now fix some concrete isomorphisms. We denote the canonical basis of $\mathbb{H}$ by $1, \mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3$. Then for $z = z_1 + z_2\mathrm{i}_1 + z_3\mathrm{i}_2 + z_4\mathrm{i}_3 \in \mathbb{H}$ with $z_j \in \mathbb{R}$ the conjugate of z is given by $\overline{z} = z_1 - z_2\mathrm{i}_1 - z_3\mathrm{i}_2 - z_4\mathrm{i}_3$ and the norm of z is given by $N(z) = z\overline{z} = z_1^2 + z_2^2 + z_3^2 + z_4^2$. The Hurwitz order is denoted by

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\mathrm{i}_1 + \mathbb{Z}\mathrm{i}_2 + \mathbb{Z}\omega, \quad \omega = \frac{1}{2}(1 + \mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_3).$$

Proposition 1.17 *Let S be one of the matrices listed in (1.2), and let l be the rank of S. Then the quadratic space $(\mathbb{R}^l, q_S)$ with lattice $\Lambda = \mathbb{Z}^l$ is isomorphic to the quadratic space $(\mathbb{H}_S, N_S)$ with lattice $\mathcal{O}_S$ where $\mathbb{H}_S$ is a subspace of the Hamilton quaternions $\mathbb{H}$, N_S is the restriction of the norm N to $\mathbb{H}_S$, and $\mathcal{O}_S = \mathcal{O} \cap \mathbb{H}_S$ is the sublattice of the Hurwitz order $\mathcal{O}$ in $\mathbb{H}_S$. The following list contains the subspaces $\mathbb{H}_S$, the corresponding lattices $\mathcal{O}_S$, one possible isomorphism $\iota_S : \mathbb{R}^l \to \mathbb{H}_S$ and the quadratic forms $q_S = N_S \circ \iota_S$.*

a) $\mathbb{H}_{D_4} = \mathbb{H}$, $\mathcal{O}_{D_4} = \mathcal{O}$,

$\iota_{D_4} : \mathbb{R}^4 \to \mathbb{H}$, $(x_1, x_2, x_3, x_4) \mapsto x_1 + x_2\mathrm{i}_1 + x_3\mathrm{i}_2 + x_4\omega$,

$q_{D_4}(x) = x_1^2 + x_1x_4 + x_2^2 + x_2x_4 + x_3^2 + x_3x_4 + x_4^2$,

$\mathrm{Dis}(\Lambda) = \langle(\frac{1}{2}, \frac{1}{2}, 0, 0) + \Lambda,\ (\frac{1}{2}, 0, \frac{1}{2}, 0) + \Lambda\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

b) $\mathbb{H}_{A_1^{(3)}} = \{x \in \mathbb{H};\ x_4 = 0\}$, $\mathcal{O}_{A_1^{(3)}} = \mathbb{Z} + \mathbb{Z}\mathrm{i}_1 + \mathbb{Z}\mathrm{i}_2$,

$\iota_{A_1^{(3)}} : \mathbb{R}^3 \to \mathbb{H}_{A_1^{(3)}}$, $(x_1, x_2, x_3) \mapsto x_1 + x_2\mathrm{i}_1 + x_3\mathrm{i}_2$,

$q_{A_1^{(3)}}(x) = x_1^2 + x_2^2 + x_3^2$,

$\mathrm{Dis}(\Lambda) = \langle(\frac{1}{2}, 0, 0) + \Lambda,\ (0, \frac{1}{2}, 0) + \Lambda,\ (0, 0, \frac{1}{2}) + \Lambda\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

c) $\mathbb{H}_{A_3} = \{x \in \mathbb{H};\ x_3 = x_4\}$, $\mathcal{O}_{A_3} = \mathbb{Z} + \mathbb{Z}\omega + \mathbb{Z}\mathrm{i}_1$,

$\iota_{A_3} : \mathbb{R}^3 \to \mathbb{H}_{A_3}$, $(x_1, x_2, x_3) \mapsto x_1 + x_2\omega + x_3\mathrm{i}_1$,

$q_{A_3}(x) = x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2$,

$\mathrm{Dis}(\Lambda) = \langle(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}) + \Lambda\rangle \cong \mathbb{Z}_4$.

d) $\mathbb{H}_{A_1^{(2)}} = \{x \in \mathbb{H};\ x_3 = x_4 = 0\}$, $\mathcal{O}_{A_1^{(2)}} = \mathbb{Z} + \mathbb{Z}\mathrm{i}_1$,

$\iota_{A_1^{(2)}} : \mathbb{R}^2 \to \mathbb{H}_{A_1^{(2)}}$, $(x_1, x_2) \mapsto x_1 + x_2\mathrm{i}_1$,

$q_{A_1^{(2)}}(x) = x_1^2 + x_2^2$,

$\mathrm{Dis}(\Lambda) = \langle(\frac{1}{2}, 0) + \Lambda,\ (0, \frac{1}{2}) + \Lambda\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

e) $\mathbb{H}_{A_2} = \{x \in \mathbb{H};\ x_2 = x_3 = x_4\}$, $\mathcal{O}_{A_2} = \mathbb{Z} + \mathbb{Z}\omega$,

$\iota_{A_2} : \mathbb{R}^2 \to \mathbb{H}_{A_2}$, $(x_1, x_2) \mapsto x_1 + x_2\omega$,

$q_{A_2}(x) = x_1^2 + x_1x_2 + x_2^2$,

$\mathrm{Dis}(\Lambda) = \langle(\frac{1}{3}, \frac{1}{3}) + \Lambda\rangle \cong \mathbb{Z}_3$.

f) $\mathbb{H}_{S_2} = \{x \in \mathbb{H};\ x_2 = x_3,\ x_4 = 0\}$, $\mathcal{O}_{S_2} = \mathbb{Z} + \mathbb{Z}(\mathrm{i}_1 + \mathrm{i}_2)$,

$\iota_{S_2} : \mathbb{R}^2 \to \mathbb{H}_{S_2}$, $(x_1, x_2) \mapsto x_1 + x_2(\mathrm{i}_1 + \mathrm{i}_2)$,

$q_{S_2}(x) = x_1^2 + 2x_2^2$,

$\mathrm{Dis}(\Lambda) = \langle(\frac{1}{2}, 0) + \Lambda,\ (0, \frac{1}{4}) + \Lambda\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

PROOF Explicit calculations show that the quadratic forms are preserved under the isomorphisms, i.e., that $q_S = N_S \circ \iota_S$. ∎

Next we will show in several steps that the orthogonal modular groups associated to the above matrices are nicely generated. We start by defining what we mean by "nicely generated".

Definition 1.18 *The orthogonal modular group Γ_S is* nicely generated *if it is generated by the inversion J, the translations T_g, $g \in \Lambda_0$, and the rotations R_A, $A \in \mathrm{O}(\Lambda)$.*

Remark *Γ_S is nicely generated in the above sense if and only if the corresponding group in the terminology of [FH00] is nicely generated in the sense of [FH00, Def. 4.7] (cf. Appendix C).*

In a first step, using results from [Bü96], we reduce the problem of determining generators of $\Gamma_S \subset \mathrm{O}(\Lambda_1)$ to the problem of determining generators of a certain subgroup of $\mathrm{O}(\Lambda_0)$.

Proposition 1.19 *Γ_S is generated by*

$$J,\ T_g\ (g \in \Lambda_0),\ \textit{and}\ \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix};\ A \in \mathrm{O}^+(\Lambda_0) \right\} \cap \Gamma_S,$$

where $\mathrm{O}^+(\Lambda_0) := \{A \in \mathrm{O}(\Lambda_0);\ A \cdot \mathcal{H}_S = \mathcal{H}_S\}$.

PROOF According to [Bü96, Satz 2.31], Γ_S is generated by

$$J \text{ and } \Gamma_{S,0} := \left\{ M \in \Gamma_S;\ M = \begin{pmatrix} 1 & {}^t a & \beta \\ 0 & A & c \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

So let $M = \begin{pmatrix} 1 & {}^t a & \beta \\ 0 & A & c \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_{S,0}$. Then by virtue of (1.1) we have $a = -{}^t A S_0 c$, $\beta = -q_0(c)$ and $S_0[A] = S_0$, and thus, in particular, $A \in \mathrm{O}(S_0;\mathbb{R}) \cap \mathrm{GL}(l+2;\mathbb{Z}) = \mathrm{O}(\Lambda_0)$. By multiplication with $T_c^{-1} = T_{-c}$ we get

$$T_{-c}M = \begin{pmatrix} 1 & {}^t c S_0 & -q_0(c) \\ 0 & I_{n+2} & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -{}^t c S_0 A & -q_0(c) \\ 0 & A & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally $(1) \times A \times (1) \in \mathrm{O}^+(S_1;\mathbb{R})$ yields $\mathcal{H}_S = ((1) \times A \times (1))\langle \mathcal{H}_S \rangle = A \cdot \mathcal{H}_S$. Hence $A \in \mathrm{O}^+(\Lambda_0)$. This completes the proof. ∎

Next we show that the lattices Λ associated to the above matrices are Euclidean. This will allow us to reduce the problem of determining generators of a subgroup of $\mathrm{O}(\Lambda_0)$ to the problem of determining generators of the finite groups $\mathrm{O}(\Lambda)$.

Proposition 1.20 *a) Given $a \in \mathbb{H}_{A_3}$ there exists $g \in \mathcal{O}_{A_3}$ such that*

$$a - g = b_1 + b_2 \mathrm{i}_1 + b_3(\mathrm{i}_2 + \mathrm{i}_3) \quad \textit{with} \quad |b_j| \le \frac{1}{2},\ 1 \le j \le 3, \quad \textit{and} \quad \sum_{j=1}^{3} |b_j| \le \frac{3}{4}.$$

b) *Given* $a \in \mathbb{H}_{A_3}$ *there exists* $g \in \mathcal{O}_{A_3}$ *such that*

$$N_{A_3}(a-g) \leq \frac{9}{16}.$$

PROOF a) Let $a = a_1 + a_2\mathrm{i}_1 + a_3(\mathrm{i}_2 + \mathrm{i}_3) \in \mathbb{H}_{A_3}$. Because of $\mathbb{Z} + \mathbb{Z}\mathrm{i}_1 + \mathbb{Z}(\mathrm{i}_2 + \mathrm{i}_3) \subset \mathcal{O}_{A_3}$ we may assume $|a_j| \leq \frac{1}{2}$ for $1 \leq j \leq 3$. If $\sum_{j=1}^{3} |a_j| > \frac{3}{4}$ then we choose $g_j = \frac{1}{2}\operatorname{sign} a_j$ for $1 \leq j \leq 3$. Then $g = g_1 + g_2\mathrm{i}_1 + g_3(\mathrm{i}_2 + \mathrm{i}_3) \in \mathcal{O}_{A_3}$ with $|a_j - g_j| = \frac{1}{2} - |a_j| \leq \frac{1}{2}$ for $1 \leq j \leq 3$ and $\sum_{j=1}^{3} |a_j - g_j| = \frac{3}{2} - \sum_{j=1}^{3} |a_j| < \frac{3}{4}$.

b) Because of a) it remains to be shown that

$$\varphi(b_1, b_2, b_3) := N_{A_3}(b_1 + b_2\mathrm{i}_1 + b_3(\mathrm{i}_2 + \mathrm{i}_3)) = b_1^2 + b_2^2 + 2b_3^2 \leq \frac{9}{16}$$

whenever $(b_1, b_2, b_3) \in A := \{(b_1, b_2, b_3) \in \mathbb{R}^3;\ 0 \leq b_j \leq \frac{1}{2},\ \sum_{j=1}^{3} b_j \leq \frac{3}{4}\}$. We choose $(b_1, b_2, b_3) \in A$ such that $\varphi(b_1, b_2, b_3)$ is maximal. Since we have $\varphi(b_1, b_2, b_3) = \varphi(b_2, b_1, b_3)$ we may assume $b_2 \geq b_1$. Furthermore, due to the choice of (b_1, b_2, b_3) we have $0 \leq \varphi(b_1, b_2, b_3) - \varphi(b_1, b_3, b_2) = b_3^2 - b_2^2$ which implies $b_3 \geq b_2$. Thus $b_2 \leq \frac{3}{8}$.
If $b_1 > 0$ then $b_2 < \frac{3}{8}$ and there exists $\varepsilon > 0$ such that $b_1 - \varepsilon > 0$ and $b_2 + \varepsilon < \frac{1}{2}$. Then

$$\varphi(b_1 - \varepsilon, b_2 + \varepsilon, b_3) - \varphi(b_1, b_2, b_3) = 2\varepsilon(b_2 - b_1) + 2\varepsilon^2 > 0$$

yields a contradiction to the choice of (b_1, b_2, b_3). Hence $b_1 = 0$.
If $b_2 > \frac{1}{4}$ then $b_3 < \frac{1}{2}$ and there exists $\varepsilon > 0$ such that $b_2 - \varepsilon > 0$ and $b_3 + \varepsilon < \frac{1}{2}$. Then

$$\varphi(b_1, b_2 - \varepsilon, b_3 + \varepsilon) - \varphi(b_1, b_2, b_3) = 2\varepsilon(2b_3 - b_2) + 3\varepsilon^2 > 0$$

yields a contradiction to the choice of (b_1, b_2, b_3). Hence $b_2 \leq \frac{1}{4}$.
Therefore

$$\max_{(b_1,b_2,b_3)\in A} \varphi(b_1, b_2, b_3) \leq 0 + \frac{1}{16} + 2 \cdot \frac{1}{4} = \frac{9}{16}.$$

■

Proposition 1.21 *Let S be one of the matrices listed in (1.2), and let Λ be the associated lattice. Then for all $x \in \mathbb{R}^l$ there exists $\lambda \in \Lambda$ such that*

$$q_S(x + \lambda) \leq c(S),$$

where $c(D_4) = c(A_1^{(2)}) = \frac{1}{2}$, $c(A_3) = \frac{9}{16}$ and $c(A_1^{(3)}) = c(S_2) = c(A_2) = \frac{3}{4}$. In particular, Λ is Euclidean.

PROOF Let $x = {}^t(x_1, \ldots, x_l) \in \mathbb{R}^l$. Because of $\Lambda = \mathbb{Z}^l$ we may assume $|x_j| \leq \frac{1}{2}$ for $1 \leq j \leq l$. Then for $S \in \{A_1^{(3)}, A_1^{(2)}, S_2, A_2\}$ the assertion is obvious. By virtue of Proposition 1.17, for $S = D_4$ the assertion follows from [Kr85, 1.7] and for $S = A_3$ it follows from Proposition 1.20. ■

Proposition 1.22 *Let*

$$\widetilde{\Gamma}_S := \left\{ M \in \mathrm{O}^+(\Lambda_0);\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_S \right\}.$$

a) We have

$$\widetilde{\Gamma}_S = \left\{ \begin{pmatrix} \alpha & {}^t a & \beta \\ b & A & c \\ \gamma & {}^t d & \delta \end{pmatrix} \in \mathrm{O}(\Lambda_0);\ \gamma + \delta > 0 \right\}.$$

In particular, the following matrices are elements of $\widetilde{\Gamma}_S$*:*

$$\widetilde{P},\ \widetilde{U}_\lambda\ (\lambda \in \Lambda),\ \textit{and}\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}\ (A \in \mathrm{O}(\Lambda)).$$

b) Suppose that Λ *is Euclidean. Then given* $\mu_0 \in \Lambda_0^\sharp$ *with* $q_0(\mu_0) = 0$ *there exists* $M \in \left\langle \widetilde{P}, \widetilde{U}_\lambda;\ \lambda \in \Lambda \right\rangle \leq \widetilde{\Gamma}_S$ *such that* $M\mu_0 = {}^t(m, 0, \ldots, 0)$ *for some* $m \in \mathbb{Z}$*.*

c) Suppose that Λ *is Euclidean. Then* $\widetilde{\Gamma}_S$ *is generated by*

$$\widetilde{P},\ \widetilde{U}_\lambda\ (\lambda \in \Lambda),\ \textit{and}\ \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix};\ A \in \mathrm{O}(\Lambda) \right\}.$$

PROOF a) Let $M = \begin{pmatrix} \alpha & {}^t a & \beta \\ b & A & c \\ \gamma & {}^t d & \delta \end{pmatrix} \in \tilde{\Gamma}_S$. Then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_S \subset \mathrm{O}^+(S_1; \mathbb{R})$ yields

$$\det\left(\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \right) = \gamma + \delta > 0.$$

$\widetilde{P}$, $\widetilde{U}_\lambda$ for all $\lambda \in \Lambda$ and $(1) \times A \times (1)$ for all $A \in \mathrm{O}(\Lambda)$ obviously satisfy this condition and are thus elements of $\widetilde{\Gamma}_S$.

b) Let $\mu_0 = (m, \mu, n) \in \Lambda_0^\sharp$, i.e., $m, n \in \mathbb{Z}$ and $\mu \in \Lambda^\sharp$. Without restriction we may assume that $|n| \leq |m|$ (otherwise we consider $\widetilde{P}\mu_0$). $q_0(\mu_0) = 0$ implies $mn = q(\mu)$. So if $\mu = 0$ then $n = 0$ and thus $\mu_0 = {}^t(m, 0, \ldots, 0)$. Otherwise, since Λ is Euclidean there exists $\lambda \in \Lambda$ such that $q(\mu + n\lambda) = n^2 q(\frac{1}{n}\mu + \lambda) < n^2$. We consider

$$\widetilde{U}_\lambda \mu_0 = \begin{pmatrix} m + {}^t\lambda S\mu + nq(\lambda) \\ \mu + n\lambda \\ n \end{pmatrix} = \begin{pmatrix} m' \\ \mu' \\ n \end{pmatrix}.$$

Due to $\widetilde{U}_\lambda \in \mathrm{O}(\Lambda_0)$ we have $\widetilde{U}_\lambda\mu_0 \in \Lambda_0^\sharp$ and $q_0(\widetilde{U}_\lambda\mu_0) = q_0(\mu_0) = 0$. Thus $m' \in \mathbb{Z}$ and

$m'n = q(\mu + n\lambda) < n^2$. This yields $|m'| < |n|$. Therefore, after finitely many steps we get the matrix $M \in \widetilde{\Gamma}_S$ we are looking for.

c) Let $M = \begin{pmatrix} \alpha & {}^t a & \beta \\ b & A & c \\ \gamma & {}^t d & \delta \end{pmatrix} \in \widetilde{\Gamma}_S$. Then $S_0 = S_0[M]$ yields $\alpha\gamma = q(b)$. Therefore, by virtue of b), there exists an $\widetilde{M} \in \left\langle \widetilde{P}, \widetilde{U}_\lambda;\ \lambda \in \Lambda \right\rangle \leq \widetilde{\Gamma}_S$ such that

$$M' = \widetilde{M}M = \begin{pmatrix} \alpha' & * & * \\ 0 & A' & * \\ 0 & {}^t d' & \delta' \end{pmatrix}.$$

Due to $M' \in \mathrm{O}(\Lambda_0)$ we have $d' = 0$ and $A' \in \mathrm{O}(\Lambda)$ (and thus also $A'^{-1} \in \mathrm{O}(\Lambda)$). Then

$$M'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A'^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} M' = \begin{pmatrix} \alpha' & * & * \\ 0 & I & c'' \\ 0 & 0 & \delta' \end{pmatrix}.$$

Now $M'' \in \mathrm{O}(\Lambda_0)$ yields $\alpha'\delta' = 1$ with $\alpha', \delta' \in \mathbb{Z}$ and a) yields $\delta' > 0$. Therefore $\alpha' = \delta' = 1$. Multiplying with $\widetilde{U}_{-c''}$ we get

$$M''' = \widetilde{U}_{-c''}M'' = \begin{pmatrix} 1 & {}^t a'' & \beta'' \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally $a'' = 0$ and $\beta'' = 0$ follow from $M''' \in \mathrm{O}(S_0;\mathbb{R})$. ■

Corollary 1.23 *If S is one of the matrices listed in (1.2) then Γ_S is nicely generated.*

PROOF Due to Proposition 1.21 $\Lambda = \Lambda_S$ is Euclidean. Therefore, Proposition 1.19 and Proposition 1.22 yield that Γ_S is generated by

$$J,\ T_g\ (g \in \Lambda_0),\ P,\ U_\lambda\ (\lambda \in \Lambda),\ \text{and}\ R_A\ (A \in \mathrm{O}(\Lambda)).$$

According to [Kr96, p. 249f], U_λ and R_g can be written as product of J and T_h for certain $h \in \Lambda_0$. Furthermore, we have $P = -R_{(0,1,0,\ldots,0)}M_{\mathrm{tr}} = R_{(1,0,\ldots,0,-1)}R_{(-I_l)}R_{(0,1,0,\ldots,0)}M_{\mathrm{tr}}$ with $M_{\mathrm{tr}} = M_{\mathrm{tr}}^S$ as defined below in (1.3). Thus only J, T_g, $g \in \Lambda_0$, and R_A, $A \in \mathrm{O}(\Lambda)$, are needed to generate Γ_S. ■

Finally we determine some properties of the finite orthogonal groups $\mathrm{O}(\Lambda)$ associated to the matrices listed in (1.2).

Proposition 1.24 *a) Let $S = D_4$. Then $\mathrm{O}(\Lambda)$ is generated by*

$$\begin{pmatrix}1&0&0&1\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ \begin{pmatrix}1&0&-1&0\\0&1&-1&0\\0&0&-1&-1\\0&0&2&1\end{pmatrix},\ \begin{pmatrix}1&0&-1&0\\0&0&-1&0\\0&1&-1&0\\0&0&2&1\end{pmatrix},\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&0&1\\-1&-1&-1&-1\end{pmatrix}.$$

$\mathrm{SO}(\Lambda) = \mathrm{O}(\Lambda) \cap \mathrm{SL}(4;\mathbb{Z})$ is generated by the last three matrices. The commutator subgroup $\mathrm{O}(\Lambda)'$ is generated by the last two matrices. The commutator factor group $\mathrm{O}(\Lambda)^{\mathrm{ab}}$ is isomorphic to $C_2 \times C_2$ where C_2 denotes the cyclic group of order 2. The first two matrices are representatives for the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$. The discriminant kernel $\mathrm{O_d}(\Lambda)$ is generated by the first, the square of the second and the last matrix, and the factor group $\mathrm{O}(\Lambda)/\mathrm{O_d}(\Lambda)$ is isomorphic to $S(3)$, the symmetric group of degree 3.

b) Let $S = A_1^{(3)}$. Then $\mathrm{O}(\Lambda)$ is generated by

$$-I_3,\quad \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\quad \textit{and}\quad \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

$\mathrm{SO}(\Lambda)$ is generated by the last four matrices and is isomorphic to $S(4)$. The commutator subgroup $\mathrm{O}(\Lambda)'$ is generated by the last three matrices and is isomorphic to $A(4)$, the alternating group of degree 4. The commutator factor group $\mathrm{O}(\Lambda)^{\mathrm{ab}}$ is isomorphic to $C_2 \times C_2$. The first two matrices are representatives for the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$. The discriminant kernel $\mathrm{O_d}(\Lambda)$ is generated by the three diagonal matrices, and the factor group $\mathrm{O}(\Lambda)/\mathrm{O_d}(\Lambda)$ is isomorphic to $S(3)$.

c) Let $S = A_3$. Then $\mathrm{O}(\Lambda)$ is generated by

$$\begin{pmatrix}-1&-1&0\\0&1&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&1&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad \begin{pmatrix}-1&-1&0\\0&1&0\\0&-1&-1\end{pmatrix}\quad \textit{and}\quad \begin{pmatrix}0&0&-1\\-1&0&1\\1&1&0\end{pmatrix}.$$

$\mathrm{SO}(\Lambda)$ is generated by the last three matrices and is isomorphic to $S(4)$. The commutator subgroup $\mathrm{O}(\Lambda)'$ is generated by the last two matrices and is isomorphic to $A(4)$. The commutator factor group $\mathrm{O}(\Lambda)^{\mathrm{ab}}$ is isomorphic to $C_2 \times C_2$. The first two matrices are representatives for the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$. The discriminant kernel $\mathrm{O_d}(\Lambda)$ is generated by the first and the last two matrices, and the factor group $\mathrm{O}(\Lambda)/\mathrm{O_d}(\Lambda)$ is isomorphic to C_2.

d) Let $S = A_1^{(2)}$. Then $\mathrm{O}(\Lambda)$ is generated by

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad \textit{and}\quad \begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

and is isomorphic to D_8, the dihedral group of order 8. $\mathrm{SO}(\Lambda)$ is generated by the second matrix and is isomorphic to C_4. The commutator subgroup $\mathrm{O}(\Lambda)'$ is generated *by $-I_2$ and is isomorphic to C_2. The commutator factor group $\mathrm{O}(\Lambda)^{\mathrm{ab}}$ is isomorphic to $C_2 \times C_2$. The two matrices are representatives for the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$. The discriminant kernel $\mathrm{O}_{\mathrm{d}}(\Lambda)$ is generated by $-I_2$ and the first matrix, and the factor group $\mathrm{O}(\Lambda)/\mathrm{O}_{\mathrm{d}}(\Lambda)$ is isomorphic to C_2.*

e) Let $S = A_2$. Then $\mathrm{O}(\Lambda)$ is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

and is isomorphic to D_{12}. $\mathrm{SO}(\Lambda)$ is generated by the second matrix and is isomorphic to C_6. The commutator subgroup $\mathrm{O}(\Lambda)'$ is generated by the square of the second matrix (i.e., by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$) and is isomorphic to C_3. The commutator factor group $\mathrm{O}(\Lambda)^{\mathrm{ab}}$ is isomorphic to $C_2 \times C_2$. The two matrices are representatives for the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$. The discriminant kernel $\mathrm{O}_{\mathrm{d}}(\Lambda)$ is generated by the first and the square of the second matrix, and the factor group $\mathrm{O}(\Lambda)/\mathrm{O}_{\mathrm{d}}(\Lambda)$ is isomorphic to C_2.

f) Let $S = S_2$. Then $\mathrm{O}(\Lambda)$ is generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad -I_2$$

and is isomorphic to $C_2 \times C_2$. $\mathrm{SO}(\Lambda)$ is generated by $-I_2$ and is isomorphic to C_2. The commutator subgroup $\mathrm{O}(\Lambda)'$ is the trivial group. The commutator factor group $\mathrm{O}(\Lambda)^{\mathrm{ab}}$ is isomorphic to $C_2 \times C_2$. The two matrices are representatives for the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$. The discriminant kernel $\mathrm{O}_{\mathrm{d}}(\Lambda)$ is generated by the first matrix, and the factor group $\mathrm{O}(\Lambda)/\mathrm{O}_{\mathrm{d}}(\Lambda)$ is isomorphic to C_2.

PROOF The generators were explicitly calculated. The rest of the assertions was verified with GAP ([GAP05]). ∎

All of the above groups $\mathrm{O}(\Lambda)$ contain an element which corresponds to the conjugation on the corresponding subspaces $\mathbb{H}_S$ of $\mathbb{H}$ (cf. Proposition 1.17), i.e., for all matrices S listed in (1.2) there is an $A_S \in \mathrm{O}(\Lambda)$ such that $\iota_S(A_S \iota_S^{-1}(z)) = \overline{z}$ for all $z \in \mathbb{H}_S$. We denote the corresponding rotations R_{A_S} by M_{tr}^S or, if it is clear which S is meant, simply by M_{tr}. We have

$$\begin{gathered} M_{\mathrm{tr}}^{D_4} = R_{\left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right)}, \qquad M_{\mathrm{tr}}^{A_1^{(3)}} = R_{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)}, \qquad M_{\mathrm{tr}}^{A_3} = R_{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)}, \\ M_{\mathrm{tr}}^{A_1^{(2)}} = R_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}, \qquad M_{\mathrm{tr}}^{A_2} = R_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)}, \qquad M_{\mathrm{tr}}^{S_2} = R_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}. \end{gathered} \tag{1.3}$$

1.5. The commutator subgroups of certain orthogonal modular groups

In this section we proof an estimation for the index of the commutator subgroup Γ'_S in Γ_S for the matrices S listed in (1.2). In the next section we show that the inequalities are in fact equalities.

Proposition 1.25 *a) If* $S \in \{A_1^{(3)}, A_1^{(2)}, S_2\}$ *then* $[\Gamma_S : \Gamma'_S] \leq 8$.
b) If $S \in \{D_4, A_3, A_2\}$ *then* $[\Gamma_S : \Gamma'_S] \leq 4$.

PROOF a) First we will show that $[\Gamma_S : \Gamma'_S] \leq 8$ for all of the matrices S listed in (1.2). So let S be one of those matrices. We start by calculating a few commutators. For $\lambda \in \Lambda$ and $g = (g_0, \widetilde{g}, g_{l+1}) \in \Lambda_0$ we get

$$[U_\lambda, T_g] = T_{({}^t\lambda S\widetilde{g}+q(\lambda)g_{l+1}, \lambda g_{l+1}, 0)}.$$

Thus for the standard basis $(e_1, \ldots, e_l)$ of Λ we get

$$[U_{e_j}, T_{(0,\ldots,0,1)}] = T_{(q(e_j),e_j,0)} = \begin{cases} T_{(2,0,1,0)} & \text{if } S = S_2 \text{ and } j = 2, \\ T_{(1,e_j,0)} & \text{otherwise,} \end{cases}$$

and

$$[U_{e_1}, T_{(0,e_1,0)}] = T_{(2q(e_1),0,\ldots,0)} = T_{(2,0,\ldots,0)}.$$

Furthermore,

$$[R_{(0,e_1,0)}, T_{(0,\ldots,0,1)}] = T_{(1,0,\ldots,0,-1)}.$$

Because of $T_gT_h = T_{g+h}$ for all $g,\ h \in \Lambda_0$ this yields

$$T_g \in \Gamma'_S \text{ for all } g = {}^t(g_0, \ldots, g_{l+1}) \in \Lambda_0 \text{ with } g_0 + g_{l+1} + \sum_{j=1}^{l} \frac{s_j}{2} g_j \equiv 0 \pmod 2,$$

where $s_j \in 2\mathbb{Z}$, $1 \leq j \leq l$, are the diagonal entries of S. So modulo Γ'_S all matrices T_g with $g \in \Lambda_0$ are equivalent either to I_{l+4} or to $T_{(1,0,\ldots,0)}$. Moreover, $(JT_{(1,0,\ldots,0,1)})^3 = 1$ yields $J = J^3 \in \Gamma'_S$, and, due to $R_AR_B = R_{AB}$ and $R_A^{-1} = R_{A^{-1}}$ for all $A, B \in \mathrm{O}(\Lambda)$, we have $R_A \in \Gamma'_S$ for all $A \in \mathrm{O}(\Lambda)'$.
According to Corollary 1.23, each element of Γ_S can be written as a product of J, T_g, $g \in \Lambda_0$, and R_A, $A \in \mathrm{O}(\Lambda)$. Since, by virtue of Proposition 1.24, $\mathrm{O}(\Lambda)^{\mathrm{ab}} \cong C_2 \times C_2$ for all S we are considering we have $[\Gamma_S : \Gamma'_S] \leq 8$.

b) If $S \in \{D_4, A_3, A_2\}$ then

$$[R_{(0,e_j,0)}, T_{(0,e_1,0)}] = T_{(0,-2e_1+e_j,0)},$$

where e_j is the vector which is mapped to ω under the isomorphisms in Proposition

1.17, i.e., $j = 4$ if $S = D_4$ and $j = 2$ if $S \in \{A_3, A_2\}$. Therefore, $T_g \in \Gamma'_S$ for all $g \in \Lambda_0$, and thus $[\Gamma_S : \Gamma'_S] \leq 4$. ∎

1.6. Abelian characters of the orthogonal modular groups

The Abelian characters of the orthogonal modular group Γ_S are in one-to-one correspondence to the elements of the corresponding commutator factor group Γ_S^{ab}. Because of this correspondence we denote the group of Abelian characters of Γ_S also by Γ_S^{ab}. According to Proposition 1.25, for all S listed in (1.2) the commutator factor groups are finite (Abelian) groups of order 4 or 8, and thus at most three different characters (and their products) occur.

1.6.1. The determinant

The determinant occurs in all cases as character of the orthogonal modular groups Γ_S. The determinant is -1 for R_A if A is the first generator of $\mathrm{O}(\Lambda)$ given in Proposition 1.24, and it is 1 for J, T_g, $g \in \Lambda_0$, and R_A, $A \in \mathrm{SO}(\Lambda)$.

1.6.2. The orthogonal character(s)

According to Proposition 1.15, Γ_S acts on the sets of elements of $\mathrm{Dis}(\Lambda_1)$ with the same value of $\overline{q}_1$, and for

$$M = \begin{pmatrix} * & * & * \\ * & A & * \\ * & * & * \end{pmatrix} \in \Gamma_S$$

the action only depends on $A \in \mathrm{Mat}(l; \mathbb{Z})$. The signs of the permutations of non-trivial sets of elements of $\mathrm{Dis}(\Lambda_1)$ with the same value of $\overline{q}_1$ are Abelian characters of Γ_S. In all cases we are considering exactly one such character occurs. We denote this character by ν_π. It is -1 for R_A if A is the second generator of $\mathrm{O}(\Lambda)$ given in Proposition 1.24, and it is 1 for J, T_g, $g \in \Lambda_0$, and R_A if A is one of the other generators of $\mathrm{O}(\Lambda)$ given in Proposition 1.24.

1.6.3. The Siegel character

Let $S \in \{A_1^{(3)}, A_1^{(2)}, S_2\}$. Then $S \equiv 0 \pmod 2$. In this case another character occurs. It corresponds to the non-trivial character of the Siegel modular group of degree 2.

Proposition 1.26 *If* $S \equiv 0 \pmod 2$ *then the map*

$$\varphi : \Gamma_S \to \mathrm{Sp}(2; \mathbb{F}_2), \quad \begin{pmatrix} \alpha & * & \beta \\ * & * & * \\ \gamma & * & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{bmatrix} H & 0 \\ 0 & I_2 \end{bmatrix} \bmod 2 = \begin{pmatrix} H\alpha H & H\beta \\ \gamma H & \delta \end{pmatrix} \bmod 2,$$

where $\alpha, \beta, \gamma, \delta \in \mathrm{Mat}(2;\mathbb{Z})$, $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\mathrm{Sp}(2;\mathbb{F}_2)$ *is the symplectic group of degree* 2 *over the field* $\mathbb{F}_2$ *of two elements, is a surjective homomorphism of groups.*

PROOF Let $S \equiv 0 \pmod 2$ and

$$M_j = \begin{pmatrix} \alpha_j & a_j & \beta_j \\ b_j & A_j & c_j \\ \gamma_j & d_j & \delta_j \end{pmatrix} \in \Gamma_S,$$

where $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathrm{Mat}(2;\mathbb{Z})$, $a_j, d_j \in \mathrm{Mat}(2,l;\mathbb{Z})$, $b_j, c_j \in \mathrm{Mat}(l,2;\mathbb{Z})$, $A_j \in \mathrm{Mat}(l;\mathbb{Z})$ for $j \in \{1,2\}$. If $a_1 \equiv a_2 \equiv d_1 \equiv d_2 \equiv 0 \pmod 2$ then

$$M_1 M_2 \equiv \begin{pmatrix} \alpha_1\alpha_2 + \beta_1\gamma_2 & 0 & \alpha_1\beta_2 + \beta_1\delta_2 \\ * & A_1A_2 & * \\ \gamma_1\alpha_2 + \delta_1\gamma_2 & 0 & \gamma_1\beta_2 + \delta_1\delta_2 \end{pmatrix} \pmod 2.$$

Since the assumption is true for the generators J, T_g, $g \in \Lambda_0$, and R_A, $A \in \mathrm{O}(\Lambda)$, of Γ_S we find $a_j \equiv d_j \equiv 0 \pmod 2$ for all $M_j \in \Gamma_S$. An easy calculation shows that the images of the generators of Γ_S under φ are in $\mathrm{Sp}(2;\mathbb{F}_2)$. Together with

$$\begin{aligned} \varphi(M_1M_2) &= \begin{pmatrix} H\alpha_1\alpha_2H + H\beta_1\gamma_2H & H\alpha_1\beta_2 + H\beta_1\delta_2 \\ \gamma_1\alpha_2H + \delta_1\gamma_2H & \gamma_1\beta_2 + \delta_1\delta_2 \end{pmatrix} \bmod 2 \\ &= \begin{pmatrix} H\alpha_1H & H\beta_1 \\ \gamma_1H & \delta_1 \end{pmatrix} \begin{pmatrix} H\alpha_2H & H\beta_2 \\ \gamma_2H & \delta_2 \end{pmatrix} \bmod 2 \\ &= \varphi(M_1)\varphi(M_2) \end{aligned}$$

for all $M_1, M_2 \in \Gamma_S$ this yields that φ is a homomorphism of groups. Finally, the surjectivity of this homomorphism follows from the fact that $\mathrm{Sp}(2;\mathbb{F}_2)$ is generated by the following four matrices

$$\varphi(J) = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},\ \varphi(T_{e_1}) = \begin{pmatrix} I_2 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \\ 0 & I_2 \end{pmatrix},\ \varphi(T_{e_2}) = \begin{pmatrix} I_2 & \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & I_2 \end{pmatrix},\ \varphi(U_{e_1}) = \begin{pmatrix} I_2 & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \\ 0 & I_2 \end{pmatrix}$$

(cf. [Fr83, A 5.4]). ∎

According to O'Meara [O'M78, 3.1.5] $\mathrm{Sp}(2;\mathbb{F}_2)$ is isomorphic to the symmetric group $S(6)$. By Igusa [Ig64, p. 398] we can explicitly describe the isomorphism of $\mathrm{Sp}(2;\mathbb{F}_2)$ and $S(6)$ in the following way: Let

$$\mathcal{C}_4 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix};\ a, b \in \mathbb{F}_2^2,\ {}^t ab \equiv 1 \pmod 2 \right\}$$

be the set of odd theta characteristics mod 2. $\mathrm{Sp}(2;\mathbb{F}_2)$ acts on this set via

$$\left(M, \begin{pmatrix} a \\ b \end{pmatrix} \right) \mapsto M \begin{Bmatrix} a \\ b \end{Bmatrix} := \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \mathrm{diag}(C\,{}^tD) \\ \mathrm{diag}(A\,{}^tB) \end{pmatrix}$$

for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2;\mathbb{F}_2)$, where $\mathrm{diag}(T)$ is the column vector consisting of the diagonal entries of a matrix T. Since $\mathcal{C}_4$ contains exactly 6 elements the mapping

$$M \mapsto \left(\mathcal{C}_4 \to \mathcal{C}_4,\ \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \mapsto M\left\{\begin{smallmatrix} a \\ b \end{smallmatrix}\right\}\right)$$

defines a homomorphism $\pi : \mathrm{Sp}(2;\mathbb{F}_2) \to S(6)$ (which is an isomorphism according to Igusa). The non-trivial character of $\mathrm{Sp}(2;\mathbb{F}_2)$ is then given by the sign of the permutation $\pi(M)$ for $M \in \mathrm{Sp}(2;\mathbb{F}_2)$.

By combining the epimorphism $\varphi : \Gamma_S \to \mathrm{Sp}(2;\mathbb{F}_2)$ from Proposition 1.26, the isomorphism $\mathrm{Sp}(2;\mathbb{F}_2) \to S(6)$ and the sign map, i.e., by

$$\Gamma_S \xrightarrow{\ \varphi\ } \mathrm{Sp}(2;\mathbb{F}_2) \xrightarrow{\ \pi\ } S(6) \xrightarrow{\ \mathrm{sign}\ } \{\pm 1\},$$

we get an explicit description for the Siegel character of the orthogonal modular group Γ_S if $S \equiv 0 \pmod 2$. We denote this character by ν_2. Obviously, all R_A, $A \in \mathrm{O}(\Lambda)$, lie in the kernel of φ and therefore also in the kernel of ν_2. Moreover, the kernel contains of course the commutator subgroup Γ_S' and thus, in particular, J. According to the proof of Proposition 1.25, all matrices T_g with $g = {}^t(g_0, \ldots, g_{l+1}) \in \Lambda_0$ are modulo Γ_S' equivalent to I_{l+4} whenever $g_0 + g_{l+1} + \sum_{j=1}^{l} s_j g_j/2 \equiv 0 \pmod 2$ where the s_j, $1 \le j \le l$, are the diagonal entries of $S \in \{A_1^{(3)}, A_1^{(2)}, S_2\}$. Otherwise T_g is equivalent to T_{e_1} modulo Γ_S'. An easy calculation shows that $\nu_2(T_{e_1}) = -1$, and so we have

$$\nu_2(T_g) = (-1)^{g_0 + g_{l+1} + \sum_{j=1}^{l} s_j g_j/2} = \begin{cases} (-1)^{\sum_{j=0}^{l+1} g_j} & \text{if } S \in \{A_1^{(3)}, A_1^{(2)}\}, \\ (-1)^{g_0 + g_1 + g_3} & \text{if } S = S_2. \end{cases} \tag{1.4}$$

By applying the above description directly to T_g for arbitrary $g = (g_0, \widetilde{g}, g_{l+1}) \in \Lambda_0$ we get a nicer and more general formula, namely

$$\nu_2(T_{(g_0, \widetilde{g}, g_{l+1})}) = (-1)^{g_0 + g_{l+1} + q(\widetilde{g})}. \tag{1.5}$$

The value of $\nu_2(M_D)$, $D \in \mathrm{SL}(2;\mathbb{Z})$, can also be explicitly calculated using the above description. We get

$$\nu_2(M_D) = (-1)^{\alpha+\beta+\gamma+\delta+\beta\gamma} \text{ for all } D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}). \tag{1.6}$$

Using representations in terms of the above matrices we can now easily determine the value of $\nu_2(M)$ for some of the other matrices M from Proposition 1.16. For the vectors $g = {}^t(g_0, \ldots, g_{l+1}) \in \Lambda_0$ with $q_0(g) = \pm 1$ we define

$$g^* := -J\langle g \rangle = q_0(g)^{-1}\, {}^t(g_{l+1}, -g_1, \ldots, -g_l, g_0) \in \Lambda_0.$$

According to [Kr96, p. 249f], $R_g = T_g J T_{g^*} J T_g J$ for every $g \in \Lambda_0$ with $q_0(g) = \pm 1$, and thus

$$\nu_2(R_g) = \nu_2(T_g J T_{g^*} J T_g J) = \nu_2(T_{g^*}) = \nu_2(T_g).$$

Moreover, for all $\lambda \in \Lambda$ we have

$$U_\lambda = M_D T_{(0,\lambda,0)} M_{D^*},$$

where $D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\nu_2(M_D) = \nu_2(M_{D^*})$ we get

$$\nu_2(U_\lambda) = \nu_2(T_{(0,\lambda,0)}) = (-1)^{q(\lambda)}. \tag{1.7}$$

Using the estimation for the index of the commutator subgroup Γ'_S in Γ_S (Proposition 1.25) and the explicit knowledge of the characters of Γ_S we can now derive the structure of Γ_S^{ab}.

Proposition 1.27 *a) If $S \in \{A_1^{(3)}, A_1^{(2)}, S_2\}$ then $\Gamma_S^{\mathrm{ab}} = \langle \det, \nu_\pi, \nu_2 \rangle \cong C_2 \times C_2 \times C_2$.*
b) If $S \in \{D_4, A_3, A_2\}$ then $\Gamma_S^{\mathrm{ab}} = \langle \det, \nu_\pi \rangle \cong C_2 \times C_2$.

Moreover, using the explicit knowledge about the generators of $\mathrm{O}(\Lambda)$ we can determine which rotations R_A, $A \in \mathrm{O}(\Lambda)$, are necessary to generate the commutator subgroup Γ'_S, the discriminant kernel $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$ and the full modular group Γ_S.

Corollary 1.28 *a) If $S \in \{A_1^{(2)}, A_2, S_2, A_3\}$ then Γ'_S is a subgroup of $\langle J, T_g;\ g \in \Lambda_0 \rangle$.*
If $S = A_1^{(3)}$ then Γ'_S is generated by J, T_g, $g \in \ker \nu_2$, and the rotation R_A, $A = \left(\begin{smallmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{smallmatrix}\right)$.
If $S = D_4$ then Γ'_S is generated by J, T_g, $g \in \Lambda_0$, and the rotation R_A, $A = \left(\begin{smallmatrix} 1&0&-1&0 \\ 0&0&-1&0 \\ 0&1&-1&0 \\ 0&0&2&1 \end{smallmatrix}\right)$.
b) The discriminant kernel $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$ is generated by J, T_g, $g \in \Lambda_0$, and $-M_{\mathrm{tr}}$.
c) Γ_S is generated by J, T_g, $g \in \Lambda_0$, the rotations R_B, where B runs over the representatives of the generators of $\mathrm{O}(\Lambda)^{\mathrm{ab}}$, and, in case of $S = A_1^{(3)}$ or $S = D_4$, additionally by the rotation R_A from a).

PROOF a) We use GAP ([GAP05]) to calculate the subgroup H of $G = \langle R_A;\ A \in \mathrm{O}(\Lambda)' \rangle$ which is generated by all $R_{g_1} \cdot R_{g_2}$ with $g_j = (0, \lambda_j, 0) \in \Lambda_0$ such that $q_0(g_j) = -1$, $j = 1, 2$. In case of $S \in \{A_1^{(2)}, A_2, S_2, A_3\}$ we get $H = G$ which implies our claim. If $S = A_1^{(3)}$ or $S = D_4$ then H is a subgroup of index 3 of G and $G = \langle H, R_A \rangle$. Since J and T_g, $g \in \Lambda_0$, act trivially on $\mathrm{Dis}(\Lambda_1)$ while R_A does not we obviously have $R_A \notin \langle J, T_g;\ g \in \Lambda_0 \rangle$. This completes the proof.

b) It is easy to check that additionally to J and T_g, $g \in \Lambda_0$, the matrix $-M_{\mathrm{tr}}$ also acts trivially on $\mathrm{Dis}(\Lambda_1)$. Note that $-M_{\mathrm{tr}}$ is not contained in $\langle J, T_g;\ g \in \Lambda_0 \rangle$ because of $\det(-M_{\mathrm{tr}}) = -1$. It remains to be verified that the given matrices generate $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$. This can be done similarly to the proof of part a) since by Proposition 1.24 we explicitly know $\mathrm{O_d}(\Lambda)$.

c) This follows from part a) and the fact that J and T_g, $g \in \Lambda_0$, act trivially on $\operatorname{Dis}(\Lambda_1)$ and have determinant 1. ∎

1.7. Parabolic subgroups

An important subgroup of $\mathrm{O}^+(S_1;\mathbb{R})$ is the parabolic subgroup

$$P_S(\mathbb{R}) := \left\{ \begin{pmatrix} D^* & * & * \\ 0 & * & * \\ 0 & 0 & D \end{pmatrix} \in \mathrm{O}^+(S_1;\mathbb{R});\ D \in \mathrm{SL}(2;\mathbb{R}) \right\}$$

where $D^* = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) {}^tD^{-1} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix}\right)$ for $D = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{R})$. It plays an important role in the theory of Jacobi forms (cf. Section 2.3). According to [Bü96, Prop. 2.5] $P_S(\mathbb{R})$ is generated by the matrices M_D, $D \in \mathrm{SL}(2;\mathbb{R})$, R_A, $A \in \mathrm{O}(S;\mathbb{R})$, U_λ, $\lambda \in \mathbb{R}^l$, and $T_{(0,\mu,0)}$, $\mu \in \mathbb{R}^l$. In fact we have the following

Proposition 1.29 *Each element M of the parabolic subgroup $P_S(\mathbb{R})$ can be written in the form*

$$M = M_D R_A U_\lambda T_{(\kappa,\mu,0)}$$

with $A \in \mathrm{O}(S;\mathbb{R})$, $D \in \mathrm{SL}(2;\mathbb{R})$, $\lambda, \mu \in \mathbb{R}^l$ and $\kappa \in \mathbb{R}$. This representation is unique.

PROOF Let $M = \begin{pmatrix} D^* & * & * \\ 0 & A & * \\ 0 & 0 & D \end{pmatrix} \in P_S(\mathbb{R})$. By virtue of [Bü96, Prop. 2.4], we have $A \in \mathrm{O}(S;\mathbb{R})$ and get

$$M' = R_{A^{-1}} M_{D^{-1}} M = \begin{pmatrix} I_2 & \begin{smallmatrix} {}^t\mu S \\ {}^t\lambda S \end{smallmatrix} & \begin{smallmatrix} -\kappa & q(\mu) \\ q(\lambda) & \kappa + {}^t\lambda S \mu \end{smallmatrix} \\ 0 & I_l & \begin{smallmatrix} \lambda & \mu \end{smallmatrix} \\ 0 & 0 & I_2 \end{pmatrix},\ \lambda, \mu \in \mathbb{R}^l,\ \kappa \in \mathbb{R}.$$

Now

$$U_{-\lambda} M' = \begin{pmatrix} I_2 & \begin{smallmatrix} {}^t\mu S \\ 0 \end{smallmatrix} & \begin{smallmatrix} -\kappa & q(\mu) \\ 0 & \kappa \end{smallmatrix} \\ 0 & I_l & \begin{smallmatrix} 0 & \mu \end{smallmatrix} \\ 0 & 0 & I_2 \end{pmatrix} = T_{(\kappa,\mu,0)}.$$

The uniqueness of the representation is obvious. ∎

The subgroup $H = \{U_\lambda T_{(\kappa,\mu,0)};\ \lambda, \mu \in \mathbb{R}^l,\ \kappa \in \mathbb{R}\}$ of $P_S(\mathbb{R})$ is normal in $P_S(\mathbb{R})$ and the center of $P_S(\mathbb{R})$ as well as the center of H are both given by the subgroup $\{T_{(\kappa,0,0)};\ \kappa \in \mathbb{R}\}$. Due to the preceding proposition, we have

$$P_S(\mathbb{R})/H \cong \mathrm{SL}(2;\mathbb{R}) \times \mathrm{O}(S;\mathbb{R}),$$

and thus

$$P_S(\mathbb{R}) \cong (\mathrm{SL}(2;\mathbb{R}) \times \mathrm{O}(S;\mathbb{R})) \ltimes H.$$

The structure of $P_S(\mathbb{R})$ and the above unique representation of elements of $P_S(\mathbb{R})$ inspire the representation of $P_S(\mathbb{R})$ in a different form. Let

$$J_S(\mathbb{R}) := \{[D, A, (\lambda,\mu), \kappa];\ D \in \mathrm{SL}(2;\mathbb{R}),\ A \in \mathrm{O}(S;\mathbb{R}),\ \lambda,\mu \in \mathbb{R}^l,\ \kappa \in \mathbb{R}\}.$$

Then, by virtue of the preceding proposition the map

$$J_S(\mathbb{R}) \to P_S(\mathbb{R}),\ [D, A, (\lambda,\mu), \kappa] \mapsto M_D R_A U_\lambda T_{(\kappa/2-{}^t\lambda S\mu,\mu,0)}, \tag{1.8}$$

is bijective. If we define the composition law on $J_S(\mathbb{R})$ by

$$g_1 g_2 = [D_1D_2,\ A_1A_2,\ (\widetilde{\lambda}_1,\widetilde{\mu}_1) + (\lambda_2,\mu_2),\ \kappa_1+\kappa_2 - {}^t\lambda_1 S\mu_1 + {}^t\widetilde{\lambda}_1 S\widetilde{\mu}_1 + 2\,{}^t\widetilde{\lambda}_1 S\mu_2]$$

for $g_j = [D_j, A_j, (\lambda_j,\mu_j), \kappa_j] \in J_S(\mathbb{R})$ where $(\widetilde{\lambda}_1,\widetilde{\mu}_1) = A_2^{-1}(\lambda_1,\mu_1)D_2$ then $J_S(\mathbb{R})$ becomes a group and the above map $J_S(\mathbb{R}) \to P_S(\mathbb{R})$ becomes an isomorphism of groups. We call $J_S(\mathbb{R})$ the Jacobi group. The Heisenberg group

$$H_S(\mathbb{R}) := \{[(\lambda,\mu),\kappa];\ \lambda,\mu \in \mathbb{R}^l,\ \kappa \in \mathbb{R}\}$$

with composition law

$$[(\lambda_1,\mu_1),\kappa_1][(\lambda_2,\mu_2),\kappa_2] = [(\lambda_1,\mu_1)+(\lambda_2,\mu_2),\ \kappa_1+\kappa_2+2\,{}^t\lambda_1 S\mu_2]$$

for $[(\lambda_j,\mu_j),\kappa_j] \in H_S(\mathbb{R})$ is obviously a subgroup of $J_S(\mathbb{R})$. It is isomorphic to the subgroup H of $P_S(\mathbb{R})$ and thus we have

$$J_S(\mathbb{R}) \cong (\mathrm{SL}(2;\mathbb{R}) \times \mathrm{O}(S;\mathbb{R})) \ltimes H_S(\mathbb{R}).$$

Since we can canonically identify any element D of $\mathrm{SL}(2;\mathbb{R})$, A of $\mathrm{O}(S;\mathbb{R})$, (λ,μ) of $\mathbb{R}^l \times \mathbb{R}^l$ and κ of $\mathbb{R}$ with the elements $[D, I_l, (0,0), 0]$, $[I_2, A, (0,0), 0]$, $[I_2, I_l, (\lambda,\mu), 0]$ and $[I_2, I_l, (0,0), \kappa]$ of $J_S(\mathbb{R})$, respectively, we will often simply write $[D]$, $[A]$, $[\lambda,\mu]$ or $[\kappa]$ instead of the corresponding element of $J_S(\mathbb{R})$.

For dealing with Jacobi forms we introduce another Jacobi group, namely the one defined by Arakawa in [Ar92]. It is given by

$$G^J := \{(D, (\lambda,\mu), \rho);\ D \in \mathrm{SL}(2;\mathbb{R}),\ \lambda,\mu \in \mathbb{R}^l,\ \rho \in \mathrm{Sym}(l;\mathbb{R})\}$$

with the composition law

$$g_1g_2 = (D_1D_2,\ (\lambda_1,\mu_1)D_2 + (\lambda_2,\mu_2),\ \rho_1+\rho_2 - \mu_1\,{}^t\lambda_1 + \widetilde{\mu}_1\,{}^t\widetilde{\lambda}_1 + \widetilde{\lambda}_1\,{}^t\mu_2 + \mu_2\,{}^t\widetilde{\lambda}_1)$$

for $g_j = (D_j, (\lambda_j, \mu_j), \rho_j) \in G^J$ where $(\widetilde{\lambda}_1, \widetilde{\mu}_1) = (\lambda_1, \mu_1)D_2$. The map

$$G^J \to J_S(\mathbb{R}),\ (D, (\lambda, \mu), \rho) \mapsto [D, I_l, (\lambda, \mu), \mathrm{trace}(S\rho)] \tag{1.9}$$

is obviously a homomorphism of groups. By abuse of notation we will also often write $[D]$ and $[\lambda, \mu]$ instead of the corresponding elements of G^J.

Next we consider the parabolic subgroup of Γ_S. Let

$$P_S(\mathbb{Z}) := P_S(\mathbb{R}) \cap \Gamma_S = P_S(\mathbb{R}) \cap \mathrm{Mat}(l+4; \mathbb{Z}).$$

Then the corresponding Jacobi group $J_S(\mathbb{Z})$, defined as preimage of $P_S(\mathbb{Z})$ under the isomorphism 1.8, is given by

$$J_S(\mathbb{Z}) = \{[D, A, (\lambda, \mu), \kappa];\ D \in \mathrm{SL}(2;\mathbb{Z}),\ A \in \mathrm{O}(\Lambda),\ \lambda, \mu \in \Lambda,\ \kappa \in 2\mathbb{Z}\}$$

and Arakawa's discrete Jacobi group is given by

$$\Gamma^J = \{(D, (\lambda, \mu), \rho);\ D \in \mathrm{SL}(2;\mathbb{Z}),\ \lambda, \mu \in \mathbb{Z}^l,\ \rho \in \mathrm{Sym}(l;\mathbb{Z})\}.$$

Note that the image of Γ^J under the above homomorphism $G^J \to J_S(\mathbb{R})$ lies in $J_S(\mathbb{Z})$.

Finally we take a look at the action of the paramodular subgroup on $\mathcal{H}_S$. Let $w = (\tau_1, z, \tau_2) \in \mathcal{H}_S$ and $M = M_D R_A U_\lambda T_{(\kappa/2 - {}^t\lambda S\mu, \mu, 0)} \in P_S(\mathbb{R})$, $D = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{R})$, $A \in \mathrm{O}(S;\mathbb{R})$, $\lambda, \mu \in \mathbb{R}^l$, $\kappa \in \mathbb{R}$. Then

$$M\langle w\rangle = \left(\tau_1 + {}^t\lambda S z + q(\lambda)\tau_2 + \kappa/2 - \frac{\gamma q(z + \lambda\tau_2 + \mu)}{\gamma\tau_2 + \delta}, A\frac{z + \lambda\tau_2 + \mu}{\gamma\tau_2 + \delta}, D\langle\tau_2\rangle\right) \tag{1.10}$$

where $D\langle\tau_2\rangle = \frac{\alpha\tau_2 + \beta}{\gamma\tau_2 + \delta}$ is the usual action of $\mathrm{SL}(2;\mathbb{R})$ on the upper half plane $\mathcal{H}$. Since the second and third component of $M\langle w\rangle$ only depend on the second and third component of $w = (*, z, \tau_2)$ the action of $P_S(\mathbb{R})$ on $\mathcal{H}_S$ induces an action of $J_S(\mathbb{R})$ on $\mathcal{H} \times \mathbb{C}^l$ which is given by

$$[D, A, (\lambda, \mu), \kappa](\tau, z) = \left(D\langle\tau\rangle, A\frac{z + \lambda\tau + \mu}{\gamma\tau + \delta}\right). \tag{1.11}$$

This action is compatible with the action of Arakawa's Jacobi group G^J on $\mathcal{H} \times \mathbb{C}^l$ ([Ar92, (3.2)]) via the homomorphism (1.9).

2. Modular Forms

2.1. Orthogonal modular forms

Let S be an even positive definite matrix of degree l. Note that

$$j : \mathrm{O}^+(S_1;\mathbb{R}) \times \mathcal{H}_S \to \mathbb{C}^\times, \quad (M,w) \mapsto M\{w\},$$

is a factor of automorphy (cf. [Bü96, La. 2.10]), i.e., $j(M,\cdot)$ is holomorphic for all $M \in \mathrm{O}^+(S_1;\mathbb{R})$, and j satisfies the cocycle condition

$$j(M_1M_2,w) = j(M_1, M_2\langle w\rangle)\, j(M_2,w) \quad \text{for all } M_1, M_2 \in \mathrm{O}^+(S_1;\mathbb{R}). \tag{2.1}$$

Given $f : \mathcal{H}_S \to \mathbb{C}$, $M \in \mathrm{O}^+(S_1;\mathbb{R})$ and $k \in \mathbb{Z}$ we define a function $f|_kM : \mathcal{H}_S \to \mathbb{C}$ by

$$(f|_kM)(w) := j(M,w)^{-k} f(M\langle w\rangle) \quad \text{for all } w \in \mathcal{H}_S.$$

Then $f|_kM$ is holomorphic whenever f is holomorphic, and, moreover,

$$(f|_kM_1)|_kM_2 = f|_k(M_1M_2) \quad \text{for all } M_1, M_2 \in \mathrm{O}^+(S_1;\mathbb{R}).$$

Thus $(M,f) \mapsto f|_kM$ defines an action of $\mathrm{O}^+(S_1;\mathbb{R})$ on the set of holomorphic functions on $\mathcal{H}_S$.

Definition 2.1 *Let $k \in \mathbb{Z}$, Γ a subgroup of Γ_S of finite index and $\nu \in \Gamma^{\mathrm{ab}}$ an Abelian character of Γ of finite order. A holomorphic function $f : \mathcal{H}_S \to \mathbb{C}$ is called an (*orthogonal*)* modular form of weight k *(*on $\mathcal{H}_S$*)* with respect to Γ and ν *if it satisfies*

$$f|_kM = \nu(M)f \qquad \textit{for all } M \in \Gamma. \tag{2.2}$$

We denote the vector space of (orthogonal) modular forms of weight k with respect to Γ and ν by $[\Gamma,k,\nu]$. If $\nu = 1$ then we sometimes simply write $[\Gamma,k]$. Moreover, we write $[\Gamma',k,1]$ or $[\Gamma',k]$ for the vector space of all modular forms of weight k with respect to Γ, i.e.,

$$[\Gamma',k] = [\Gamma',k,1] = \bigoplus_{\nu\in\Gamma^{\mathrm{ab}}} [\Gamma,k,\nu],$$

where Γ^{ab} is the group of Abelian characters of Γ.

The constant functions are obviously modular forms of weight 0 with respect to the trivial character. Moreover, given two modular forms $f \in [\Gamma, k, \nu]$ and $g \in [\Gamma, k', \nu']$ we have

$$fg \in [\Gamma, k + k', \nu\nu'].$$

Thus the modular forms with respect to the trivial character and some subgroup Γ of Γ_S of finite index form a graded ring (which is graded by the weight). We denote this graded ring by

$$\mathcal{A}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} [\Gamma, k, 1].$$

If $-I \in \Gamma$ then we get a first necessary condition for the existence of non-trivial modular forms.

Proposition 2.2 *If $-I \in \Gamma$ and $\nu(-I) \neq (-1)^k$ then $[\Gamma, k, \nu] = \{0\}$.*

PROOF This follows immediately from $f|_k(-I) = (-1)^k f$ and (2.2). ∎

This result allows us to derive some conditions on the weight and/or the characters for the existence of non-trivial modular forms in the cases we are mainly interested in.

Corollary 2.3 *Let $k, l, m, n \in \mathbb{Z}$.*

a) If $S \in \{D_4, A_1^{(2)}\}$ and k is odd then $[\Gamma_S, k, \nu] = \{0\}$ for all Abelian characters $\nu \in \Gamma_S^{\mathrm{ab}}$.

b) $[\Gamma_{A_1^{(3)}}, k, \det^l \nu_\pi^m \nu_2^n] = \{0\}$ if $k + l \equiv 1 \pmod 2$.

c) $[\Gamma_{A_3}, k, \det^l \nu_\pi^m] = \{0\}$ if $k + l + m \equiv 1 \pmod 2$.

d) $[\Gamma_{A_2}, k, \det^l \nu_\pi^m] = \{0\}$ if $k + m \equiv 1 \pmod 2$.

e) $[\Gamma_{S_2}, k, \det^l \nu_\pi^m \nu_2^n] = \{0\}$ if $k + m \equiv 1 \pmod 2$.

Since our modular forms with respect to the full modular group Γ_S and the trivial character are also modular forms in the sense of [Kr96] we can apply some of Krieg's results.

Theorem 2.4 *Let $\nu \in \Gamma_S^{\mathrm{ab}}$ be an Abelian character of Γ_S of order h, and let $k \in \mathbb{Z}$. Then*

$$[\Gamma_S, 0, \nu] = \begin{cases} \mathbb{C}, & \text{if } \nu = 1, \\ \{0\}, & \text{if } \nu \neq 1, \end{cases}$$

and

$$[\Gamma_S, k, \nu] = \{0\}, \quad \text{if } k < \frac{l}{2h},\ k \neq 0,$$

where l is the rank of S.

PROOF If $f \in [\Gamma_S, 0, 1]$ then f is a modular form of weight 0 in the sense of Krieg, and thus a constant function by virtue of [Kr96, Cor. 4]. If $\nu \in \Gamma_S^{\mathrm{ab}}$ is of order $h > 1$ and $f \in [\Gamma_S, 0, \nu]$ then $f^h \in [\Gamma_S, 0, 1] = \mathbb{C}$ and hence also $f \in \mathbb{C}$. Due to $\nu \neq 1$ there is an $M \in \Gamma_S$ such that $\nu(M) \neq 1$. Then $f = f|_0 M = \nu(M) f$ yields $f = 0$.

Now let $k \in \mathbb{Z}$, $k < l/(2h)$, $k \neq 0$, and let $f \in [\Gamma_S, k, \nu]$. In case of the trivial character $f = 0$ follows immediately from [Kr96, Cor. 4]. Otherwise, we again have to consider $f^h \in [\Gamma_S, hk, 1] = \{0\}$. ■

Lemma 2.5 *Let $k \in \mathbb{Z}$, Γ a subgroup of Γ_S of finite index and $\nu \in \Gamma^{\mathrm{ab}}$ an Abelian character of Γ of finite order. Then each $f \in [\Gamma, k, \nu]$ possesses an absolutely convergent Fourier expansion of the form*

$$f(w) = \sum_{\mu \in \Lambda_0^\sharp} \alpha_f(\mu)\, e^{2\pi i\, {}^t\mu S_0 w/h} \qquad \textit{for all } w \in \mathcal{H}_S$$

for some $h \in \mathbb{N}$ which depends on Γ and the order of ν.

If $\widetilde{M} \in \mathrm{O}^+(\Lambda_0)$ such that $M = (1) \times \widetilde{M} \times (1) \in \Gamma$ then we have

$$\alpha_f(\widetilde{M}\mu) = \nu(M)\alpha_f(\mu) \qquad \textit{for all } \mu \in \Lambda_0^\sharp.$$

PROOF Since Γ is of finite index in Γ_S and ν is of finite order there is a $h \in \mathbb{N}$ such that $T_g^h = T_{hg} \in \Gamma$ and $\nu(T_g^h) = \nu(T_g)^h = 1$ for all $g \in \Lambda_0$. Then

$$f(w) = (f|_k T_{hg})(w) = f(w + hg) \qquad \text{for all } g \in \Lambda_0$$

yields the existence of an absolutely convergent Fourier expansion of the form

$$f(w) = \sum_{\mu \in \Lambda_0^\sharp} \alpha_f(\mu)\, e^{2\pi i\, {}^t\mu S_0 w/h} \qquad \text{for all } w \in \mathcal{H}_S.$$

The property of the Fourier coefficients follows from $f(Mw) = (f|_k M)(w) = \nu(M)f(w)$ and the uniqueness of the Fourier expansion. ■

Definition 2.6 *For $a \in \mathbb{R}^{l+2}$ we write $a > 0$, if a belongs to $\mathcal{P}_S$, and we write $a \geq 0$, if a belongs to the closure*

$$\overline{\mathcal{P}_S} = \{v = (v_0, \ldots, v_{l+1}) \in \mathbb{R}^{l+2};\ q_0(v) \geq 0,\ v_0 \geq 0\}$$

of $\mathcal{P}_S$. Moreover, given $a, b \in \mathbb{R}^{l+2}$ we define as usual

$$\begin{aligned} a > b &\iff a - b > 0, \\ a \geq b &\iff a - b \geq 0. \end{aligned}$$

A few properties of positive and semi-positive elements of $\mathbb{R}^{l+2}$ are given in the following

Proposition 2.7 *Let $u, v \in \mathbb{R}^{l+2}$ with $u \geq 0$ and $v > 0$.*
a) There exists $M \in \mathrm{O}^+(S_0;\mathbb{R})$ such that $Mv = {}^t(v_0', 0, v_{l+1}')$ with $v_0', v_{l+1}' > 0$.
b) There exists $M \in \mathrm{O}^+(S_0;\mathbb{R})$ such that $Mu = {}^t(u_0', 0, u_{l+1}')$ with $u_0', u_{l+1}' \geq 0$.
c) If $u \neq 0$ then we have ${}^t u S_0 v > 0$.

PROOF Let $u = (u_0, \widetilde{u}, u_{l+1}) \geq 0$ and $v = (v_0, \widetilde{v}, v_{l+1}) > 0$.

a) Due to $v > 0$ we have $v_{l+1} > 0$. Therefore, with $M = \widetilde{U}_{-\widetilde{v}/v_{l+1}} \in \mathrm{O}^+(S_0; \mathbb{R})$ we get $Mv = {}^t(v_0', 0, v_{l+1}') > 0$ which, in particular, implies $v_0', v_{l+1}' > 0$.
b) If $u_0 = 0$ or $u_{l+1} = 0$ then $\widetilde{u} = 0$. So we only have to consider the case $u_0, u_{l+1} > 0$. In this case we can just as in a) choose $M = \widetilde{U}_{-\widetilde{u}/u_{l+1}} \in \mathrm{O}^+(S_0; \mathbb{R})$.
c) By virtue of b) we can find $M \in \mathrm{O}^+(S_0; \mathbb{R})$ such that $Mu = {}^t(u_0', 0, u_{l+1}') =: u'$. Then

$$ {}^t uS_0v = {}^t u\, {}^tMS_0Mv = {}^t u'S_0(Mv) = u_0'v_{l+1}' + u_{l+1}'v_0' $$

where $Mv = (v_0', *, v_{l+1}') > 0$. Now $u \neq 0$ implies $u_0' > 0$ or $u_{l+1}' > 0$. This yields the assertion. ■

Theorem 2.8 (Koecher's principle) *Let $k \in \mathbb{Z}$, $\nu \in \Gamma_S^{\mathrm{ab}}$ an Abelian character of Γ_S of order $h \in \mathbb{N}$ and $f \in [\Gamma_S, k, \nu]$ a modular form with Fourier expansion*

$$ f(w) = \sum_{\mu \in \Lambda_0^\sharp} \alpha_f(\mu)\, e^{2\pi i\, {}^t\mu S_0 w/h} \qquad \textit{for all } w \in \mathcal{H}_S. $$

Then $\alpha_f(\mu) = 0$ unless $\mu \geq 0$. Furthermore, given $\beta > 0$ then f is bounded in the domain $\{w \in \mathcal{H}_S;\ \mathrm{Im}(w) \geq \beta\mathrm{e}\}$, where $\mathrm{e} = {}^t(1, 0, \dots, 0, 1)$, and its Fourier series converges uniformly in this domain.

PROOF Bühler proved this for $\nu = 1$ in [Bü96, Satz 3.7]. The proof can easily be extended to the case of non-trivial characters. Let $\nu \in \Gamma_S^{\mathrm{ab}}$ be a non-trivial character of order $h \in \mathbb{N}$. Then, due to Lemma 2.5, we have

$$ \alpha_f({}_{h\lambda}\widetilde{U}\mu_0) = \nu({}_{h\lambda}U)\alpha_f(\mu_0) = \nu({}_{\lambda}U^h)\alpha_f(\mu_0) = \alpha_f(\mu_0) \qquad \text{for all } \lambda \in \Lambda, \mu_0 \in \Lambda_0^\sharp. $$

If one now replaces ${}_\lambda U$ by ${}_{h\lambda}U$ in Bühler's proof then the assertion follows, i.e., we have

$$ f(w) = \sum_{\substack{\mu_0 \in \Lambda_0^\sharp \\ \mu_0 \geq 0}} \alpha_f(\mu_0)\, e^{2\pi i\, {}^t\mu_0 S_0 w/h} \qquad \text{for all } w \in \mathcal{H}_S. $$

Since the Fourier series converges in $w = i\frac{1}{2}\beta\mathrm{e} \in \mathcal{H}_S$ there exists $c > 0$ such that

$$ |\alpha_f(\mu_0)e^{2\pi i\, {}^t\mu_0 S_0 w/h}| = |\alpha_f(\mu_0)|\, e^{-\pi\beta(m+n)/h} \leq c $$

for all $\mu_0 = (m, \mu, n) \in \Lambda_0^\sharp$, $\mu_0 \geq 0$. If $v \geq \beta\mathrm{e}$ and $\mu_0 \geq 0$ then ${}^t\mu_0 S_0 v \geq {}^t\mu_0 S_0 \beta\mathrm{e} =$

$\beta(m+n)$. Thus for $w = u + iv \in \mathcal{H}_S$ with $v \geq \beta\mathrm{e}$ we have

$$\begin{aligned} |f(w)| &\leq \sum_{\substack{\mu_0 \in \Lambda_0^\sharp \\ \mu_0 \geq 0}} |\alpha_f(\mu_0)| \, e^{-2\pi \, {}^t\mu_0 S_0 v/h} \\ &\leq c \sum_{\substack{\mu_0 \in \Lambda_0^\sharp \\ \mu_0 \geq 0}} e^{-\pi\beta(m+n)/h}. \end{aligned}$$

In order to further estimate this sum we determine an upper bound for the number of vectors $\mu_0 = (m, \mu, n) \in \Lambda_0^\sharp$ with $\mu_0 \geq 0$ and $m + n = t \in \mathbb{N}_0$. Due to $S^{-1} > 0$ there exists an $r > 0$ such that $S^{-1} - rI_l > 0$. If $\lambda \in \Lambda = \mathbb{Z}^l$ with $||\lambda||_\infty > t^2/r$ then for $\mu = S^{-1}\lambda$ we have $S[\mu] = S^{-1}[\lambda] > r\,{}^t\lambda\lambda > t^2$. But $\mu_0 = (m, \mu, n) \geq 0$ yields $S[\mu] \leq 2mn \leq (m+n)^2 = t^2$. Thus there are at most $(2\lfloor t^2/r \rfloor + 1)^l$ vectors $\mu_0 = (m, \mu, n) \in \Lambda_0^\sharp$ with $\mu_0 \geq 0$ and $m + n = t$. The convergence of the series

$$\sum_{t=0}^{\infty} \left(2 \left\lfloor \frac{t^2}{r} \right\rfloor + 1 \right)^l e^{-\pi\beta t/h}$$

completes the proof. ∎

Definition 2.9 *A modular form $f \in [\Gamma_S, k, \nu]$ with Fourier expansion*

$$f(w) = \sum_{\substack{\mu \in \Lambda_0^\sharp \\ \mu \geq 0}} \alpha_f(\mu) e^{2\pi i \, {}^t\mu S_0 w/h} \qquad \textit{for all } w \in \mathcal{H}_S$$

*is called an (*orthogonal*)* cusp form *if $\alpha_f(\mu) \neq 0$ implies $\mu > 0$. We denote the subspace of cusp forms in $[\Gamma_S, k, \nu]$ by $[\Gamma_S, k, \nu]_0$.*

In the theory of symplectic modular forms the space of cusp forms is sometimes defined as kernel of a certain operator, namely Siegel's Φ-operator (cf. [Kr85]). We can define Siegel's Φ-operator also for orthogonal modular forms, and, if Λ is Euclidean, then just as in the symplectic theory the space of cusp forms turns out to be the kernel of this operator.

Proposition 2.10 *Let $\nu \in \Gamma_S^{\mathrm{ab}}$ be an Abelian character of Γ_S such that $\nu(T_g) = 1$ for all $g \in \Lambda_0$. Then for $k \in \mathbb{Z}$ the map*

$$\Phi \;:\; [\Gamma_S, k, \nu] \to [\mathrm{SL}(2;\mathbb{Z}), k], \;\; f \mapsto f|\Phi,$$

$$(f|\Phi)(\tau) := \lim_{y \to \infty} f(iy, 0, \tau) \quad \textit{for } \tau \in \mathcal{H},$$

where $[\mathrm{SL}(2;\mathbb{Z}), k]$ is the space of elliptic modular forms of weight k, is a homomorphism. We call this map Siegel's Φ-operator.

If Λ is Euclidean then we have

$$[\Gamma_S, k, \nu]_0 = \ker \Phi,$$

i.e., $f \in [\Gamma_S, k, \nu]$ is a cusp form if and only if $f|\Phi = 0$.

PROOF Due to the condition on the character all $f \in [\Gamma_S, k, \nu]$ have a Fourier expansion of the form

$$f(w) = \sum_{\substack{\mu_0 \in \Lambda_0^\sharp \\ \mu_0 \geq 0}} \alpha_f(\mu_0)\, e^{2\pi i\, {}^t\mu_0 S_0 w} \qquad \text{for all } w \in \mathcal{H}_S.$$

Since the Fourier series is locally uniformly convergent we have

$$\begin{aligned} \lim_{y\to\infty} f(iy, 0, \tau) &= \sum_{\substack{\mu_0 = (m,\mu,n) \in \Lambda_0^\sharp \\ \mu_0 \geq 0}} \alpha_f(\mu_0)\, e^{2\pi i m\tau} \lim_{y\to\infty} e^{-2\pi n y} \\ &= \sum_{m \in \mathbb{N}_0} \alpha_f(m, 0, 0)\, e^{2\pi i m \tau}. \end{aligned}$$

Thus $f|\Phi$ is well-defined. The linearity of Φ is obvious, and $f|\Phi \in [\mathrm{SL}(2;\mathbb{Z}), k]$ follows from $f|_k M_D = \nu(M_D) f = f$ for all $D \in \mathrm{SL}(2;\mathbb{Z})$. Note that $\nu(M_D) = 1$ is a consequence of $\nu(T_g) = 1$.

If f is a cusp form then $\alpha_f(m, 0, 0) = 0$ for all $m \in \mathbb{N}_0$ yields $f|\Phi = 0$. Conversely, $f|\Phi = 0$ implies $\alpha_f(m, 0, 0) = 0$ for all $m \in \mathbb{N}_0$. Now suppose that Λ is Euclidean. Then, by virtue of Proposition 1.22, for each $\mu_0 \in \Lambda_0^\sharp$ with $q_0(\mu_0) = 0$ there exists an $M \in \mathrm{O}^+(\Lambda_0)$ such that $(1) \times M \times (1) \in \Gamma_S$ and $M\mu_0 = {}^t(m, 0, \dots, 0)$. Due to $|\alpha_f(\mu_0)| = |\alpha_f(M\mu_0)| = 0$ we conclude that f is a cusp form. ∎

Using the above characterization of cusp forms and common knowledge about elliptic modular forms we can show that the subspace of cusp forms often coincides with the space of modular forms.

Corollary 2.11 *Suppose that S is one of the matrices listed in (1.2). Let $\nu \in \Gamma_S^{\mathrm{ab}}$ be an Abelian character of Γ_S such that $\nu(T_g) = 1$ for all $g \in \Lambda_0$, and let $k \in \mathbb{N}_0$. If k is odd or $k = 2$ or $\nu \neq 1$ then*

$$[\Gamma_S, k, \nu] = [\Gamma_S, k, \nu]_0.$$

PROOF Let $f \in [\Gamma_S, k, \nu]$. If k is odd or $k = 2$ then $f|\Phi \in [\mathrm{SL}(2;\mathbb{Z}), k] = \{0\}$, and thus f is a cusp form. If $\nu \neq 1$ then because of the condition on ν there exists $A \in \mathrm{O}(\Lambda)$ such that $\nu(R_A) = -1$. Then $\alpha_f(\mu_0) = \alpha_f(R_A\mu_0) = \nu(R_A)\alpha_f(\mu_0) = -\alpha_f(\mu_0)$ for all $\mu_0 = (m, 0, 0)$, $m \in \mathbb{N}_0$, yields $f|\Phi = 0$. Hence f is a cusp form. ∎

2.2. Rankin-Cohen type differential operators

In this section we introduce a certain holomorphic differential operator for orthogonal modular forms. The interesting property of this differential operator is that it produces a new modular form from several given modular forms. In the case of Siegel modular forms differential operators with this property were studied by Ibukiyama in [Ib99a]. We restrict ourselves to considering the equivalent of the Rankin-Cohen type differential operator that was used by Aoki and Ibukiyama in [AI05].

Let S be an even positive definite matrix of degree l. We write $w \in \mathcal{H}_S$ either, as usual, in the form $w = (\tau_1, z, \tau_2)$, $\tau_1, \tau_2 \in \mathcal{H}$, $z = (z_1, \ldots, z_l) \in \mathbb{C}^l$, or simply in the form $w = (w_0, \ldots, w_{l+1})$. First determine the Jacobian of the modular transformations. Recall that the Jacobian (determinant) of a function $F : \mathbb{C}^n \to \mathbb{C}^n$ is given by

$$\det\left(\frac{\partial F}{\partial z}\right) = \det\left(\frac{\partial(F_1, \ldots, F_n)}{\partial(z_1, \ldots, z_n)}\right) = \det\begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial z_1} & \cdots & \frac{\partial F_n}{\partial z_n} \end{pmatrix}.$$

Proposition 2.12 *Let Γ_S be nicely generated. Then*

$$\det\left(\frac{\partial M\langle w\rangle}{\partial w}\right) = (\det M) \cdot j(M, w)^{-l-2}$$

for all $M \in \Gamma_S$ and all $w \in \mathcal{H}_S$.

PROOF Let $M_1, M_2 \in \Gamma_S$. Due to the chain rule we have

$$\det\left(\frac{\partial(M_1M_2)\langle w\rangle}{\partial w}\right) = \det\left(\frac{\partial M_1\langle M_2\langle w\rangle\rangle}{\partial M_2\langle w\rangle}\right)\det\left(\frac{\partial M_2\langle w\rangle}{\partial w}\right).$$

Moreover, j satisfies the cocycle condition (2.1). Therefore it suffices to prove the assertion for generators J, T_g, $g \in \Lambda_0$, and R_A, $A \in \mathrm{O}(\Lambda)$, of Γ_S. For the translations the assertion is trivial, and for the rotations R_A, $A \in \mathrm{O}(\Lambda)$, we have

$$\det\left(\frac{\partial R_A\langle w\rangle}{\partial w}\right) = \det\left(\frac{\partial(\tau_1, Az, \tau_2)}{\partial(\tau_1, z_1, \ldots, z_l, \tau_2)}\right) = \det A = (\det R_A) \cdot j(R_A, w)^{-l-2}.$$

It remains to prove the assertion for $M = J$. Instead we show the assertion for $M_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$ and $M^*_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$. We have

$$M_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\langle w\rangle = \left(\tau_1 - \frac{q(z)}{\tau_2}, \frac{z}{\tau_2}, -\frac{1}{\tau_2}\right) \quad \text{and} \quad M^*_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\langle w\rangle = \left(-\frac{1}{\tau_1}, \frac{z}{\tau_1}, \tau_2 - \frac{q(z)}{\tau_1}\right).$$

Therefore

$$\det\left(\frac{\partial M_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}\langle w\rangle}{\partial w}\right)=\det\begin{pmatrix}1&*&*&*&*\\0&\tau_2^{-1}&0&0&*\\0&0&\ddots&0&*\\0&0&0&\tau_2^{-1}&*\\0&0&0&0&\tau_2^{-2}\end{pmatrix}=\tau_2^{-l-2}=j(M_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)},w)^{-l-2}$$

and analogously

$$\det\left(\frac{\partial M^*_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}\langle w\rangle}{\partial w}\right)=\tau_1^{-l-2}=j(M^*_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)},w)^{-l-2}.$$

In view of $\det M_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}=\det M^*_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}=1$ and $J=M_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}M^*_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}$ this completes the proof. ■

Now we define the differential operator.

Definition 2.13 *Let Γ be a subgroup of Γ_S of finite index. Given $l+3$ orthogonal modular forms $f_j\in[\Gamma,k_j,\chi_j]$ of weight k_j with respect to an Abelian character $\chi_j\in\Gamma^{\mathrm{ab}}$, $1\le j\le l+3$, and with respect to Γ, we define a function $\{f_1,\dots,f_{l+3}\}:\mathcal{H}_S\to\mathbb{C}$ by*

$$\{f_1,\dots,f_{l+3}\}=\det\begin{pmatrix}k_1f_1&\cdots&k_{l+3}f_{l+3}\\ \frac{\partial f_1}{\partial w_0}&\cdots&\frac{\partial f_{l+3}}{\partial w_0}\\ \vdots&&\vdots\\ \frac{\partial f_1}{\partial w_{l+1}}&\cdots&\frac{\partial f_{l+3}}{\partial w_{l+1}}\end{pmatrix}.$$

Under certain conditions this function turns out to be a modular form. We restrict our considerations to nicely generated modular groups.

Proposition 2.14 *Let Γ_S be nicely generated. Given $f_j\in[\Gamma_S,k_j,\chi_j]$ with $k_j\in\mathbb{Z}$ and $\chi_j\in\Gamma_S^{\mathrm{ab}}$, $1\le j\le l+3$, the function $\{f_1,\dots,f_{l+3}\}$ is a modular form of weight $k_1+\ldots+k_{l+3}+l+2$ with respect to Γ_S and the Abelian character $\chi=\chi_1\chi_2\cdots\chi_{l+3}\det$. If $f_1,\dots,f_{l+3}$ are algebraically independent, then $\{f_1,\dots,f_{l+3}\}$ does not vanish identically.*

PROOF We closely follows the proof of [AI05, Prop. 2.1]. For $2\le n\le l+3$ we define functions F_n by $F_n:=f_n^{k_1}/f_1^{k_n}$. Let $M\in\Gamma_S$. Then

$$\begin{aligned}F_n(M\langle w\rangle)&=\frac{f_n^{k_1}(M\langle w\rangle)}{f_1^{k_n}(M\langle w\rangle)}\cdot\frac{\left(j(M,w)^{-k_n}\right)^{k_1}}{\left(j(M,w)^{-k_1}\right)^{k_n}}=\frac{\left(f_n|_{k_n}M\right)^{k_1}(w)}{\left(f_1|_{k_1}M\right)^{k_n}(w)}\\&=\frac{\chi_n^{k_1}(M)\ f_n^{k_1}(w)}{\chi_1^{k_n}(M)\ f_1^{k_n}(w)}=\left(\chi_n^{k_1}\chi_1^{-k_n}\right)(M)\ F_n(w).\end{aligned}$$

Hence the F_n are modular functions, that is meromorphic modular forms of weight 0, with respect to the Abelian characters $\widetilde{\chi}_n := \chi_n^{k_1}\chi_1^{-k_n}$. Next we consider the Jacobian of $(F_2,\ldots,F_{l+3})$. We set

$$F := \det\left(\frac{\partial(F_2,\ldots,F_{l+3})}{\partial(w_0,\ldots,w_{l+1})}\right).$$

Then for $M\in\Gamma_S$ we have

$$\begin{aligned}
F(w) &= \det\left(\frac{\partial(F_2(w),\ldots,F_{l+3}(w))}{\partial(w_0,\ldots,w_{l+1})}\right)\\
&= \det\left(\frac{\partial\left(\widetilde{\chi}_2^{-1}(M)\ F_2(M\langle w\rangle),\ldots,\widetilde{\chi}_{l+3}^{-1}(M)\ F_{l+3}(M\langle w\rangle)\right)}{\partial(w_0,\ldots,w_{l+1})}\right)\\
&= \det\left(\frac{\partial(F_2(M\langle w\rangle),\ldots,F_{l+3}(M\langle w\rangle))}{\partial((M\langle w\rangle)_0,\ldots,(M\langle w\rangle)_{l+1})}\right)\times\\
&\qquad\qquad\times\det\left(\frac{\partial((M\langle w\rangle)_0,\ldots,(M\langle w\rangle)_{l+1})}{\partial(w_0,\ldots,w_{l+1})}\right)\cdot\prod_{n=2}^{l+3}\widetilde{\chi}_n^{-1}(M)\\
&= F(M\langle w\rangle)\cdot(\det M)\cdot j(M,w)^{-l-2}\cdot\left(\widetilde{\chi}_2\widetilde{\chi}_3\cdots\widetilde{\chi}_{l+3}\right)^{-1}(M).
\end{aligned}$$

Thus F is a meromorphic modular form of weight $l+2$ with respect to Γ_S and the Abelian character $\widetilde{\chi} := \widetilde{\chi}_2\widetilde{\chi}_3\cdots\widetilde{\chi}_{l+3}\det$. (Note that $\det M = \det^{-1} M$ for all $M\in\Gamma_S$). Moreover, we have

$$\begin{aligned}
\frac{\partial F_n}{\partial w_i} &= \frac{\partial}{\partial w_i}(f_n^{k_1}f_1^{-k_n}) = k_1(f_n^{k_1-1}f_1^{-k_n})\frac{\partial f_n}{\partial w_i} - k_n(f_n^{k_1}f_1^{-k_n-1})\frac{\partial f_1}{\partial w_i}\\
&= \left(\frac{k_1f_n^{k_1-1}}{f_1^{k_n}}\right)\left(\frac{\partial f_n}{\partial w_i}-\frac{k_nf_n}{k_1f_1}\cdot\frac{\partial f_1}{\partial w_i}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
\{f_1,\ldots,f_{l+3}\} &= \det\begin{pmatrix} k_1f_1 & 0 & \cdots & 0\\ \frac{\partial f_1}{\partial w_0} & \frac{\partial F_2}{\partial w_0} & \cdots & \frac{\partial F_{l+3}}{\partial w_0}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_1}{\partial w_{l+1}} & \frac{\partial F_2}{\partial w_{l+1}} & \cdots & \frac{\partial F_{l+3}}{\partial w_{l+1}}\end{pmatrix}\cdot\prod_{n=2}^{l+3}\frac{f_1^{k_n}}{k_1f_n^{k_1-1}}\\
&= k_1f_1\det\left(\frac{\partial(F_2,\ldots,F_{l+3})}{\partial(w_0,\ldots,w_{l+1})}\right)\cdot\prod_{n=2}^{l+3}\frac{f_1^{k_n}}{k_1f_n^{k_1-1}}\\
&= \frac{f_1^{k_2+\ldots+k_{l+3}+1}}{k_1^{l+1}(f_2\cdot\ldots\cdot f_{l+3})^{k_1-1}}\cdot F.
\end{aligned}$$

Inserting $M\langle w\rangle$, $M\in\Gamma_S$, in $\{f_1,\dots,f_{l+3}\}$ we get

$$\begin{aligned}
\{f_1,\dots,f_{l+3}\}(M\langle w\rangle) &= \frac{\left(f_1(M\langle w\rangle)\right)^{k_2+\ldots+k_{l+3}+1}}{k_1^{l+1}\left(f_2(M\langle w\rangle)\cdot\ldots\cdot f_{l+3}(M\langle w\rangle)\right)^{k_1-1}}\cdot F(M\langle w\rangle)\\
&= \frac{\left(j(M,w)^{k_1}\right)^{k_2+\ldots+k_{l+3}+1}}{\left(j(M,w)^{k_2}\cdot\ldots\cdot j(M,w)^{k_{l+3}}\right)^{k_1-1}}\cdot j(M,w)^{l+2}\times\\
&\qquad\times\frac{\chi_1^{k_2+\ldots+k_{l+3}+1}}{\left(\chi_2\chi_3\cdots\chi_{l+3}\right)^{k_1-1}}(M)\cdot\widetilde{\chi}(M)\cdot\{f_1,\dots,f_{l+3}\}(w)\\
&= j(M,w)^{k_1+\ldots+k_{l+3}+l+2}\times\\
&\qquad\times\left(\chi_1\chi_2\cdots\chi_{l+3}\det\right)(M)\cdot\{f_1,\dots,f_{l+3}\}(w).
\end{aligned}$$

We conclude that $\{f_1,\dots,f_{l+3}\}$ is a holomorphic modular form of weight $k_1+\ldots+k_{l+3}+l+2$ with respect to Γ_S and the Abelian character $\chi=\chi_1\chi_2\cdots\chi_{l+3}\det$.

The second part of the assertion, that is $\{f_1,\dots,f_{l+3}\}\neq 0$ if $f_1,\dots,f_{l+3}$ are algebraically independent, follows just as in the proof of [AI05, Prop. 2.1]. ∎

We will use this differential operator in order to give an alternative realization for some of the generators of the graded rings of modular forms.

2.3. Jacobi forms

Let S be an even positive definite matrix of degree l. As usual we will write $w\in\mathcal{H}_S$ in the form $w=(\tau_1,z,\tau_2)$, $\tau_1,\tau_2\in\mathcal{H}$, $z\in\mathbb{C}^l$. According to [Kr96, Th. 2] each $f\in[\Gamma_S,k,1]$, $k\in\mathbb{Z}$, possesses a Fourier-Jacobi expansion of the form

$$f(w)=\sum_{m=0}^{\infty}\varphi_m(\tau_2,z)\,e^{2\pi i m\tau_1}\qquad\text{for } w=(\tau_1,z,\tau_2)\in\mathcal{H}_S \tag{2.3}$$

where

$$\varphi_m(\tau,z)=\sum_{n=0}^{\infty}\sum_{\substack{\mu\in\Lambda^\sharp\\ q(\mu)\leq mn}}\alpha_f(n,\mu,m)\,e^{2\pi i(n\tau+{}^t\mu Sz)}. \tag{2.4}$$

This result can easily be generalized to orthogonal modular forms with respect to an Abelian character of finite order. We restrict our considerations to the cases we are mainly interested in. **So for the rest of this section we assume that S is one of the matrices listed in (1.2).**

Proposition 2.15 *Let $k\in\mathbb{Z}$, $\nu\in\Gamma_S^{\mathrm{ab}}$ and $f\in[\Gamma_S,k,\nu]$. If $S\in\{A_1^{(3)},A_1^{(2)},S_2\}$ and*

$\nu \in \nu_2 \cdot \langle \nu_\pi, \det \rangle$ *then* f *possesses a Fourier-Jacobi expansion of the form*

$$f(w) = \sum_{m \in \frac{1}{2}+\mathbb{N}_0} \varphi_m(\tau_2, z)\, e^{2\pi i m \tau_1} \qquad \textit{for } w = (\tau_1, z, \tau_2) \in \mathcal{H}_S$$

where

$$\varphi_m(\tau, z) = \sum_{n \in \frac{1}{2}+\mathbb{N}_0} \sum_{\substack{\mu \in \gamma + \Lambda^\sharp \\ q(\mu) \le mn}} \alpha_f(n, \mu, m)\, e^{2\pi i (n\tau + {}^t\mu S z)} \tag{2.5}$$

with $\gamma = S^{-1} \operatorname{diag}(S)/4$ *where* $\operatorname{diag}(S)$ *is the column vector consisting of the diagonal entries of* S*. Otherwise the Fourier-Jacobi expansion of* f *is of the form (2.3).*

PROOF If $\nu \in \langle \nu_\pi, \det \rangle$ then $f(w+g) = f(w)$ for all $g \in \Lambda_0$. Thus f has a Fourier expansion of the form

$$f(w) = \sum_{\substack{\mu_0 \in \Lambda_0^\sharp \\ \mu_0 \ge 0}} \alpha_f(\mu_0)\, e^{2\pi i\, {}^t\mu_0 S_0 w}$$

and consequently a Fourier-Jacobi expansion of the form (2.3). On the other hand, if $S \in \{A_1^{(3)}, A_1^{(2)}, S_2\}$ and $\nu \in \nu_2 \cdot \langle \nu_\pi, \det \rangle$ then $f(w+2g) = f(w)$ for all $g \in \Lambda_0$. Hence f has a Fourier expansion of the form

$$f(w) = \sum_{\substack{\mu_0 \in \frac{1}{2}\Lambda_0^\sharp \\ \mu_0 \ge 0}} \alpha_f(\mu_0)\, e^{2\pi i\, {}^t\mu_0 S_0 w}.$$

Now $f(w+g) = \nu_2(T_g)\ f(w)$ for all $g \in \Lambda_0$ yields $\alpha_f(\mu_0) = 0$ for $\mu_0 \in \frac{1}{2}\Lambda_0^\sharp$ whenever $\nu_2(T_g) \neq e^{2\pi i\, {}^t\mu_0 S_0 g}$ for some $g \in \Lambda_0$. In view of $2\,{}^t\mu_0 S_0 g = (m, \mu, n) S_0 g = n g_0 + (g_1, \ldots, g_l)\lambda + m g_{l+1}$ for $2\mu_0 = (m, \mu, n) = S_0^{-1}(n, \lambda, m) \in \Lambda_0^\sharp = S_0^{-1}\Lambda_0$ and $g = (g_0, \ldots, g_{l+1}) \in \Lambda_0$ the claim follows from (1.4). ∎

Remark 2.16 *Note that* $\varphi_0(\tau, z)$ *is independent of* z*. In fact we have*

$$\varphi_0(\tau, z) = \sum_{n=0}^{\infty} \alpha_f(n, 0, 0)\, e^{2\pi i n \tau} = (f|\Phi)(\tau) \quad \textit{for } \tau \in \mathcal{H}, z \in \mathbb{C}^l.$$

So if $\nu \in \Gamma_S^{\mathrm{ab}}$ *with* $\nu(T_g) = 1$ *for all* $g \in \Lambda_0$ *and if additionally* Λ *is Euclidean then* f *is a cusp form if and only if the* 0*-th Fourier-Jacobi coefficient vanishes.*

The functions $\varphi_m : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ which occur in the Fourier-Jacobi expansion are so called Jacobi forms. We will give a formal definition a bit further down. First we show how the action of $\mathrm{O}^+(S_1; \mathbb{R})$ on the set of holomorphic functions on $\mathcal{H}_S$ induces an action of the Jacobi group $J_S(\mathbb{R})$ on the set of holomorphic functions on $\mathcal{H} \times \mathbb{C}^l$.

Let $\varphi : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ be a holomorphic function. Then for each $m \in \mathbb{Q}$, $m > 0$, we define the function $\varphi^*_m : \mathcal{H}_S \to \mathbb{C}$ by

$$\varphi^*_m(\tau_1, z, \tau_2) = e^{2\pi i m \tau_1} \varphi(\tau_2, z),$$

and for each $k \in \mathbb{Z}$, $m \in \mathbb{Q}$, $m > 0$, and $g \in J_S(\mathbb{R})$ we define the function $\varphi|_{k,m,S}g : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ by

$$(\varphi|_{k,m,S}g)(\tau, z) = e^{-2\pi i m \tau'} (\varphi^*_m|_k M_g)(\tau', z, \tau)$$

where M_g is the element of $P_S(\mathbb{R})$ which corresponds to g and τ' is an arbitrary element of $\mathcal{H}$. For $g = [D, A, (\lambda, \mu), \kappa] \in J_S(\mathbb{R})$, $D = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, we have $M_g = M_D R_A U_\lambda T_{(\kappa/2 - {}^t\lambda S \mu, \mu, 0)}$ and the above translates to

$$\begin{aligned}
(\varphi|_{k,m,S}g)(\tau, z) &= e^{-2\pi i m \tau'} (\gamma\tau + \delta)^{-k} \varphi^*_m(M_g\langle(\tau', z, \tau)\rangle) \\
&= e^{-2\pi i m \tau'} (\gamma\tau + \delta)^{-k} \varphi^*_m \left(\tau' + {}^t\lambda S z + q(\lambda)\tau + \tfrac{\kappa}{2} - \tfrac{\gamma q(z+\lambda\tau+\mu)}{\gamma\tau+\delta}, A\tfrac{z+\lambda\tau+\mu}{\gamma\tau+\delta}, D\langle\tau\rangle\right) \\
&= (\gamma\tau + \delta)^{-k} e^{2\pi i m ({}^t\lambda S z + q(\lambda)\tau + \kappa/2 - \gamma q(z+\lambda\tau+\mu)/(\gamma\tau+\delta))} \varphi\left(D\langle\tau\rangle, A\frac{z + \lambda\tau + \mu}{\gamma\tau + \delta}\right).
\end{aligned}$$

In particular, we see that the definition of $\varphi|_{k,m,S}g$ is independent of the choice of $\tau' \in \mathcal{H}$.

Moreover, due to the definition the map $(\varphi, g) \mapsto \varphi|_{k,m,S}g$ obviously defines an action of $J_S(\mathbb{R})$ on the set of holomorphic functions on $\mathcal{H} \times \mathbb{C}^l$ and

$$j_{k,m,S}(g, (\tau, z)) = (\gamma\tau + \delta)^k e^{-2\pi i m ({}^t\lambda S z + q(\lambda)\tau + \kappa/2 - \gamma q(z+\lambda\tau+\mu)/(\gamma\tau+\delta))}$$

defines a factor of automorphy on $J_S(\mathbb{R}) \times (\mathcal{H} \times \mathbb{C}^l)$. Note that by virtue of (1.9), $j_{k,m,S}$ corresponds to the factor of automorphy $J_{k,\frac{1}{2}mS}$ on $G^J \times (\mathcal{H} \times \mathbb{C}^l)$ defined by Arakawa in [Ar92]. In view of (1.11), the action of $J_S(\mathbb{R})$ on a holomorphic function $\varphi : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ can also be written in the form

$$(\varphi|_{k,m,S}g)(\tau, z) = j_{k,m,S}(g, (\tau, z))^{-1} \varphi(g(\tau, z)).$$

Now we define Jacobi forms on $\mathcal{H} \times \mathbb{C}^l$.

Definition 2.17 *Let $k \in \mathbb{Z}$, $m \in \mathbb{Q}$, $m > 0$, and $\nu \in \Gamma^{\mathrm{ab}}_S$ an Abelian character of Γ_S. A holomorphic function $\varphi : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ is called a* Jacobi form of index (m, S) and weight k with respect to ν *if it satisfies the following conditions:*

(i) For all $g \in J_S(\mathbb{Z})$ we have

$$\varphi|_{k,m,S}g = \nu(g)\ \varphi \tag{2.6}$$

where ν is considered as character of $J_S(\mathbb{Z})$ via the correspondence of $J_S(\mathbb{Z})$ and $P_S(\mathbb{Z})$.

(ii) φ *has a Fourier expansion of the form*

$$\varphi(\tau, z) = \sum_{\substack{n\in\mathbb{Q}\\ n\geq 0}} \sum_{\substack{\mu\in\Lambda_{\mathbb{Q}}\\ q(\mu)\leq mn}} \alpha_\varphi(n,\mu)\, e^{2\pi i(n\tau + {}^t\mu S z)} \tag{2.7}$$

where $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

If $\alpha_\varphi(n,\mu) = 0$ *whenever* $q(\mu) = mn$ *then we call* φ *a* Jacobi cusp form.

We denote the space of Jacobi forms of index (m, S) *and weight* k *with respect to* ν *by* $J_k(m, S, \nu)$ *and the corresponding space of Jacobi cusp forms by* $J_k^0(m, S, \nu)$*. If* $\nu = 1$ *then we simply write* $J_k(m, S)$ *and* $J_k^0(m, S)$*.*

As we already mentioned above the functions φ_m, $m > 0$, appearing in the Fourier-Jacobi expansion of a modular form are Jacobi forms.

Proposition 2.18 *Let* $k \in \mathbb{Z}$*,* $\nu \in \Gamma_S^{\mathrm{ab}}$ *and* $f \in [\Gamma_S, k, \nu]$ *with Fourier-Jacobi expansion*

$$f(w) = \sum_{m\in\mathbb{Q}} \varphi_m(\tau_2, z)\, e^{2\pi i m\tau_1} \qquad \text{for } w = (\tau_1, z, \tau_2) \in \mathcal{H}_S.$$

Then $\varphi_m \in J_k(m, S, \nu)$ *for all* $m \in \mathbb{Q}$*,* $m > 0$*.*

PROOF Let $g = [D, A, (\lambda, \mu), \kappa] \in J_S(\mathbb{Z})$, $D = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Then the corresponding element of $P_S(\mathbb{Z})$ is given by $M_g = M_D R_A U_\lambda T_{(\kappa/2 - {}^t\lambda S\mu, \mu, 0)}$ and we have

$$\nu(M_g) f(w) = (f|_k M_g)(w) = j(M_g, w)^{-k} f(M_g\langle w\rangle) = (\gamma\tau_2 + \delta)^{-k} f(M_g\langle w\rangle)$$

for $w = (\tau_1, z, \tau_2) \in \mathcal{H}_S$. Replacing f by its Fourier-Jacobi expansion, using (1.10) and taking into account the uniqueness of the Fourier expansion of f with respect to τ_1, we get

$$\begin{aligned}\nu(M_g)\varphi_m(\tau_2, z) &= (\gamma\tau_2 + \delta)^{-k} e^{2\pi i m({}^t\lambda S z + q(\lambda)\tau_2 + \kappa/2 - \frac{\gamma q(z+\lambda\tau_2+\mu)}{\gamma\tau_2+\delta})} \varphi_m\left(D\langle\tau_2\rangle, A\tfrac{z+\lambda\tau_2+\mu}{\gamma\tau_2+\delta}\right)\\ &= j_{k,m,S}(g, (\tau_2, z))^{-1}\, \varphi_m(g(\tau_2, z))\\ &= (\varphi_m|_{k,m,S}\, g)(\tau_2, z)\end{aligned}$$

for $(\tau_2, z) \in \mathcal{H} \times \mathbb{C}^l$. Moreover, by virtue of Proposition 2.15, the φ_m have a Fourier expansion of the form (2.7). This completes the proof. ■

In view of the structure of $J_S(\mathbb{Z})$, a function $\varphi : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ satisfies (2.6) if and only if it satisfies the following conditions:

(i) $\nu([D])\, \varphi(\tau, z) = (\gamma\tau+\delta)^{-k} e^{-2\pi i m \frac{\gamma q(z)}{\gamma\tau+\delta}} \varphi\left(D\langle\tau\rangle, \frac{z}{\gamma\tau+\delta}\right)$ for all $D = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{Z})$,

(ii) $\nu([A])\, \varphi(\tau, z) = \varphi(\tau, Az)$ for all $A \in \mathrm{O}(\Lambda)$,

(iii) $\nu([\lambda, \mu])\, \varphi(\tau, z) = e^{2\pi i m({}^t\lambda S z + q(\lambda)\tau)} \varphi(\tau, z + \lambda\tau + \mu)$ for all $\lambda, \mu \in \mathbb{Z}^l$,

(iv) $\nu([\kappa])\, \varphi(\tau, z) = e^{\pi i m\kappa} \varphi(\tau, z)$ for all $\kappa \in 2\mathbb{Z}$.

In particular, we see that in case of $\nu([2]) = \nu(T_{e_1}) = 1$ there are non-trivial Jacobi forms only if $m \in \mathbb{Z}$. Moreover, if $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $\nu \in \Gamma_S^{\mathrm{ab}}$ with $\nu(g) = 1$ for all $g = [*, I_l, (*, *), *] \in J_S(\mathbb{Z})$ then the elements of $J_k(m, S, \nu)$ are elements of the space $J^{\text{Arakawa}}_{k,\frac{1}{2}mS}(\mathrm{SL}(2;\mathbb{Z}))$ of Jacobi forms for $\mathrm{SL}(2;\mathbb{Z})$ of index $\frac{1}{2}mS$ and weight k in the sense of Arakawa ([Ar92]) and also elements of the space $J_k^{\text{Krieg}}(\mathbb{Z}^l, \sigma_{mS})$ of Jacobi forms of weight k with respect to $(\mathbb{Z}^l, \sigma_{mS})$ in the sense of Krieg ([Kr96]). Conversely, as we will later see, for $S \in \{A_1^{(2)}, A_2, S_2, A_3\}$ we have

$$J_k^{\text{Krieg}}(\mathbb{Z}^l, \sigma_S) = J^{\text{Arakawa}}_{k,\frac{1}{2}S}(\mathrm{SL}(2;\mathbb{Z})) = \bigoplus_{\nu} J_k(1, S, \nu)$$

where the sum runs over all Abelian characters $\nu \in \langle \nu_\pi, \det \rangle \leq \Gamma_S^{\mathrm{ab}}$. Thus we can apply Arakawa's results in order to determine the dimensions of certain spaces of Jacobi cusp forms.

Proposition 2.19 *Let $S = A_3$. If $k \geq 4$ then*

$$\sum_{\nu \in \Gamma_S^{\mathrm{ab}}} \dim J_k^0(1, S, \nu) = \begin{cases} \left\lfloor \frac{k}{4} \right\rfloor - 1 & \text{if } k \text{ is even}, \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \text{ is odd}, k \not\equiv 9 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \text{ is odd}, k \equiv 9 \pmod{12}. \end{cases}$$

PROOF Apply [Ar92, Thm. 5.2]. ∎

2.4. Maaß spaces

In this section we introduce the Maaß space which consists of modular forms with particularly nice Fourier expansion. Let S be an arbitrary even positive definite matrix of degree l.

Definition 2.20 *Let $k \in \mathbb{Z}$ and $\nu \in \Gamma_S^{\mathrm{ab}}$ an Abelian character of Γ_S. A modular form $f \in [\Gamma_S, k, \nu]$ is called a* Maaß form *of weight k with respect to ν if its Fourier expansion*

$$f(w) = \sum_{\substack{\mu_0 \in \Lambda_0^\sharp \\ \mu_0 \geq 0}} \alpha_f(\mu_0) e^{2\pi i \, {}^t\mu_0 S_0 w}$$

satisfies

$$\alpha_f(\mu_0) = \sum_{d \,|\, \gcd(S_0\mu_0)} d^{k-1} \alpha_f(mn/d^2, \mu/d, 1) \quad \text{for all } 0 \neq \mu_0 = (m, \mu, n) \in \Lambda_0^\sharp. \tag{2.8}$$

The subspace of $[\Gamma_S, k, \nu]$ consisting of Maaß forms is called the Maaß space. *We denote it by $\mathcal{M}(\Gamma_S, k, \nu)$. If $\nu = 1$ then we simply write $\mathcal{M}(\Gamma_S, k)$.*

The Maaß space considered by Krieg in [Kr96] corresponds to the space $\mathcal{M}(\widetilde{\Gamma}_S, k)$ where $\widetilde{\Gamma}_S = \langle J, T_g;\ g \in \Lambda_0\rangle$ is the subgroup of Γ_S which is generated by the inversion J and the translations T_g, $g \in \Lambda_0$. Note that for all S in (1.2) we have $\widetilde{\Gamma}_S = \Gamma_S \cap \mathrm{O_d}(\Lambda_1) \cap \mathrm{SO}(\Lambda_1)$. If Γ_S is nicely generated and Γ'_S is a subgroup of $\widetilde{\Gamma}_S$ then we can decompose the space $\mathcal{M}(\widetilde{\Gamma}_S, k)$ into a direct sum of certain spaces $\mathcal{M}(\Gamma_S, k, \nu)$. By virtue of Corollary 1.28 these conditions are fulfilled if $S \in \{A_1^{(2)}, A_2, S_2, A_3\}$.

Proposition 2.21 *Suppose that Γ_S is nicely generated and that Γ'_S is a subgroup of $\widetilde{\Gamma}_S = \langle J, T_g;\ g \in \Lambda_0\rangle$. Then*

$$\mathcal{M}(\widetilde{\Gamma}_S, k) = \bigoplus_{\nu} \mathcal{M}(\Gamma_S, k, \nu)$$

for all $k \in \mathbb{Z}$ where the sum runs over all Abelian characters ν of Γ_S for which $\widetilde{\Gamma}_S \leq \ker \nu$.

Proof Let $G := \{\nu \in \Gamma_S^{\mathrm{ab}};\ \widetilde{\Gamma}_S \leq \ker \nu\}$. If $\nu \in G$ then we obviously have $\mathcal{M}(\Gamma_S, k, \nu) \subset \mathcal{M}(\widetilde{\Gamma}_S, k)$. It remains to be shown that all $f \in \mathcal{M}(\widetilde{\Gamma}_S, k)$ can be written as a linear combination of functions $f_\nu \in \mathcal{M}(\Gamma_S, k, \nu)$, $\nu \in G$.

G is an Abelian group. Therefore there exist $\nu_j \in G$, $1 \leq j \leq r$, such that $G = \prod_{j=1}^r \langle \nu_j \rangle$. Let s_j be the order of ν_j and let $\zeta_j \in \mathbb{C}$ be a primitive s_j-th root of unity for $1 \leq j \leq r$. Since Γ_S is generated by $\widetilde{\Gamma}_S$ and the rotations R_A, $A \in \mathrm{O}(\Lambda)$, and since $\widetilde{\Gamma}_S \leq \ker \nu_j$, $1 \leq j \leq r$, we actually have $G \cong \mathrm{O}(\Lambda)^{\mathrm{ab}}$. Therefore we find $A_1, \ldots, A_r \in \mathrm{O}(\Lambda)$ such that $\nu_j(R_{A_j}) = \zeta_j$ and $\nu_i(R_{A_j}) = 1$ for all $1 \leq i, j \leq r$, $i \neq j$. Note that $\Gamma_S = \langle \widetilde{\Gamma}_S, R_{A_1}, \ldots, R_{A_r}\rangle$.

Now let $f \in \mathcal{M}(\widetilde{\Gamma}_S, k)$ and $A \in \mathrm{O}(\Lambda)$. If f has Fourier coefficients $\alpha_f(\mu_0)$, $\mu_0 \in \Lambda_0^\sharp$, then $f|_k R_A$ has Fourier coefficients $\alpha_{f|_k R_A}(m, \mu, n) = \alpha_f(m, A\mu, n)$, $(m, \mu, n) \in \Lambda_0^\sharp$. One easily checks that the Fourier coefficients of $f|_k R_A$ satisfy the Maaß condition. Moreover, we have $f|_k R_A \in [\widetilde{\Gamma}_S, k]$ because R_A commutes modulo Γ'_S with all elements of $\widetilde{\Gamma}_S$ so that for all $M \in \widetilde{\Gamma}_S$ we have $(f|_k R_A)|_k M = (f|_k M')|_k R_A$ with some $M' \in \widetilde{\Gamma}_S$ and thus $(f|_k R_A)|_k M = f|_k R_A$ for all $M \in \widetilde{\Gamma}_S$. So for all $A \in \mathrm{O}(\Lambda)$ we have $f|_k R_A \in \mathcal{M}(\widetilde{\Gamma}_S, k)$, and, in particular, $f|_k R_{A_j} \in \mathcal{M}(\widetilde{\Gamma}_S, k)$, $1 \leq j \leq r$.

We define functions g_i, $0 \leq i \leq s_1 - 1$, by

$$\begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{s_1-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{s_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{s_1-1} & x_1^{s_1-1} & \cdots & x_{s_1-1}^{s_1-1} \end{pmatrix} \begin{pmatrix} f \\ f|_k R_{A_1} \\ \vdots \\ f|_k R_{A_1}^{s_1-1} \end{pmatrix}$$

where $x_i = \zeta_1^i$, $0 \leq i \leq s_1 - 1$. Obviously, we have $g_i \in \mathcal{M}(\widetilde{\Gamma}_S, k)$ for all $1 \leq i \leq s_1 - 1$. Furthermore, those functions satisfy

$$g_i|_k R_{A_1} = \sum_{s=0}^{s_1-1} (\zeta_1^s)^i\, f|_k R_{A_1}^{s+1} = \zeta_1^{(s_1-1)i} f + \sum_{s=1}^{s_1-1} \zeta_1^{(s-1)i}\, f|_k R_{A_1}^s = \zeta_1^{-i} g_i = \nu_1^{-i}(R_{A_1})\, g_i,$$

so that we actually have $g_i \in \mathcal{M}(\langle \widetilde{\Gamma}_S, R_{A_1}\rangle, k, \nu_1^{-i})$, $0 \le i \le s_1 - 1$. Note that f is recoverable as linear combination of the g_i since the transformation matrix is a Vandermonde matrix and thus invertible.

In the second step we use the functions g_i instead of f as input and get functions

$$h_j^{(i)} = \sum_{s=0}^{s_2-1} (\zeta_2^s)^j \, g_i|_k R_{A_2}^s \in \mathcal{M}(\langle \widetilde{\Gamma}_S, R_{A_1}, R_{A_2}\rangle, k, \nu_1^{-i}\nu_2^{-j}),$$

for $0 \le i \le s_1 - 1$, $0 \le j \le s_2 - 1$. After r iterations we finally get functions $f_\nu \in \mathcal{M}(\Gamma_S, k, \nu)$, $\nu \in G$. Due to the construction we have $f \in \operatorname{span}\{f_\nu;\ \nu \in G\}$. This completes the proof. ■

We can now prove that certain spaces of Maaß forms are isomorphic to certain spaces of Jacobi forms.

Corollary 2.22 *Suppose that* Γ_S *is nicely generated and that* $\Gamma_S' \le \widetilde{\Gamma}_S = \langle J, T_g;\ g \in \Lambda_0\rangle$. *Given* $k \in \mathbb{N}$ *and an Abelian character* ν *of* Γ_S *with* $\widetilde{\Gamma}_S \le \ker \nu$ *the map*

$$\mathcal{M}(\Gamma_S, k, \nu) \to J_k(1, S, \nu), \quad f \mapsto \varphi_1(f),$$

where $\varphi_1(f)$ *is the first Fourier-Jacobi coefficient of* f*, is an isomorphism of vector spaces.*

PROOF We consider the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_\nu \mathcal{M}(\Gamma_S, k, \nu) & \xrightarrow{(f_\nu)_\nu \mapsto \sum_\nu f_\nu} & \mathcal{M}(\widetilde{\Gamma}_S, k) \\
{\scriptstyle (f_\nu)_\nu \mapsto (\varphi_1(f_\nu))_\nu} \big\downarrow & & \big\downarrow {\scriptstyle f \mapsto \varphi_1(f)} \\
\bigoplus_\nu J_k(1, S, \nu) & \xrightarrow{(\varphi_\nu)_\nu \mapsto \sum_\nu \varphi_\nu} & J_k^{\text{Krieg}}(\mathbb{Z}^l, \sigma_S)
\end{array}$$

The right map and the upper map are isomorphisms of vector spaces according to [Kr96, Thm. 3] and Proposition 2.21, respectively. Consequently, the lower map has to be surjective. Since the lower map is the canonical injection it is also injective and thus an isomorphism. Therefore the left map also has to be an isomorphism. This completes the proof. ■

Note that the above isomorphism obviously maps cusp forms to cusp forms. Moreover, by considering the Fourier-Jacobi expansion of Maaß forms we can show that the dimension of the Maaß space is at most one greater than the dimension of the space of Maaß cusp forms. This will allow us to calculate the exact dimension of certain Maaß spaces using Arakawa's formulas for the dimension of spaces of Jacobi cusp forms.

Corollary 2.23 *Suppose that* $S \in \{A_1^{(2)}, A_2, S_2, A_3\}$. *Given* $k \in \mathbb{N}$ *and an Abelian character* ν *of* Γ_S *with* $\nu(T_g) = 1$ *for all* $g \in \Lambda_0$ *we have*

$$\dim \mathcal{M}(\Gamma_S, k, \nu) = \begin{cases} \dim J_k^0(1, S, \nu) & \text{if } k \text{ is odd or } k = 2 \text{ or } \nu \ne 1, \\ \dim J_k^0(1, S, \nu) + 1 & \text{if } k > 2 \text{ is even and } \nu = 1. \end{cases}$$

PROOF If k is odd or $k = 2$ or $\nu \neq 1$ then according to Corollary 2.11 all Maaß forms are cusp forms, and therefore $\mathcal{M}(\Gamma_S, k, \nu) \cong J_k^0(1, S, \nu)$. Now suppose that $k > 2$ is even and that $\nu = 1$. Assume we have two non-cusp forms $f, g \in \mathcal{M}(\Gamma_S, k, 1)$. Let $\varphi_0(f)$ and $\varphi_0(g)$ be the 0-th Fourier-Jacobi coefficient of f and g, respectively. According to the proof of [Kr96, Thm. 3] we have $\varphi_0(f) = \frac{1}{\gamma_k}\alpha_f(e_1)G_k$ and $\varphi_0(g) = \frac{1}{\gamma_k}\alpha_g(e_1)G_k$ where

$$G_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, e^{2\pi i n\tau}$$

is the normalized elliptic Eisenstein series in $[\mathrm{SL}(2;\mathbb{Z}), k]$ for $k > 2$ even. Now $\alpha_g(e_1)f - \alpha_f(e_1)g \in [\Gamma_S, k, 1]_0$ implies

$$\dim \mathcal{M}(\Gamma_S, k, 1) \leq \dim J_k^0(1, S, 1) + 1.$$

For $S \in \{A_1^{(2)}, A_2, S_2\}$ the existence of a non-cusp form $f \in \mathcal{M}(\Gamma_S, k, 1)$ for $k \geq 4$ even follows from [DK03, Thm. 1] (cf. Section 2.6). For $S = A_3$ non-cusp forms are given by the Eisenstein series $E_k^{A_3} \in \mathcal{M}(\Gamma_{A_3}, k, 1)$, $k \geq 4$ even, which will be defined in Section 2.5. ■

Since the preceding result is applicable in case of $S = A_3$ we get

Corollary 2.24 *Let* $S = A_3$*. If* $k \geq 4$ *then*

$$\dim \mathcal{M}(\Gamma'_S, k) = \sum_{\nu \in \Gamma_S^{\mathrm{ab}}} \dim \mathcal{M}(\Gamma_S, k, \nu) = \begin{cases} \left\lfloor \frac{k}{4} \right\rfloor & \text{if } k \text{ is even},\\ \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \text{ is odd}, k \not\equiv 9 \pmod{12},\\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \text{ is odd}, k \equiv 9 \pmod{12}.\end{cases}$$

PROOF Apply Proposition 2.19 and Corollary 2.23. ■

2.5. Restrictions of modular forms to submanifolds

In this section we examine the restrictions of orthogonal modular forms living on $\mathcal{H}_S$ to submanifolds of $\mathcal{H}_S$.

2.5.1. The general case

Let $\Lambda = \mathbb{Z}^l$ with bilinear form $(\cdot,\cdot)_S$ be the lattice associated to an even positive definite matrix S of rank $l \geq 2$. Suppose that $\Lambda_T = \mathbb{Z}^{l'}$ with bilinear form $(\cdot,\cdot)_T$ is the lattice associated to an even positive definite matrix T of rank $l' < l$ which can be considered as sublattice of Λ via an isometric embedding

$$\iota_T^S : \Lambda_T \to \Lambda,\ \lambda_T \mapsto M_T^S \lambda_T,$$

with $M_T^S \in \mathrm{Mat}(l, l'; \mathbb{Z})$ satisfying $(a,b)_T = (\iota_T^S(a), \iota_T^S(b))_S$ for all $a, b \in \Lambda_T$ (and by $\mathbb{C}$-linearity also for all $a, b \in \mathbb{C}^{l'}$). This embedding obviously induces an embedding of $\Lambda_{T_1} = \mathbb{Z}^{l'+4}$ with bilinear form $(\cdot,\cdot)_{T_1}$ in $\Lambda_1 = \mathbb{Z}^{l+4}$ with bilinear form $(\cdot,\cdot)_1$. Analogously, the corresponding half-space $\mathcal{H}_T$ can be embedded in $\mathcal{H}_S$ as submanifold. (Actually those embeddings are induced by the embedding of Λ_{T_1} in Λ_1; cf. Section 4.2.) By abuse of notation we denote those induced embeddings of Λ_{T_1} in Λ_1 and of $\mathcal{H}_T$ in $\mathcal{H}_S$ also by ι_T^S. Now those elements of Γ_S which stabilize the embedded lattice $\iota_T^S(\Lambda_{T_1})$ can be viewed as elements of Γ_T. This yields a homomorphism

$$\varphi_T : \mathrm{Stab}_{\Gamma_S}(\iota_T^S(\Lambda_{T_1})) \to \Gamma_T.$$

A priori, it is not clear whether this homomorphism is surjective, but if Γ_S and Γ_T are both nicely generated then we only have to check whether

$$\mathrm{Stab}_{\mathrm{O}(\Lambda)}(\iota_T^S(\Lambda_T)) \to \mathrm{O}(\Lambda_T)$$

is surjective (cf. also [FH00, Sec. 4]). This can be easily verified (at least for the cases we are interested in). In some cases φ_T is not injective. In those cases $\varphi_T^{-1}(I_{l'})$ contains non-trivial elements of Γ_S of the form R_A, $A \in \mathrm{O}(\Lambda)$. Those non-trivial elements can be used to show that certain modular forms on $\mathcal{H}_S$ vanish on the submanifold $\mathcal{H}_T$. Moreover, we will see that in those cases not all Abelian characters of Γ_S are the continuation of Abelian characters of Γ_T.

Now we consider the restriction of modular forms.

Theorem 2.25 *Let S and T be two even positive definite matrices of rank $l \geq 2$ and rank $l' < l$, respectively, such that an isometric embedding $\iota_T^S : \Lambda_T \to \Lambda, \lambda_T \mapsto B\lambda_T$, of $\Lambda_T = \mathbb{Z}^{l'}$ with bilinear form $(\cdot,\cdot)_T$ in $\Lambda = \mathbb{Z}^l$ with bilinear form $(\cdot,\cdot)_S$ exists. Moreover, suppose that Γ_S and Γ_T are both nicely generated and that φ_T is surjective.*

Let $k \in \mathbb{Z}$. If $\chi \in \Gamma_S^{\mathrm{ab}}$ is the continuation of an Abelian character of Γ_T and $f \in [\Gamma_S, k, \chi]$ then

$$f|\mathcal{H}_T \in [\Gamma_T, k, \chi|\Gamma_T].$$

If f has Fourier expansion

$$f(w) = \sum_{m,n \in \mathbb{N}_0} \sum_{\substack{\mu \in \Lambda^\sharp \\ q_S(\mu) \leq mn}} \alpha_f(m, \mu, n)\, e^{2\pi i (n\tau_1 + m\tau_2 - (\mu, z)_S)}$$

for $w = (\tau_1, z, \tau_2) \in \mathcal{H}_S$ then the Fourier expansion of $f|\mathcal{H}_T$ is given by

$$(f|\mathcal{H}_T)(w_T) = \sum_{m,n \in \mathbb{N}_0} \sum_{\substack{\mu_T \in \Lambda_T^\sharp \\ q_T(\mu_T) \leq mn}} \beta_f(m, \mu_T, n)\, e^{2\pi i (n\tau_1 + m\tau_2 - (\mu_T, z_T)_T)}$$

for $w_T = (\tau_1, z_T, \tau_2) \in \mathcal{H}_T$ *where*

$$\beta_f(m, \mu_T, n) = \sum_{\substack{\mu \in \Lambda^\sharp, q_S(\mu) \le mn \\ T^{-1}\,{}^tBS\mu = \mu_T}} \alpha_f(m, \mu, n)$$

for $m, n \in \mathbb{N}_0$ *and* $\mu_T \in \Lambda_T^\sharp$ *with* $q_T(\mu_T) \le mn$.

PROOF Let $f \in [\Gamma_S, k, \chi]$. We have to show that $f|\mathcal{H}_T$ transforms like a modular form for Γ_T. Since we only consider characters of Γ_S for which the restriction to Γ_T exists we only have to check that

$$j(M^{(S)}, \iota_T^S(w)) = j(M, w) \tag{2.9}$$

for all $w \in \mathcal{H}_T$ and all $M \in \Gamma_T$ where $M^{(S)} \in \varphi_T^{-1}(M)$ is an element of Γ_S which corresponds to M. Since $\varphi_T^{-1}(I_{l'+4})$ only contains elements of Γ_S of the form R_A, $A \in \mathrm{O}(\Lambda)$, we note that $j(M^{(S)}, \iota_T^S(w))$ is independent of the choice of the preimage $M^{(S)}$ of M. Moreover, φ_T is a homomorphism, and thus it suffices to verify (2.9) for the generators of Γ_T. For T_g, $g \in \Lambda_{T_0}$, and R_A, $A \in \mathrm{O}(\Lambda_T)$, this is trivial, and for $M = J$ the fact that ι_T^S is an isometric embedding implies

$$j(J^{(S)}, \iota_T^S(w)) = q_{S_0}(\iota_T^S(w)) = q_{T_0}(w) = j(J, w)$$

for all $w \in \mathcal{H}_T$. So for all $M \in \Gamma_T$ and all $w \in \mathcal{H}_T$ we have

$$\begin{aligned}
((f|\mathcal{H}_T)|_k M)(w) &= j(M, w)^{-k}\ (f|\mathcal{H}_T)(M\langle w\rangle) \\
&= j(M^{(S)}, \iota_T^S(w))^{-k}\ f(M^{(S)}\langle \iota_T^S(w)\rangle) \\
&= (f|_k M^{(S)})(\iota_T^S(w)) \\
&= \chi(M^{(S)})\ f(\iota_T^S(w)) \\
&= (\chi|\Gamma_T)(M)\ (f|\mathcal{H}_T)(w)
\end{aligned}$$

where $M^{(S)}$ is an arbitrary preimage of M in Γ_S. Hence $f|\mathcal{H}_T \in [\Gamma_T, k, \chi|\Gamma_T]$.

Since ι_T^S is an isometric embedding we have $T = {}^tBSB$. Given $\mu \in \Lambda^\sharp$ we observe that $(\mu, \iota_T^S(z_T))_S = (T^{-1}\,{}^tBS\mu, z_T)_T$ for all $z_T \in \mathbb{C}^{l'}$. Moreover, if $\mu \in \Lambda^\sharp$ then $(\mu, \lambda)_S \in \mathbb{Z}$ for all $\lambda \in \Lambda$. So, in particular, $(\mu, \iota_T^S(\lambda_T))_S = (T^{-1}\,{}^tBS\mu, \lambda_T)_T \in \mathbb{Z}$ for all $\lambda_T \in \Lambda_T$ which implies $\mu_T = T^{-1}\,{}^tBS\mu \in \Lambda_T^\sharp$. It remains to be shown that $q_T(\mu_T) \le mn$ whenever $q_S(\mu) \le mn$. Obviously, it suffices to show

$$q_S(\mu) - q_T(\mu_T) = {}^t\mu(S - SBT^{-1}\,{}^tBS)\mu \ge 0 \qquad \text{for all } \mu \in \Lambda^\sharp.$$

Let $\mu \in \Lambda^\sharp$. There exist $x \in \Lambda_T \otimes \mathbb{R}$ and $y \in \iota_T^S(\Lambda_T)^\perp$ (the orthogonal complement of $\iota_T^S(\Lambda_T)$ in $\Lambda_S \otimes \mathbb{R}$) such that

$$\mu = \iota_T^S(x) + y = Bx + y.$$

Then

$$\begin{aligned}
&{}^t\mu(S - SBT^{-1}\,{}^tBS)\mu \\
&\qquad = {}^t(Bx+y)(S - SBT^{-1}\,{}^tBS)(Bx+y) \\
&\qquad = {}^tx(\underbrace{{}^tBSB - {}^tBSBT^{-1}\,{}^tBSB}_{=T-TT^{-1}T=0})x + 2(\underbrace{{}^t(Bx)Sy}_{=0} - \underbrace{{}^t(Bx)SBT^{-1}\,{}^tBSy}_{={}^t(Bx)Sy=0}) \\
&\qquad\qquad + {}^tySy - \underbrace{{}^tySB}_{=0}T^{-1}\underbrace{{}^tBSy}_{=0} \\
&\qquad = {}^tySy \geq 0
\end{aligned}$$

because due to the choice of y we have ${}^t(Bx)Sy = 0$ for all $x \in \Lambda_T \otimes \mathbb{R}$ and thus ${}^tBSy = 0$. This completes the proof. ∎

Since we explicitly know how the Fourier expansion of the restriction of a modular form arises from the Fourier expansion of the restricted modular form we can easily show that restrictions of Maaß forms are Maaß forms.

Corollary 2.26 *Let S, T, ι_T^S, B and χ be given as in the preceding theorem. Moreover, let $k \in \mathbb{N}$. If $f \in \mathcal{M}(\Gamma_S, k, \chi)$ then $f|\mathcal{H}_T \in \mathcal{M}(\Gamma_T, k, \chi|\Gamma_T)$.*

PROOF Let $f \in \mathcal{M}(\Gamma_S, k, \chi)$. In view of Theorem 2.25 it remains to be shown that the Fourier coefficients of $f|\mathcal{H}_T$ satisfy the Maaß condition (2.8). Let $0 \neq (m, \mu_T, n) \in \Lambda_{T_0}^\sharp$ with $m, n \in \mathbb{N}_0$ and $\mu_T \in \Lambda_T^\sharp$ such that $q_T(\mu_T) \leq mn$. We set $g = \gcd(m, T\mu_T, n)$. Note that $\gcd(m, S\mu, n)$ divides g whenever ${}^tBS\mu = T\mu_T$ because $d|S\mu$ implies $d|\,{}^tBS\mu$ due to $B \in \mathrm{Mat}(l, l'; \mathbb{Z})$. Thus we have

$$\begin{aligned}
\beta_f(m, \mu_T, n) &= \sum_{\substack{\mu\in\Lambda^\sharp, q_S(\mu)\leq mn \\ T^{-1}\,{}^tBS\mu=\mu_T}} \alpha_f(m, \mu, n) \\
&= \sum_{t|g} \sum_{\substack{\mu\in\Lambda^\sharp, q_S(\mu)\leq mn \\ T^{-1}\,{}^tBS\mu=\mu_T \\ \gcd(m,S\mu,n)=t}} \alpha_f(m, \mu, n) \\
&= \sum_{t|g} \sum_{\substack{\mu\in\Lambda^\sharp, q_S(\mu)\leq mn \\ T^{-1}\,{}^tBS\mu=\mu_T \\ \gcd(m,S\mu,n)=t}} \sum_{d|t} d^{k-1}\alpha_f(mn/d^2, \mu/d, 1) \\
&= \sum_{d|g} d^{k-1} \sum_{\substack{\mu\in\Lambda^\sharp, q_S(\mu)\leq mn \\ T^{-1}\,{}^tBS\mu=\mu_T \\ d|\gcd(m,S\mu,n)}} \alpha_f(mn/d^2, \mu/d, 1) \\
&= \sum_{d|g} d^{k-1}\beta_f(mn/d^2, \mu_T/d, 1).
\end{aligned}$$

Hence $f|\mathcal{H}_T \in \mathcal{M}(\Gamma_T, k, \chi|\Gamma_T)$. ∎

2.5.2. Restrictions of modular forms living on $\mathcal{H}_{D_4}$

We consider restrictions of modular forms living on $\mathcal{H}_{D_4}$ to the submanifolds $\mathcal{H}_{A_3}$ and $\mathcal{H}_{A_1^{(3)}}$. For $T \in \{A_3, A_1^{(3)}\}$ the lattices $\Lambda_T = \mathbb{Z}^3$ with bilinear form $(\cdot,\cdot)_T$ can be viewed as sublattices of $\Lambda = \mathbb{Z}^4$ with bilinear form $(\cdot,\cdot)_{D_4}$ via the isometric embeddings

$$\begin{aligned} \iota_{A_3}^{D_4} &: \Lambda_{A_3} \to \Lambda, \quad && (x_1,x_2,x_3) \mapsto (x_1,x_3,0,x_2),\\ \iota_{A_1^{(3)}}^{D_4} &: \Lambda_{A_1^{(3)}} \to \Lambda, \quad && (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,0). \end{aligned}$$

Correspondingly, the half-spaces $\mathcal{H}_{A_3}$ and $\mathcal{H}_{A_1^{(3)}}$ can be considered as submanifolds of $\mathcal{H}_{D_4}$ via the embeddings

$$\begin{aligned} \iota_{A_3}^{D_4} &: \mathcal{H}_{A_3} \to \mathcal{H}_{D_4}, \quad && (\tau_1,z_1,z_2,z_3,\tau_2) \mapsto (\tau_1,z_1,z_3,0,z_2,\tau_2),\\ \iota_{A_1^{(3)}}^{D_4} &: \mathcal{H}_{A_1^{(3)}} \to \mathcal{H}_{D_4}, \quad && (\tau_1,z_1,z_2,z_3,\tau_2) \mapsto (\tau_1,z_1,z_2,z_3,0,\tau_2). \end{aligned}$$

First we consider restrictions to $\mathcal{H}_{A_3}$.

Proposition 2.27 *Let $k \in 2\mathbb{Z}$.*
a) If $f \in [\Gamma_{D_4}, k, \chi]$ with $\chi \in \{\nu_\pi, \det\}$ then f vanishes on $\mathcal{H}_{A_3}$.
b) If $f \in [\Gamma_{D_4}, k, (\nu_\pi \det)^m]$, $m \in \{0,1\}$, then $f|\mathcal{H}_{A_3} \in [\Gamma_{A_3}, k, (\nu_\pi \det)^m]$.
c) If $f \in \mathcal{M}(\Gamma_{D_4}, k, 1)$ then $f|\mathcal{H}_{A_3} \in \mathcal{M}(\Gamma_{A_3}, k, 1)$.

PROOF Let $M = R_{\left(\begin{smallmatrix}1&0&-1&0\\0&1&-1&0\\0&0&-1&0\\0&0&2&1\end{smallmatrix}\right)} \in \Gamma_{D_4}$. For all $w \in \iota_{A_3}^{D_4}(\mathcal{H}_{A_3})$ we have $w = M\langle w\rangle$.

a) Let $f \in [\Gamma_{D_4}, k, \chi]$, $\chi \in \{\nu_\pi, \det\}$. Due to $\chi(M) = -1$ we have $f(w) = (f|_k M)(w) = \chi(M) f(w) = -f(w)$ for all $w \in \iota_{A_3}^{D_4}(\mathcal{H}_{A_3})$. Thus f vanishes on $\mathcal{H}_{A_3}$.
b) Let $\chi = (\nu_\pi \det)^m \in \Gamma_{D_4}^{\mathrm{ab}}$, $m \in \{0,1\}$. We have to show that $\chi|\Gamma_{A_3} = (\nu_\pi \det)^m \in \Gamma_{A_3}^{\mathrm{ab}}$. It is easy to check that the above matrix M is the only non-trivial element of Γ_{D_4} acting trivially on $\iota_{A_3}^{D_4}(\mathcal{H}_{A_3})$. Due to $\chi(M) = 1$ the restriction of χ to Γ_{A_3} is well defined. By explicit calculation of some character values we can verify that $\chi|\Gamma_{A_3} = (\nu_\pi \det)^m$ holds.Thus we can apply Theorem 2.25 which proves the assertion.
c) Apply Corollary 2.26. ∎

Next we show similar results for restrictions of modular forms to $\mathcal{H}_{A_1^{(3)}}$.

Proposition 2.28 *Let $k \in 2\mathbb{Z}$.*
a) If $f \in [\Gamma_{D_4}, k, \nu_\pi^m \det]$, $m \in \{0,1\}$, then f vanishes on $\mathcal{H}_{A_1^{(3)}}$.
b) If $f \in [\Gamma_{D_4}, k, \nu_\pi^m]$, $m \in \{0,1\}$, then $f|\mathcal{H}_{A_1^{(3)}} \in [\Gamma_{A_1^{(3)}}, k, \nu_\pi^m]$.
c) If $f \in \mathcal{M}(\Gamma_{D_4}, k, 1)$ then $f|\mathcal{H}_{A_1^{(3)}} \in \mathcal{M}(\Gamma_{A_1^{(3)}}, k, 1)$.

PROOF Note that $R_{\left(\begin{smallmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&-1\end{smallmatrix}\right)} \in \Gamma_{D_4}$ acts trivially on $\iota_{A_1^{(3)}}^{D_4}(\mathcal{H}_{A_1^{(3)}})$. Now the assertions can be proved analogously to Proposition 2.27.

2.5.3. Restrictions of modular forms living on $\mathcal{H}_{A_3}$

Now we look at restrictions of modular forms living on $\mathcal{H}_{A_3}$ to the submanifolds $\mathcal{H}_T$, $T \in \{A_1^{(2)}, A_2, S_2\}$. The lattices $\Lambda_T = \mathbb{Z}^2$ with bilinear form $(\cdot,\cdot)_T$ can be considered as sublattice of $\Lambda = \mathbb{Z}^3$ with bilinear form $(\cdot,\cdot)_{A_3}$ via the isometric embeddings

$$\begin{aligned} \iota^{A_3}_{A_1^{(2)}} &: \Lambda_{A_1^{(2)}} \to \Lambda, \quad (x_1,x_2) \mapsto (x_1, 0, x_2),\\ \iota^{A_3}_{A_2} &: \Lambda_{A_2} \to \Lambda, \quad (x_1,x_2) \mapsto (x_1, x_2, 0),\\ \iota^{A_3}_{S_2} &: \Lambda_{S_2} \to \Lambda, \quad (x_1,x_2) \mapsto (x_1 - x_2, 2x_2, -x_2). \end{aligned}$$

The corresponding embeddings of the half-spaces $\mathcal{H}_T$ in $\mathcal{H}_{A_3}$ are given by

$$\begin{aligned} \iota^{A_3}_{A_1^{(2)}} &: \mathcal{H}_{A_1^{(2)}} \to \mathcal{H}_{A_3}, \quad (\tau_1, z_1, z_2, \tau_2) \mapsto (\tau_1, z_1, 0, z_2, \tau_2),\\ \iota^{A_3}_{A_2} &: \mathcal{H}_{A_2} \to \mathcal{H}_{A_3}, \quad (\tau_1, z_1, z_2, \tau_2) \mapsto (\tau_1, z_1, z_2, 0, \tau_2),\\ \iota^{A_3}_{S_2} &: \mathcal{H}_{S_2} \to \mathcal{H}_{A_3}, \quad (\tau_1, z_1, z_2, \tau_2) \mapsto (\tau_1, z_1 - z_2, 2z_2, -z_2, \tau_2). \end{aligned}$$

For $T \in \{A_1^{(2)}, S_2\}$ each element of Γ_T is restriction of two elements of Γ_{A_3} while in case of $T = A_2$ each element of Γ_T is restriction of exactly one element of Γ_{A_3}, i.e., the homomorphism φ_{A_2} is injective. This allows us to derive a first

Proposition 2.29 *Let $k \in \mathbb{Z}$.*
a) If k is odd and $f \in [\Gamma'_{A_3}, k, 1]$ then f vanishes on $\mathcal{H}_{A_1^{(2)}}$.
b) If k is odd and $f \in [\Gamma_{A_3}, k, \det]$ or k is even and $f \in [\Gamma_{A_3}, k, \nu_\pi \det]$ then f vanishes on $\mathcal{H}_{S_2}$.

PROOF a) Let $f \in [\Gamma'_{A_3}, k, 1]$, k odd. Then $f = f_{\nu_\pi} + f_{\det}$ for some $f_\chi \in [\Gamma_{A_3}, k, \chi]$ since $[\Gamma'_{A_3}, k, 1] = [\Gamma_{A_3}, k, \nu_\pi] \oplus [\Gamma_{A_3}, k, \det]$ according to Corollary 2.3. For all $w \in \iota^{A_3}_{A_1^{(2)}}(\mathcal{H}_{A_1^{(2)}})$ we have

$$w = M\langle w \rangle \quad \text{for } M = R_{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)},$$

and thus for $\chi \in \{\nu_\pi, \det\}$

$$f_\chi(w) = (f_\chi|_k M)(w) = \chi(M) f_\chi(w) = -f_\chi(w) \quad \text{for all } w \in \iota^{A_3}_{A_1^{(2)}}(\mathcal{H}_{A_1^{(2)}}).$$

b) Let $f \in [\Gamma_{A_3}, k, \chi]$, $\chi = \nu_\pi^{k+1} \det$. Then for all $w \in \iota^{A_3}_{S_2}(\mathcal{H}_{S_2})$ we have

$$w = M\langle w \rangle \quad \text{for } M = R_{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{smallmatrix}\right)},$$

and thus

$$f(w) = (f|_k M)(w) = \chi(M) f(w) = -f(w) \quad \text{for all } w \in \iota^{A_3}_{S_2}(\mathcal{H}_{S_2}). \qquad \blacksquare$$

Now we examine how the Abelian characters of Γ_{A_3} and Γ_T are related to each other since this is important to identify the characters of the restrictions of the orthogonal modular forms.

Proposition 2.30 *The following table shows for $T \in \{A_1^{(2)}, A_2, S_2\}$ which Abelian characters of Γ_T the nontrivial Abelian characters of Γ_{A_3} correspond to. Those Abelian characters of Γ_T which do not occur in this table do not possess a continuation on Γ_{A_3}.*

$\chi \in \Gamma_{A_3}^{\mathrm{ab}}$	$\chi\|\Gamma_{A_1^{(2)}}$	$\chi\|\Gamma_{A_2}$	$\chi\|\Gamma_{S_2}$
ν_π	-	ν_π	ν_π
$\det$	-	$\nu_\pi \det$	-
$\nu_\pi \det$	$\nu_\pi \det$	$\det$	-

PROOF This can be verified by explicit calculation. In particular, note that the restriction of χ to Γ_T does not exist if the value of χ is not independent of the choice of the preimage of $M \in \Gamma_T$ in Γ_{A_3}. ■

Finally we take a look at the restrictions of orthogonal modular forms.

Theorem 2.31 *Let $k \in \mathbb{Z}$ and $m \in \{0, 1\}$.*
a) If k is even and $f \in [\Gamma_{A_3}, k, \nu_\pi^m \det^m]$ then $f|\mathcal{H}_{A_1^{(2)}} \in [\Gamma_{A_1^{(2)}}, k, \nu_\pi^m \det^m]$.
b) If $f \in [\Gamma_{A_3}, k, \nu_\pi^m \det^{m+k}]$ then $f|\mathcal{H}_{A_2} \in [\Gamma_{A_2}, k, \nu_\pi^k \det^{m+k}]$.
c) If $f \in [\Gamma_{A_3}, k, \nu_\pi^k]$ then $f|\mathcal{H}_{S_2} \in [\Gamma_{S_2}, k, \nu_\pi^k]$.

PROOF Apply Theorem 2.25 and Proposition 2.30. ■

2.5.4. Restrictions of modular forms living on $\mathcal{H}_{A_1^{(3)}}$

Finally we look at the restrictions of modular forms living on $\mathcal{H}_{A_1^{(3)}}$ to the submanifolds $\mathcal{H}_T$, $T \in \{A_1^{(2)}, S_2\}$. The lattices $\Lambda_T = \mathbb{Z}^2$ with bilinear form $(\cdot,\cdot)_T$ can be considered as sublattice of $\Lambda = \mathbb{Z}^3$ with bilinear form $(\cdot,\cdot)_{A_1^{(3)}}$ via the isometric embeddings

$$\begin{aligned} \iota_{A_1^{(2)}}^{A_3} &: \Lambda_{A_1^{(2)}} \to \Lambda, \quad (x_1, x_2) \mapsto (x_1, x_2, 0), \\ \iota_{S_2}^{A_3} &: \Lambda_{S_2} \to \Lambda, \quad (x_1, x_2) \mapsto (x_1, x_2, x_2). \end{aligned}$$

The corresponding embeddings of the half-spaces $\mathcal{H}_T$ in $\mathcal{H}_{A_3}$ are given by

$$\begin{aligned} \iota_{A_1^{(2)}}^{A_3} &: \mathcal{H}_{A_1^{(2)}} \to \mathcal{H}_{A_3}, \quad (\tau_1, z_1, z_2, \tau_2) \mapsto (\tau_1, z_1, z_2, 0, \tau_2), \\ \iota_{S_2}^{A_3} &: \mathcal{H}_{S_2} \to \mathcal{H}_{A_3}, \quad (\tau_1, z_1, z_2, \tau_2) \mapsto (\tau_1, z_1, z_2, z_2, \tau_2). \end{aligned}$$

Each element of Γ_T is restriction of two elements of $\Gamma_{A_1^{(3)}}$. Therefore we get

Proposition 2.32 *Let $k \in \mathbb{Z}$ and $m \in \{0, 1\}$.*

a) *If k is odd and $f \in [\Gamma'_{A_1^{(3)}}, k, 1]$ then f vanishes on $\mathcal{H}_{A_1^{(2)}}$.*

b) *If k is odd and $f \in [\Gamma_{A_1^{(3)}}, k, \nu_2^m \det]$ or k is even and $f \in [\Gamma_{A_1^{(3)}}, k, \nu_2^m \nu_\pi]$ then f vanishes on $\mathcal{H}_{S_2}$.*

PROOF a) Let $f \in [\Gamma_{A_1^{(3)}}, k, \chi]$, k odd, $\chi \in \Gamma^{\mathrm{ab}}_{A_1^{(3)}}$. For all $w \in \iota_{A_1^{(2)}}^{A_1^{(3)}}(\mathcal{H}_{A_1^{(2)}})$ we have

$$w = M\langle w\rangle \quad \text{for } M = R_{\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right)},$$

and thus

$$f(w) = (f|_k M)(w) = \chi(M) f(w) = -f(w) \quad \text{for all } w \in \iota_{A_1^{(2)}}^{A_1^{(3)}}(\mathcal{H}_{A_1^{(2)}})$$

whenever $\chi \cdot \det \in \langle \nu_2, \nu_\pi\rangle$. On the other hand, if $\chi \cdot \det \notin \langle \nu_2, \nu_\pi\rangle$ then, according to Corollary 2.3, f vanishes identically on $\mathcal{H}_{A_1^{(3)}}$. Hence f vanishes on $\mathcal{H}_{A_1^{(2)}}$.

Now let $f \in [\Gamma'_{A_1^{(3)}}, k, 1]$. Then there exist $f_\chi \in [\Gamma_{A_1^{(3)}}, k, \chi]$, $\chi \in \Gamma^{\mathrm{ab}}_{A_1^{(3)}}$, such that $f = \sum_\chi f_\chi$. Due to the above all f_χ vanish on $\mathcal{H}_{A_1^{(2)}}$, and consequently f also vanishes.

b) Let $f \in [\Gamma_{A_1^{(3)}}, k, \chi]$, $\chi = \nu_2^m \nu_\pi^{k+1} \det^k$, $m \in \{0,1\}$. For all $w \in \iota_{S_2}^{A_1^{(2)}}(\mathcal{H}_{S_2})$ we have

$$w = M\langle w\rangle \quad \text{for } M = R_{\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)},$$

and thus

$$f(w) = (f|_k M)(w) = \chi(M) f(w) = -f(w) \quad \text{for all } w \in \iota_{S_2}^{A_1^{(3)}}(\mathcal{H}_{S_2}).$$

Hence f vanishes on $\mathcal{H}_{S_2}$. ∎

Next we examine how the Abelian characters of $\Gamma_{A_1^{(3)}}$ and Γ_T are related to each other.

Proposition 2.33 *The following table shows for $T \in \{A_1^{(2)}, S_2\}$ which Abelian characters of Γ_T the Abelian characters of $\Gamma_{A_1^{(3)}}$ correspond to (ν_2^* stands for an arbitrary power of ν_2). Those Abelian characters of Γ_T which do not occur in this table do not possess a continuation on $\Gamma_{A_1^{(3)}}$.*

$\chi \in \Gamma^{\mathrm{ab}}_{A_1^{(3)}}$	$\chi\vert\Gamma_{A_1^{(2)}}$	$\chi\vert\Gamma_{S_2}$
ν_2^*	ν_2^*	ν_2^*
$\nu_2^*\nu_\pi$	$\nu_2^*\nu_\pi$	-
$\nu_2^* \det$	-	-
$\nu_2^*\nu_\pi \det$	-	$\nu_2^*\nu_\pi$

PROOF This can be proved analogously to Proposition 2.30. ∎

Finally we again take a look at the restrictions of orthogonal modular forms.

Theorem 2.34 *Let* $k \in \mathbb{Z}$ *and* $m, n \in \{0, 1\}$.
a) If k *is even and* $f \in [\Gamma_{A_1^{(3)}}, k, \nu_2^m \nu_\pi^n]$ *then* $f|\mathcal{H}_{A_1^{(2)}} \in [\Gamma_{A_1^{(2)}}, k, \nu_2^m \nu_\pi^n]$.
b) If $f \in [\Gamma_{A_1^{(3)}}, k, \nu_2^m \nu_\pi^k \det^k]$ *then* $f|\mathcal{H}_{S_2} \in [\Gamma_{S_2}, k, \nu_2^m \nu_\pi^k]$.

PROOF Apply Theorem 2.25 and Proposition 2.33. ∎

2.6. Hermitian modular forms of degree 2

The orthogonal modular forms which live on $\mathcal{H}_S$, $S \in \{A_1^{(2)}, A_2, S_2\}$, can also be considered as Hermitian modular forms. Since we will later need results about graded rings of orthogonal modular forms for the aforementioned S in order to derive our results about the graded rings of orthogonal modular forms for $\mathrm{O}(2,5)$ we briefly show how the results about graded rings of Hermitian modular forms of degree 2 stated in [De01], [DK03] and [DK04] can be translated to our setting. For details confer [De01].

The *Hermitian half-space* $H(2;\mathbb{C})$ of degree 2 is given by

$$H(2;\mathbb{C}) = \left\{ Z = \begin{pmatrix} \tau_1 & z_1 \\ z_2 & \tau_2 \end{pmatrix} \in \mathrm{Mat}(2;\mathbb{C});\ \frac{1}{2i}(Z - {}^t\overline{Z}) > 0 \right\}.$$

Let $\mathbb{K} = \mathbb{Q}(\sqrt{-\Delta_\mathbb{K}})$ be an imaginary quadratic number field with discriminant $-\Delta_\mathbb{K}$ and class number $h(-\Delta_\mathbb{K}) = 1$, and let

$$\mathfrak{o}_\mathbb{K} = \mathbb{Z} + \mathbb{Z}\omega_\mathbb{K},\ \omega_\mathbb{K} = \begin{cases} i\sqrt{\Delta_\mathbb{K}}/2 & \text{if } \Delta_\mathbb{K} \equiv 0 \pmod 4, \\ (1 + i\sqrt{\Delta_\mathbb{K}})/2 & \text{if } \Delta_\mathbb{K} \equiv 3 \pmod 4, \end{cases}$$

be its ring of integers. The *unitary group* of degree 2 over $\mathbb{K}$ is defined by $\mathrm{U}(2;\mathbb{K}) = \{M \in \mathrm{Mat}(4;\mathbb{K});\ {}^t\overline{M} J_{\mathrm{Her}} M = J_{\mathrm{Her}}\}$ where $J_{\mathrm{Her}} = \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right)$, the *special unitary group* is defined by $\mathrm{SU}(2;\mathbb{K}) = \mathrm{U}(2;\mathbb{K}) \cap \mathrm{SL}(4;\mathbb{K})$, and the *Hermitian modular group* is defined by $\Gamma(2;\mathbb{K}) = \mathrm{U}(2;\mathfrak{o}_\mathbb{K}) = \mathrm{U}(2;\mathbb{K}) \cap \mathrm{Mat}(4;\mathfrak{o}_\mathbb{K})$. The unitary group acts on $H(2;\mathbb{C})$ as group of biholomorphic automorphisms via

$$(M, Z) \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Obviously, scalar matrices act trivially on $H(2;\mathbb{C})$. The group of all biholomorphic automorphisms $\mathrm{Bihol}(H(2;\mathbb{C}))$ is generated by $\mathrm{SU}(2;\mathbb{C})$ and the additional biholomorphic automorphism

$$I_{\mathrm{tr}} : H(2;\mathbb{C}) \to H(2;\mathbb{C}),\ Z \mapsto {}^tZ.$$

To be precise, we have

$$\mathrm{Bihol}(H(2;\mathbb{C})) \cong \mathrm{PSU}(2;\mathbb{C}) \rtimes \langle I_{\mathrm{tr}}\rangle,$$

where $\mathrm{PSU}(2;\mathbb{C}) = \mathrm{U}(2;\mathbb{C})/(\mathbb{C}^\times \cdot I_4)$ (cf. [Kr85, Thm. II.1.8]). Therefore in case of $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ we only need to consider elements of $\Gamma(2;\mathbb{K})$ of determinant 1. We set

$$\widetilde{\Gamma(2;\mathbb{K})} := \begin{cases} \Gamma(2;\mathbb{K}) \cap \mathrm{SL}(4;\mathbb{K}), & \text{if } \Delta_\mathbb{K} = 3, \\ \Gamma(2;\mathbb{K}), & \text{if } \Delta_\mathbb{K} \neq 3, \end{cases}$$

and we define the *extended Hermitian modular group* $\Gamma_\mathbb{K}$ as subgroup of $\mathrm{Bihol}(H(2;\mathbb{C}))$ by

$$\Gamma_\mathbb{K} := \left\langle \{Z \mapsto M\langle Z\rangle;\ M \in \widetilde{\Gamma(2;\mathbb{K})}\},\ I_{\mathrm{tr}} \right\rangle.$$

A *Hermitian modular form* of weight $k \in \mathbb{Z}$ with respect to $\Gamma_\mathbb{K}$ and with respect to an Abelian character $\chi \in \Gamma_\mathbb{K}^{\mathrm{ab}}$ is a holomorphic function $f : H(2;\mathbb{C}) \to \mathbb{C}$ satisfying

$$(f|_k M)(Z) := \det(CZ+D)^{-k}\ f(M\langle Z\rangle) = \chi(M)\ f(Z)$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \widetilde{\Gamma(2,\mathbb{K})}$ and, additionally,

$$f \circ I_{\mathrm{tr}} = \chi(I_{\mathrm{tr}})\ f.$$

If f satisfies $f \circ I_{\mathrm{tr}} = f$, i.e., we have $\chi(I_{\mathrm{tr}}) = 1$, then we call f *symmetric*, otherwise we call f *skew-symmetric*. We denote the vector space of those forms by $[\Gamma_\mathbb{K}, k, \chi]$. The subspace of cusp forms which is as usual defined as kernel of Siegel's Φ-operator (note that this relies on $h(-\Delta_\mathbb{K}) = 1$) is denoted by $[\Gamma_\mathbb{K}, k, \chi]_0$.

Examples of Hermitian modular forms are given by the Hermitian Eisenstein series

$$E_k^\mathbb{K}(Z) := \sum_{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma(2;\mathbb{K})_0 \backslash \Gamma(2;\mathbb{K})} (\det M)^{k/2} \det(CZ+D)^{-k}$$

for even $k > 4$ where $\Gamma(2;\mathbb{K})_0 = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma(2;\mathbb{K});\ C = 0\}$. Additionally, we define $E_4^\mathbb{K}$ as Maaß lift (cf. [Kr91]) with constant Fourier coefficient equal to 1. According to [DK03] we have $E_k^\mathbb{K} \in [\Gamma_\mathbb{K}, k, \det^{-k/2}]$ for all even $k \geq 4$. In particular, the Eisenstein series are symmetric modular forms.

Let

$$S^\mathbb{K} = \begin{pmatrix} 2 & 2\,\mathrm{Re}(\omega) \\ 2\,\mathrm{Re}(\omega) & 2|\omega|^2 \end{pmatrix}.$$

Then

$$\varphi_\mathbb{K} : \mathcal{H}_{S^\mathbb{K}} \to H(2;\mathbb{C}),\ (x_1,u_1,u_2,x_2) + i(y_1,v_1,v_2,y_2) \mapsto \begin{pmatrix} x_1 + iy_1 & (u_1+\omega u_2) + i(v_1 + \omega v_2) \\ (u_1 + \overline{\omega} u_2) + i(v_1 + \overline{\omega} v_2) & x_2 + iy_2 \end{pmatrix}$$

biholomorphically maps the orthogonal half-space $\mathcal{H}_{S^\mathbb{K}}$ to the Hermitian half-space $H(2;\mathbb{C})$. Via this map we can identify $\mathrm{Bihol}(\mathcal{H}_{S^\mathbb{K}})$ and $\mathrm{Bihol}(H(2;\mathbb{C}))$. Thus according to Remark

1.12 we have

$$\mathrm{PO}^+(S_1^{\mathbb{K}};\mathbb{R}) \cong \mathrm{PSU}(2;\mathbb{C}) \rtimes \langle I_{\mathrm{tr}}\rangle. \tag{2.10}$$

Now we consider the three imaginary quadratic number fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-2})$ of discriminant -4, -3 and -8, respectively. The corresponding matrices $S^{\mathbb{K}}$ are $A_1^{(2)}$, A_2 and S_2, respectively. The isomorphism (2.10) allows us to identify the extended Hermitian modular group $\Gamma_{\mathbb{K}}$ with a subgroup of $\mathrm{PO}^+(S_1^{\mathbb{K}};\mathbb{R})$. We get

$$\Gamma_{\mathbb{Q}(\sqrt{-1})} \cong \Gamma_{A_1^{(2)}}/\{\pm I_6\}, \qquad \Gamma_{\mathbb{Q}(\sqrt{-3})} \cong \Gamma_{A_2}/\{\pm I_6\}, \qquad \Gamma_{\mathbb{Q}(\sqrt{-2})} \cong \Gamma_{S_2}/\{\pm I_6\}.$$

In Appendix B we list the generators of $\Gamma_{\mathbb{K}}$ and the elements of $\Gamma_{S^{\mathbb{K}}}$ those generators correspond to, and we determine the characters of $\Gamma_{\mathbb{K}}$. We have

$$\Gamma^{\mathrm{ab}}_{\mathbb{Q}(\sqrt{-1})} = \langle \det, \nu_\wp, \nu_{\mathrm{skew}}\rangle\,, \qquad \Gamma^{\mathrm{ab}}_{\mathbb{Q}(\sqrt{-3})} = \langle \nu_{\mathrm{skew}}\rangle\,, \qquad \Gamma^{\mathrm{ab}}_{\mathbb{Q}(\sqrt{-2})} = \langle \nu_\wp, \nu_{\mathrm{skew}}\rangle\,.$$

Theorem 2.35 *Let $k \in \mathbb{Z}$ and $l, m, n \in \{0,1\}$.*
a) If k is even and $f \in [\Gamma_{\mathbb{Q}(\sqrt{-1})}, k, \det^l \nu_\wp^m \nu_{\mathrm{skew}}^n]$ then $f \circ \varphi_{\mathbb{Q}(\sqrt{-1})} \in [\Gamma_{A_1^{(2)}}, k, \nu_\pi^{l+k/2}\nu_2^m \det^n]$.
b) If $f \in [\Gamma_{\mathbb{Q}(\sqrt{-3})}, k, \nu_{\mathrm{skew}}^n]$ then $f \circ \varphi_{\mathbb{Q}(\sqrt{-3})} \in [\Gamma_{A_2}, k, \nu_\pi^k \det^{n+k}]$.
c) If $f \in [\Gamma_{\mathbb{Q}(\sqrt{-2})}, k, \nu_\wp^m \nu_{\mathrm{skew}}^n]$ then $f \circ \varphi_{\mathbb{Q}(\sqrt{-2})} \in [\Gamma_{S_2}, k, \nu_2^m \nu_\pi^k \det^{n+k}]$.

PROOF Let $f \in [\Gamma_{\mathbb{K}}, k, \chi]$. We have to show that $\widetilde{f} := f \circ \varphi_{\mathbb{K}} : \mathcal{H}_{S^{\mathbb{K}}} \to \mathbb{C}$ transforms like a modular form with respect to $\Gamma_{S^{\mathbb{K}}}$ and the character $\widetilde{\chi}$ given above. Let $\widetilde{N} \in \Gamma_{S^{\mathbb{K}}}$. Then $\widetilde{N} = M_{\mathrm{tr}}^r \widetilde{M}$ for some $\widetilde{M \in \Gamma_{S^{\mathbb{K}}}} \cap \mathrm{SO}(S_1^{\mathbb{K}};\mathbb{R})$ and $r \in \{0,1\}$. Let $\gamma = I_{\mathrm{tr}}^r \circ (Z \mapsto M\langle Z\rangle)$ with $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma(2,\mathbb{K})$, be the corresponding element of $\Gamma_{\mathbb{K}}$, i.e.,

$$\gamma(Z) = I_{\mathrm{tr}}^r(M\langle Z\rangle) = \varphi_{\mathbb{K}}(\widetilde{N}\langle \varphi_{\mathbb{K}}^{-1}(Z)\rangle) \quad \text{for all } Z \in H(2;\mathbb{C}).$$

Then for $w \in \mathcal{H}_{S^{\mathbb{K}}}$ and $Z = \varphi_{\mathbb{K}}(w)$ we have

$$\begin{aligned}\widetilde{f}(\widetilde{N}\langle w\rangle) &= f(\varphi_{\mathbb{K}}(\widetilde{N}\langle w\rangle)) = f(I_{\mathrm{tr}}^r(M\langle Z\rangle)) = \chi(I_{\mathrm{tr}}^r)\ f(M\langle Z\rangle)\\ &= \chi(I_{\mathrm{tr}}^r)\ \det(CZ+D)^k\ \chi(M)\ f(Z)\\ &= \chi(I_{\mathrm{tr}}^r)\ \det(CZ+D)^k\ \chi(M)\ \widetilde{f}(w),\end{aligned}$$

and thus

$$\left(\widetilde{f}|_k\widetilde{N}\right)(w) = j(\widetilde{N},w)^{-k}\ \widetilde{f}(\widetilde{N}\langle w\rangle) = j(\widetilde{N},w)^{-k}\ \chi(I_{\mathrm{tr}}^r)\ \det(C\varphi_{\mathbb{K}}(w)+D)^k\ \chi(M)\ \widetilde{f}(w).$$

So we have to show that

$$j(\widetilde{N},w)^{-k}\ \chi(I_{\mathrm{tr}}^r)\ \det(C\varphi_{\mathbb{K}}(w)+D)^k\ \chi(M) = \widetilde{\chi}(\widetilde{N}) = \widetilde{\chi}(M_{\mathrm{tr}}^r)\ \widetilde{\chi}(\widetilde{M})$$

for all $\widetilde{N} \in \Gamma_{S^{\mathbb{K}}}$ and all $w \in \mathcal{H}_{S^{\mathbb{K}}}$. Since $j(\widetilde{N},w)$ and $j_{\mathrm{Her}}(M,Z) := \det(CZ+D)$ are factors of automorphy and, moreover, $j(M_{\mathrm{tr}}, w) = 1$ we only have to verify that

$$j(\widetilde{M}, w)^{-k}\ \det(C\varphi_{\mathbb{K}}(w) + D)^k\ \chi(M) = \widetilde{\chi}(\widetilde{M})$$

holds for the generators of $\Gamma_{S^{\mathbb{K}}} \cap \mathrm{SO}(S_1^{\mathbb{K}}; \mathbb{R})$ and that

$$\chi(I_{\mathrm{tr}}) = \widetilde{\chi}(M_{\mathrm{tr}}).$$

The second equation is true, and the first equation can easily be checked for T_g, $g \in \Lambda_0^{\mathbb{K}}$, and for R_A, $A \in \mathrm{SO}(\Lambda^{\mathbb{K}})$. Finally, for J we have

$$j(J, w) = q_{S_0^{\mathbb{K}}}(w) = \det(\varphi_{\mathbb{K}}(w)) = j_{\mathrm{Her}}(J_{\mathrm{Her}}, \varphi_{\mathbb{K}}(w)).$$

This completes the proof. ∎

For $k \geq 4$, k even, we define orthogonal Eisenstein series $E_k^{S_{\mathbb{K}}}$ by

$$E_k^{S_{\mathbb{K}}} := E_k^{\mathbb{K}} \circ \varphi_{\mathbb{K}}.$$

According to the above theorem we have $E_k^{S_{\mathbb{K}}} \in [\Gamma_{S^{\mathbb{K}}}, k, 1]$.

Using the above theorem we can now translate the results about graded rings of Hermitian modular forms of degree 2 stated in [De01], [DK03] and [DK04] to our setting.

Theorem 2.36 *a) Let $S = A_1^{(2)}$. The graded ring $\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_S, k, 1]$ is generated by*

$$E_4,\ \phi_4,\ E_6,\ E_{10},\ \phi_{10},\ E_{12}\ \textit{and}\ \phi_{30},$$

where the $E_k = E_k^S$ are orthogonal Eisenstein series of weight k, $\phi_4 \in [\Gamma_S, 4, \nu_2\nu_\pi \det]_0$, $\phi_{10} \in [\Gamma_S, 10, \nu_\pi]_0$ and $\phi_{30} \in [\Gamma_S, 30, \nu_2]_0$.
Moreover, the subring $\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_S, 2k, 1]$ is a polynomial ring in

$$E_4,\ E_6,\ \phi_4^2,\ E_{10}\ \textit{and}\ E_{12},$$

i.e., E_4, E_6, ϕ_4^2, E_{10} and E_{12} are algebraically independent.

b) Let $S = A_2$. The graded ring $\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_S, k, 1]$ is generated by

$$E_4,\ E_6,\ \phi_9,\ E_{10},\ E_{12}\ \textit{and}\ \phi_{45},$$

where the $E_k = E_k^S$ are the orthogonal Eisenstein series of weight k, $\phi_9 \in [\Gamma_S, 9, \nu_\pi]_0$ and $\phi_{45} \in [\Gamma_S, 45, \nu_\pi \det]_0$.
Moreover, the subring $\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_S, 2k, 1]$ is a polynomial ring in

$$E_4,\ E_6,\ E_{10},\ E_{12}\ \textit{and}\ \phi_9^2,$$

i.e., E_4, E_6, E_{10}, E_{12} and ϕ_9 are algebraically independent.

PROOF [DK03, Thm. 10, Cor. 9] and [DK03, Thm. 6, Thm. 7]. ∎

2.7. Quaternionic modular forms of degree 2

Similar to Hermitian modular forms quaternionic modular forms of degree 2 can also be considered as orthogonal modular forms. We consider the case $S = D_4$ which corresponds to the case of quaternionic modular forms with respect to the extended modular group for the Hurwitz integers. Since we only need this case in order to define certain examples of modular forms for $\mathrm{O}(2,5)$ we only state the necessary facts. For details confer [Kr85].

Recall that we denote the canonical basis of the skew field $\mathbb{H}$ of Hamilton quaternions by $1, \mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3$. For $z = z_1 + z_2\mathrm{i}_1 + z_3\mathrm{i}_2 + z_4\mathrm{i}_3 \in \mathbb{H}$ with $z_j \in \mathbb{R}$ the conjugate of z is given by $\overline{z} = z_1 - z_2\mathrm{i}_1 - z_3\mathrm{i}_2 - z_4\mathrm{i}_3$ and the norm of z is given by $N(z) = z\overline{z} = z_1^2 + z_2^2 + z_3^2 + z_4^2$.

The *half-space of quaternions* $H(2;\mathbb{H})$ of degree 2 is given by

$$H(2;\mathbb{H}) = \left\{ Z = X + iY \in \mathrm{Mat}(2;\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C};\ Z = {}^t\overline{Z} := {}^t\overline{X} + i\,{}^t\overline{Y},\ Y > 0 \right\}.$$

Let

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\mathrm{i}_1 + \mathbb{Z}\mathrm{i}_2 + \mathbb{Z}\omega,\ \omega = \tfrac{1}{2}(1 + \mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_3)$$

be the Hurwitz order, and let

$$\wp = (1 + \mathrm{i}_1)\mathcal{O} = \mathcal{O}(1 + \mathrm{i}_1) = \{a \in \mathcal{O};\ N(a) \in 2\mathbb{Z}\}$$

be the ideal of even Hurwitz quaternions. The *symplectic group* of degree 2 over $\mathbb{H}$ is defined by

$$\mathrm{Sp}(2;\mathbb{H}) = \{M \in \mathrm{Mat}(4;\mathbb{H});\ {}^t\overline{M} J_{\mathbb{H}} M = J_{\mathbb{H}}\}, \qquad J_{\mathbb{H}} = \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right).$$

It acts on $H(2;\mathbb{H})$ as group of biholomorphic automorphisms via the symplectic transformations

$$(M, Z) \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The group of all biholomorphic automorphisms $\mathrm{Bihol}(H(2;\mathbb{H}))$ is generated by $\mathrm{Sp}(2;\mathbb{H})$ and the additional biholomorphic automorphism

$$I_{\mathrm{tr}} : H(2;\mathbb{H}) \to H(2;\mathbb{H}),\ Z \mapsto {}^tZ.$$

We define the *extended quaternionic modular group* $\Gamma_{\mathbb{H}}$ as subgroup of $\mathrm{Bihol}(H(2;\mathbb{H}))$ by

$$\Gamma_{\mathbb{H}} := \langle \{Z \mapsto M\langle Z\rangle;\ M \in \mathrm{Sp}(2;\mathcal{O}) \text{ or } M = \rho I\},\ I_{\mathrm{tr}}\rangle$$

where $\mathrm{Sp}(2;\mathcal{O}) = \mathrm{Sp}(2;\mathbb{H}) \cap \mathrm{Mat}(4;\mathcal{O})$ and $\rho = (1 + \mathrm{i}_1)/\sqrt{2}$.

A *quaternionic modular form* of weight $k \in 2\mathbb{Z}$ with respect to $\Gamma_{\mathbb{H}}$ and an Abelian character $\chi \in \Gamma_{\mathbb{H}}^{\mathrm{ab}}$ is a holomorphic function $f : H(2;\ \mathbb{H}) \to \mathbb{C}$ satisfying

$$(f|_k M)(Z) := (\det(CZ + D)^{\vee})^{-k/2}\ f(M\langle Z\rangle) = \chi(M)\ f(Z) \quad \text{and} \quad f \circ I_{\mathrm{tr}} = \chi(I_{\mathrm{tr}})\ f$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \langle \mathrm{Sp}(2;\mathcal{O}),\ \rho I\rangle$ where $^\vee$ denotes the representation of quaternions as complex 2×2 matrices. We denote the space of all those functions by $[\Gamma_{\mathbb{H}}, k, \chi]$.

Let

$$\mathcal{S} := \left\{ \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix};\ m, n \in \mathbb{N}_0,\ t \in \mathcal{O}^\sharp,\ N(t) = t\bar{t} \le mn \right\}$$

where $\mathcal{O}^\sharp$ is the dual lattice of $\mathcal{O}$ with respect to the bilinear form $(a, b)_{\mathbb{H}} = 2\,\mathrm{Re}(\bar{a}b)$ for $a, b \in \mathbb{H}$ which is $\mathbb{C}$-linearly extended to $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. Each quaternionic modular form $f \in [\Gamma_{\mathbb{H}}, k, \chi]$ has a Fourier expansion of the form

$$f(Z) = \sum_{T \in \mathcal{S}} \alpha_f(T)\ e^{\pi i\,\mathrm{trace}(T\,{}^t\!\bar{Z} + Z\,{}^t\!\bar{T})} = \sum_{n,m \in \mathbb{N}_0} \sum_{\substack{t \in \mathcal{O}^\sharp \\ N(t) \le mn}} \alpha_f \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix}\ e^{2\pi i (n\tau_1 + m\tau_2 + (t,z)_{\mathbb{H}})}$$

for $Z = \left(\begin{smallmatrix} \tau_1 & z \\ * & \tau_2 \end{smallmatrix}\right) \in H(2;\mathbb{H})$. If the Fourier coefficients of $f \in [\Gamma'_{\mathbb{H}}, k, 1]$ satisfy the condition

$$\alpha_f(T) = \sum_{d \mid \varepsilon(T)} d^{k-1} \alpha_f \begin{pmatrix} 1 & t/d \\ \bar{t}/d & mn/d^2 \end{pmatrix} \quad \text{for all } T = \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix} \in \mathcal{S}, T \ne 0, \tag{2.11}$$

where $\varepsilon(T) = \max\{d \in \mathbb{N};\ d^{-1}T \in \mathcal{S}\}$ then f belongs to the *Maaß space* $\mathcal{M}(\Gamma_{\mathbb{H}}, k)$ (cf. [Kr87]). Note that, according to Krieg, $f \in [\Gamma'_{\mathbb{H}}, k, 1]$ satisfies the Maaß condition (2.11) if and only if a function $\alpha_f^* : \mathbb{N}_0 \to \mathbb{C}$ exists such that

$$\alpha_f(T) = \sum_{d \mid \varepsilon(T)} d^{k-1} \alpha_f^*(4 \det T / d^2) \quad \text{for all } T = \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix} \in \mathcal{S}, T \ne 0. \tag{2.12}$$

Due to $\det({}^t\!T) = \det(T) = \det(\rho T \bar{\rho})$ and $\varepsilon({}^t\!T) = \varepsilon(T) = \varepsilon(\rho T \bar{\rho})$ for all $T \in \mathcal{S}$ the alternative Maaß condition (2.12) implies $\alpha_f({}^t\!T) = \alpha_f(T) = \alpha_f(\rho T \bar{\rho})$ and thus $f({}^t\!Z) = f(Z) = (f|_k(\rho I))(Z)$ for all $f \in \mathcal{M}(\Gamma_{\mathbb{H}}, k)$. Hence $\mathcal{M}(\Gamma_{\mathbb{H}}, k) \subset [\Gamma_{\mathbb{H}}, k, 1]$.

Examples of quaternionic modular forms are given by the quaternionic Siegel-Eisenstein series

$$E_k^{\mathbb{H}}(Z) := \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2;\mathcal{O})_0 \backslash \mathrm{Sp}(2;\mathcal{O})} \left(\det(CZ + D)^\vee\right)^{-k/2}$$

for even $k > 6$ where $\mathrm{Sp}(2;\mathcal{O})_0 = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2;\mathcal{O});\ C = 0\}$. The Fourier expansion of the Eisenstein series can be explicitly calculated (cf. [Kr90, Thm. 3]). In particular, the Eisenstein series are normalized, i.e., the constant term in the Fourier expansion equals 1. Additionally, we define $E_4^{\mathbb{H}}$ and $E_6^{\mathbb{H}}$ as Maaß lifts (cf. [Kr90]) with constant Fourier coefficient equal to 1. According to [Kr90], we have $E_k^{\mathbb{H}} \in \mathcal{M}(\Gamma_{\mathbb{H}}, k)$ for all even $k \ge 4$, and thus, in particular, $E_k^{\mathbb{H}} \in [\Gamma_{\mathbb{H}}, k, 1]$ for all even $k \ge 4$.

The orthogonal half-space $\mathcal{H}_{D_4}$ is biholomorphically mapped to $H(2;\mathbb{H})$ by

$$\varphi_{\mathbb{H}} : \mathcal{H}_{D_4} \to H(2;\mathbb{H}),\ (x_1, u, x_2) + i(y_1, v, y_2) \mapsto \begin{pmatrix} x_1 + iy_1 & \iota_{D_4}(u) + i\ \iota_{D_4}(v) \\ \overline{\iota_{D_4}(u) + i\ \iota_{D_4}(v)} & x_2 + iy_2 \end{pmatrix},$$

where $\iota_{D_4} : \mathbb{R}^4 \to \mathbb{H}$ is defined as in Proposition 1.17. This map allows us to identify $\mathrm{Bihol}(\mathcal{H}_{D_4})$ and $\mathrm{Bihol}(H(2;\mathbb{H}))$. In particular, we get

$$\Gamma_{\mathbb{H}} \cong \Gamma_{D_4}/\{\pm I_8\}.$$

In Appendix A we list the generators of $\Gamma_{\mathbb{H}}$ and the elements of Γ_{D_4} they correspond to. According to [KW98] we have

$$\Gamma_{\mathbb{H}}^{\mathrm{ab}} = \langle \nu_\rho, \nu_{\mathrm{tr}} \rangle$$

where

$$\begin{aligned} \nu_\rho(\rho I) &= -1, \quad \nu_\rho(I_{\mathrm{tr}}) = 1, \qquad \nu_\rho(M) = 1 \text{ for all } M \in \mathrm{Sp}(2;\mathcal{O}), \\ \nu_{\mathrm{tr}}(\rho I) &= 1, \qquad \nu_{\mathrm{tr}}(I_{\mathrm{tr}}) = -1, \quad \nu_{\mathrm{tr}}(M) = 1 \text{ for all } M \in \mathrm{Sp}(2;\mathcal{O}). \end{aligned}$$

Theorem 2.37 *Let $k \in 2\mathbb{Z}$ and $r, s \in \{0,1\}$. If $f \in [\Gamma_{\mathbb{H}}, k, \nu_\rho^r \nu_{\mathrm{tr}}^s]$ with Fourier expansion*

$$f(Z) = \sum_{n,m \in \mathbb{N}_0} \sum_{\substack{t \in \mathcal{O}^\sharp \\ N(t) \le mn}} \alpha_f \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix} e^{2\pi i (n\tau_1 + m\tau_2 + (t,z)_{\mathbb{H}})}, \qquad Z = \left(\begin{smallmatrix} \tau_1 & z \\ * & \tau_2 \end{smallmatrix} \right) \in H(2;\mathbb{H}),$$

then $g := f \circ \varphi_{\mathbb{H}} \in [\Gamma_{D_4}, k, \nu_\pi^r \det^s]$ with Fourier expansion

$$g(w) = \sum_{m,n \in \mathbb{N}_0} \sum_{\substack{\mu \in \Lambda^\sharp \\ q(\mu) \le mn}} \alpha_g(m,\mu,n)\, e^{2\pi i (n\tau_1 + m\tau_2 + (\mu, \widetilde{w})_S)}, \qquad w = (\tau_1, \widetilde{w}, \tau_2) \in \mathcal{H}_S,$$

where

$$\alpha_g(m,\mu,n) = \alpha_f \begin{pmatrix} n & \iota_{D_4}(\mu) \\ * & m \end{pmatrix}. \tag{2.13}$$

PROOF The assertion $f \circ \varphi_{\mathbb{H}} \in [\Gamma_{D_4}, k, \nu_\pi^r \det^s]$ can be proved analogously to Theorem 2.35 if one notes that $j_{\mathbb{H}}(M,Z) := (\det(CZ+D)^\vee)^{k/2}$ is a factor of automorphy of weight k (cf. [KW98]) and that

$$j(J,w)^k = q_{(D_4)_0}(w)^k = (\det(\varphi_{\mathbb{H}}(w)^\vee))^{k/2} = j_{\mathbb{H}}(J_{\mathbb{H}}, \varphi_{\mathbb{H}}(w)).$$

Since we have $\mathcal{O}^\sharp = \iota_{D_4}(\Lambda^\sharp)$ and $(a,b)_{D_4} = (\iota_{D_4}(a), \iota_{D_4}(b))_{\mathbb{H}}$ the Fourier expansion of $f \circ \varphi_{\mathbb{H}}$ can easily be derived from the expansion of f. ∎

Since we explicitly know how the Fourier expansion of $f \circ \varphi_{\mathbb{H}}$ arises from the Fourier expansion of $f \in [\Gamma_{\mathbb{H}}, k, \chi]$ we can show that Maaß forms are mapped to Maaß forms.

Corollary 2.38 *Given an even $k > 0$, the map*

$$\mathcal{M}(\Gamma_{\mathbb{H}}, k) \to \mathcal{M}(\Gamma_{D_4}, k), \; f \mapsto f \circ \varphi_{\mathbb{H}},$$

is an isomorphism. In particular, we have

$$\dim \mathcal{M}(\Gamma_{D_4}, k) = \left\lfloor \frac{k+2}{6} \right\rfloor .$$

PROOF The map $[\Gamma_{\mathbb{H}}, k] \longrightarrow [\Gamma_{D_4}, k]$, $f \mapsto f \circ \varphi_{\mathbb{H}}$, is obviously an isomorphism. Moreover, by virtue of (2.13) the validity of the Maaß condition (2.8) for the Fourier coefficients of $f \circ \varphi_{\mathbb{H}}$ follows immediately from the validity of the Maaß condition (2.11) for the Fourier coefficients of f and vice versa. According to [Kr87, Thm. 1], we have $\dim \mathcal{M}(\Gamma_{\mathbb{H}}, k) = \left\lfloor \frac{k+2}{6} \right\rfloor$. This completes the proof. ∎

For $k \geq 4$, k even, we define the orthogonal Eisenstein series $E_k^{D_4}$ by

$$E_k^{D_4} := E_k^{\mathbb{H}} \circ \varphi_{\mathbb{H}},$$

and for $T \in \{A_3, A_1^{(3)}\}$ we define the orthogonal Eisenstein series $E_k^T : \mathcal{H}_T \longrightarrow \mathbb{C}$ as restrictions of the Eisenstein series $E_k^{D_4}$ to $\mathcal{H}_T$, i.e.,

$$E_k^{A_3} := E_k^{D_4} | \mathcal{H}_{A_3} \qquad \text{and} \qquad E_k^{A_1^{(3)}} := E_k^{D_4} | \mathcal{H}_{A_1^{(3)}}.$$

According to the above corollary, we have $E_k^{D_4} \in \mathcal{M}(\Gamma_{D_4}, k)$ for all even $k \geq 4$. By virtue of Proposition 2.27 and Proposition 2.28 this implies $E_k^{A_3} \in \mathcal{M}(\Gamma_{A_3}, k)$ and $E_k^{A_1^{(3)}} \in \mathcal{M}(\Gamma_{A_1^{(3)}}, k)$ for all even $k \geq 4$. Since the $E_k^{\mathbb{H}}$ are normalized the same is true for the $E_k^{D_4}$. Moreover, note that the E_k^T are no cusp forms (and consequently do not vanish identically) since the constant term in the Fourier expansion equals 1.

2.8. Quaternionic theta series

In this section we consider the theta series

$$Y_1 = \Theta_{\binom{0}{0}}, \quad Y_2 = \Theta_{\binom{0}{2}}, \quad Y_3 = \Theta_{\binom{2}{0}}, \quad Y_4 = \Theta_{\binom{2}{2}}, \quad Y_5 = \Theta_{\binom{2}{2\omega}}, \quad Y_6 = \Theta_{\binom{2}{2\overline{\omega}}}$$

introduced in [FH00, Sect. 10], where for all $a \in \mathcal{O}^2$ the theta series Θ_a are defined by

$$\Theta_a(Z) := \vartheta_1(a)(Z) = \sum_{g \in \wp^2} e^{\pi i Z[g + a/2]} \quad \text{for all } Z \in H(2; \mathbb{H}).$$

According to [FH00], the Y_j are modular forms of weight 2 with respect to the principal congruence subgroup

$$\mathrm{Sp}(2; \mathcal{O})[\wp] := \{M \in \mathrm{Sp}(2; \mathcal{O});\ M \equiv I_4 \pmod{\wp}\}$$

and the trivial character, i.e., we have $Y_j \in [\mathrm{Sp}(2;\mathcal{O})[\wp], 2, 1]$ for $1 \leq j \leq 6$. Moreover, according to [Krb], for all $Z \in H(2;\mathbb{H})$ we have

$$\begin{aligned} \Theta_{a+2b}(Z) &= \Theta_a(Z) &&\text{for all } b \in \wp^2, \\ \Theta_a(Z[U]) &= \Theta_{Ua}(Z) &&\text{for all } U \in \mathrm{GL}(2;\mathcal{O}), \\ \Theta_{a\varepsilon}(Z) &= \Theta_a(Z) &&\text{for all } \varepsilon \in \mathcal{O}^\times, \end{aligned}$$

where $\mathcal{O}^\times = \{\varepsilon \in \mathcal{O};\ N(\varepsilon) = 1\}$ is the unit group of $\mathcal{O}$. Additionally, Krieg showed that

$$Y_j(Z[\rho I]) = Y_j({}^tZ) = Y_{\pi(j)}(Z), \quad 1 \leq j \leq 6,$$

where $\pi = (1)(2)(3)(4)(5\ 6) \in S(6)$.

We are particularly interested in the restrictions of the Y_j to the submanifold

$$H(2;\mathbb{H}_{A_1^{(3)}}) = \left\{ \begin{pmatrix} \tau_1 & z \\ * & \tau_2 \end{pmatrix} \in H(2;\mathbb{H});\ z = z_1 + z_2\mathrm{i}_1 + z_3\mathrm{i}_2 + z_4\mathrm{i}_3,\ z_4 = 0 \right\}$$

of $H(2;\mathbb{H})$ which corresponds via $\varphi_{\mathbb{H}} \circ \iota^{D_4}_{A_1^{(3)}} : \mathcal{H}_{A_1^{(3)}} \to H(2;\mathbb{H})$ to the orthogonal half-space $\mathcal{H}_{A_1^{(3)}}$. We denote those restrictions by $\widetilde{Y}_j$. Note that for all $Z \in H(2;\mathbb{H}_{A_1^{(3)}})$ we have ${}^tZ = Z[\mathrm{i}_3 I]$. Thus, by applying the above transformation formulas it is easy to check that

$$Y_6(Z) = Y_5({}^tZ) = Y_5(Z[\mathrm{i}_3 I]) = Y_5(Z)$$

holds for all $Z \in H(2;\mathbb{H}_{A_1^{(3)}})$. Hence $\widetilde{Y}_5$ and $\widetilde{Y}_6$ coincide.

We want to examine how the $\widetilde{Y}_j$ behave under the generators of

$$\Gamma^* := \mathrm{Bihol}(H(2;\mathbb{H}_{A_1^{(3)}})) \cap \Gamma_{\mathbb{H}} \cong \mathrm{Bihol}(\mathcal{H}_{A_1^{(3)}}) \cap \mathrm{O}(\Lambda_1) \cong (\Gamma_{A_1^{(3)}}/\{\pm I\}),$$

where $\Lambda_1 = \mathbb{Z}^7$ is the lattice associated with $A_1^{(3)}$. Due to Corollary 1.28 and the table in Section A.2 we know that Γ^* is generated by the modular transformations corresponding to

$$J_{\mathbb{H}},\ \mathrm{Trans}(H) := \begin{pmatrix} I_2 & H \\ 0 & I_2 \end{pmatrix}\ (H \in \mathrm{Her}(2;\mathcal{O}_{A_1^{(3)}})),\ R_1 := \mathrm{Rot}\begin{pmatrix} \overline{\omega} + \mathrm{i}_2 & 0 \\ 0 & \overline{\omega} + \mathrm{i}_1 \end{pmatrix} \text{ and}$$
$$R_2 := \mathrm{Rot}\begin{pmatrix} (\mathrm{i}_2 - \mathrm{i}_1)/\sqrt{2} & 0 \\ 0 & (\mathrm{i}_1 - \mathrm{i}_2)/\sqrt{2} \end{pmatrix} = \mathrm{Rot}\begin{pmatrix} -\overline{\omega} - \mathrm{i}_1 & 0 \\ 0 & \overline{\omega} + \mathrm{i}_1 \end{pmatrix} \mathrm{Rot}\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix},$$

where $\mathcal{O}_{A_1^{(3)}} = \mathbb{Z} + \mathbb{Z}\mathrm{i}_1 + \mathbb{Z}\mathrm{i}_2$ and $\mathrm{Rot}(U) = \begin{pmatrix} {}^t\overline{U} & 0 \\ 0 & U^{-1} \end{pmatrix}$.

Theorem 2.39 *Let* $\Theta = {}^t(\widetilde{Y}_1, \ldots, \widetilde{Y}_5) : H(2;\mathbb{H}_{A_1^{(3)}}) \to \mathbb{C}^5$. *There exists a unique homomorphism of groups*

$$\Psi : \Gamma^* \to \mathrm{GL}(5;\mathbb{C})$$

given by

$$\Theta|_2 M = \Psi(M)\,\Theta, \quad M \in \Gamma^*.$$

We have

$$\Psi(R_1) = \Psi(R_2) = I_5,$$

$$\Psi(\mathrm{Trans}(H)) = \begin{cases} [1,1,-1,-1,-1], & \textit{if } H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \\ [1,-1,1,-1,-1], & \textit{if } H = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \\ [1,1,1,1,-1], & \textit{if } H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \end{cases}$$

and

$$\Psi(J_{\mathbb{H}}) = \frac{1}{4}\begin{pmatrix} 1 & 3 & 3 & 3 & 6 \\ 1 & -1 & 3 & -1 & -2 \\ 1 & 3 & -1 & -1 & -2 \\ 1 & -1 & -1 & 3 & -2 \\ 1 & -1 & -1 & -1 & -2 \end{pmatrix}.$$

PROOF Using the above transformation formulas we can easily verify that Θ is invariant under the two rotations R_1 and R_2. Moreover, Θ transforms under the translations $\mathrm{Trans}(H)$ and under $J_{\mathbb{H}}$ as stated above according to [Kra]. In view of their Fourier expansions the $\widetilde{Y}_1, \ldots, \widetilde{Y}_5$ are obviously linearly independent. Thus Ψ is uniquely determined. ∎

Note that $\mathrm{Trans}(H) \in \ker\Psi$ whenever $H \in \mathrm{Her}(2; \wp_{A_1^{(3)}})$, where

$$\wp_{A_1^{(3)}} = \wp \cap \mathcal{O}_{A_1^{(3)}} = \mathbb{Z}2 + \mathbb{Z}(1+\mathrm{i}_1) + \mathbb{Z}(1+\mathrm{i}_2).$$

Obviously we have $\mathcal{O}_{A_1^{(3)}}/\wp_{A_1^{(3)}} \cong \mathbb{Z}/2\mathbb{Z}$. Thus in view of the above theorem the map Ψ defines a five-dimensional representation of the finite group

$$\Gamma^*/\ker\Psi \cong \mathrm{Sp}(2;\mathbb{F}_2) \cong S(6).$$

We denote the orthogonal theta series $Y_j \circ \varphi_{\mathbb{H}} \circ \iota^{D_4}_{A_1^{(3)}}$ again by Y_j. Moreover, we denote the r-th elementary symmetric polynomial in Y_2^n, Y_3^n, Y_4^n by $e_r(Y^n)$. Using MAGMA (cf. [BCP97]) we compute the invariant ring of the representation Ψ.

Theorem 2.40 *There are* 5 *algebraically independent modular forms*

$$h_k \in [\Gamma_{A_1^{(3)}}, k, 1], \quad k = 4, 6, 8, 10, 12,$$

given by

$$\begin{aligned} h_4 &= Y_1^2 + 3e_1(Y^2) + 6Y_5^2 = Y_1^2 + 3(Y_2^2 + Y_3^2 + Y_4^2 + 2Y_5^2), \\ h_6 &= Y_1^3 - 9Y_1(e_1(Y^2) - 4Y_5^2) + 54e_3(Y), \\ h_8 &= Y_1^4 + 6Y_1^2 e_1(Y^2) + 24Y_1 e_3(Y) + 6e_2(Y^2) + 9e_1(Y^4) + 48e_1(Y^2)Y_5^2 + 24Y_5^4, \end{aligned}$$

$$\begin{aligned}h_{10} &= Y_1^5 - 6Y_1^3 e_1(Y^2) + 12Y_1^2 e_3(Y) + 3Y_1(10e_2(Y^2) - 9e_1(Y^4) + 32e_1(Y^2)Y_5^2 + 16Y_5^4)\\&\quad + 36e_1(Y^2)e_3(Y) + 576e_3(Y)Y_5^2,\\h_{12} &= Y_1^6 + 45Y_1^2(e_1(Y^4) + 2Y_5^4) + 1080Y_1 e_3(Y)Y_5^2 + 18e_1(Y^6) + 270e_3(Y^2)\\&\quad + 540e_2(Y^2)Y_5^2 + 270e_1(Y^2)Y_5^4 + 36Y_5^6.\end{aligned}$$

The restrictions of those modular forms to $\mathcal{H}_{A_1^{(2)}}$ generate the graded ring $\mathcal{A}(\Gamma_{A_1^{(2)}})$.

PROOF The h_k are primary invariants of the representation Ψ. This implies that they are elements of $[\Gamma_{A_1^{(3)}}, k, 1]$. In order to show that they are algebraically independent we consider their restrictions $\widetilde{h}_k := h_k|\mathcal{H}_{A_1^{(2)}}$ to $\mathcal{H}_{A_1^{(2)}}$. Due to Theorem 2.34 we have $\widetilde{h}_k \in [\Gamma_{A_1^{(2)}}, k, 1]$. According to Theorem 2.36, the graded ring $\mathcal{A}(\Gamma_{A_1^{(2)}})$ is a polynomial ring in five modular forms of weight 4, 6, 8, 10 and 12. Thus $\dim[\Gamma_{A_1^{(2)}}, 4, 1] = \dim[\Gamma_{A_1^{(2)}}, 6, 1] = 1$, $\dim[\Gamma_{A_1^{(2)}}, 8, 1] = \dim[\Gamma_{A_1^{(2)}}, 10, 1] = 2$ and $\dim[\Gamma_{A_1^{(2)}}, 12, 1] = 3$. By calculating the Fourier expansion of the $\widetilde{h}_k$ we can easily verify that the vector spaces $[\Gamma_{A_1^{(2)}}, k, 1]$, $k = 4, 6, 8, 10, 12$, are spanned by suitable products of the $\widetilde{h}_k$. So, in particular, the five generators of $\mathcal{A}(\Gamma_{A_1^{(2)}})$ can be written as polynomials in the $\widetilde{h}_k$ which implies that the $\widetilde{h}_k$ generate the graded ring $\mathcal{A}(\Gamma_{A_1^{(2)}})$. Since the five generators of $\mathcal{A}(\Gamma_{A_1^{(2)}})$ are algebraically independent the same must be true for the $\widetilde{h}_k$ and thus of course also for the h_k. ■

Since the invariants h_k are polynomials in the theta series $Y_1, \ldots, Y_5$ the algebraic independence of the h_k implies the algebraic independence of the theta series (on $\mathcal{H}_{A_1^{(3)}}$). According to the proof of the above theorem, we even have the following

Corollary 2.41 *The theta series $Y_1, \ldots, Y_5$ are algebraically independent on $\mathcal{H}_{A_1^{(3)}}$ and also on $\mathcal{H}_{A_1^{(2)}}$.*

3. Vector-valued Modular Forms

In this chapter we introduce vector-valued elliptic modular forms of half-integral weight. They will be used as input for the construction of Borcherds products. The facts presented in this chapter are mostly well-known so we will not go into too much detail.

3.1. The metaplectic group

As usual, the action of $\mathrm{SL}(2;\mathbb{R})$ on $\mathcal{H}$ (or $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) is defined by

$$M\langle\tau\rangle = \frac{a\tau + b}{c\tau + d}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{R}).$$

Since we will have to consider modular forms of half-integral weight we have to introduce the metaplectic group $\mathrm{Mp}(2;\mathbb{R})$ which is the double cover of $\mathrm{SL}(2;\mathbb{R})$. Its elements can be written in the form

$$(M, \varphi),$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{R})$, and φ is a holomorphic function on $\mathcal{H}$ such that

$$\varphi^2(\tau) = c\tau + d \quad \text{for all } \tau \in \mathcal{H},$$

i.e., φ is a holomorphic root of $\tau \mapsto c\tau + d$. We define the action of $(M, \varphi) \in \mathrm{Mp}(2;\ \mathbb{R})$ on $\mathcal{H}$ (or $\widehat{\mathbb{C}}$) to be the same as that of M. The product of two elements $(M_1, \varphi_1),\ (M_2, \varphi_2) \in \mathrm{Mp}(2;\mathbb{R})$ is given by

$$(M_1, \varphi_1)(M_2, \varphi_2) = (M_1 M_2, \varphi_1(M_2\langle\cdot\rangle)\varphi_2).$$

As in [Br02] we define the embedding of $\mathrm{SL}(2;\mathbb{R})$ into $\mathrm{Mp}(2;\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} := \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right).$$

Let $\mathrm{Mp}(2;\mathbb{Z})$ be the inverse image of $\mathrm{SL}(2;\mathbb{Z})$ under the covering map $\mathrm{Mp}(2;\mathbb{R}) \to \mathrm{SL}(2;\mathbb{R})$. It is well known that $\mathrm{Mp}(2;\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad J = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right)$$

and that the center of $\mathrm{Mp}(2;\mathbb{Z})$ is generated by

$$C := J^2 = (JT)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right).$$

For $N \in \mathbb{N}$ we denote the principal congruence subgroup of $\mathrm{Mp}(2;\mathbb{Z})$ of level N by

$$\mathrm{Mp}(2;\mathbb{Z})[N] := \{(M,\varphi) \in \mathrm{Mp}(2;\mathbb{Z});\ M \equiv I_2 \pmod N\}.$$

Moreover, we set

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix};\ n \in \mathbb{Z} \right\} \leq \mathrm{SL}(2;\mathbb{Z})$$

and

$$\widetilde{\Gamma}_\infty := \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right);\ n \in \mathbb{Z} \right\} = \langle T \rangle \leq \mathrm{Mp}(2;\mathbb{Z}).$$

3.2. Vector-valued modular forms

Let V be a finite dimensional vector space over $\mathbb{C}$, and let $k \in \frac{1}{2}\mathbb{Z}$. For vector-valued functions $f : \mathcal{H} \to V$ and $(M,\varphi) \in \mathrm{Mp}(2;\mathbb{Z})$ we define the Petersson slash operator by

$$(f|_k(M,\varphi))(\tau) = \varphi(\tau)^{-2k}\ f(M\langle\tau\rangle).$$

This defines an action of $\mathrm{Mp}(2;\mathbb{Z})$ on functions $f : \mathcal{H} \to V$.

Definition 3.1 *Suppose that ρ is a finite representation of $\mathrm{Mp}(2;\mathbb{Z})$ on a finite dimensional complex vector space V, and let $k \in \frac{1}{2}\mathbb{Z}$. A* (holomorphic) modular form *of weight k with respect to ρ and $\mathrm{Mp}(2;\mathbb{Z})$ is a function $f : \mathcal{H} \to V$ satisfying*

(i) $f|_k g = \rho(g) f$ for all $g \in \mathrm{Mp}(2;\mathbb{Z})$,

(ii) f is holomorphic on $\mathcal{H}$,

(iii) f is bounded on $\{\tau \in \mathbb{C};\ \mathrm{Im}(\tau) > y_0\}$ for all $y_0 > 0$.

If f additionally satisfies $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau) = 0$ then f is called a cusp form. *We denote the space of (holomorphic) modular forms of weight k with respect to ρ and $\mathrm{Mp}(2;\mathbb{Z})$ by $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$ and the subspace of cusp forms by $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]_0$.*

Remark 3.2 *a) As $\mathrm{Mp}(2;\mathbb{Z})$ is generated by S and T condition (i) is equivalent to*

(i') $f(\tau+1) = \rho(T) f(\tau)$ and $f(-\tau^{-1}) = \sqrt{\tau}^{2k} \rho(J) f(\tau)$.

b) Since ρ is a finite representation there exists an $N \in \mathbb{Z}$ such that $T^N \in \ker\rho$, and thus $f(\tau+N) = f(\tau)$, i.e., f is periodic with period N. Let $\mathcal{B}$ be a basis of V. We denote the components of f by f_v, so that $f = \sum_{v\in\mathcal{B}} f_v\ v$. Obviously, f is holomorphic if and only if all its components f_v are holomorphic. Therefore each f_v has a Fourier expansion of the form

$$f_v(\tau) = \sum_{n\in\mathbb{Z}} c_v(n/N) e^{2\pi i n\tau/N}.$$

Condition (iii) is then equivalent to
(iii') f has a Fourier expansion of the form

$$f(\tau) = \sum_{v \in \mathcal{B}} \sum_{\substack{n \in \mathbb{Z} \\ n \geq 0}} c_v(n/N) e^{2\pi i n \tau / N} \ v.$$

f is a cusp form if $c_v(0) = 0$ for all $v \in \mathcal{B}$.

Example 3.3 *The Dedekind eta function $\eta : \mathcal{H} \to \mathbb{C}$ defined by*

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

is a cusp form of weight $\frac{1}{2}$ with respect to $\mathrm{Mp}(2;\mathbb{Z})$ and the Abelian character ν_η with

$$\nu_\eta(T) = e^{\pi i/12} \qquad \text{and} \qquad \nu_\eta(J) = e^{-\pi i/4}$$

(cf. [Ap90, Ch. 3] or [Le64, Thm. XI.1C]).

We note a few simple facts about vector-valued modular forms.

Proposition 3.4 *Suppose that ρ_1 and ρ_2 are two finite representations of $\mathrm{Mp}(2;\mathbb{Z})$ on finite dimensional complex vector spaces V_1 and V_2, respectively. If $f_j \in [\mathrm{Mp}(2;\mathbb{Z}), k_j, \rho_j]$, $j = 1, 2$, then $f_1 \otimes f_2 : \mathcal{H} \to V_1 \otimes V_2,\ \tau \mapsto f_1(\tau) \otimes f_2(\tau)$ is a modular form of weight $k_1 + k_2$ with respect to $\rho_1 \otimes \rho_2$. In particular, if f_1 is scalar-valued then $f_1 f_2 \in [\mathrm{Mp}(2;\mathbb{Z}), k_1 + k_2, \rho_1 \rho_2]$.*

Proposition 3.5 *Suppose that ρ is a representation of $\mathrm{Mp}(2;\mathbb{Z})$ on a finite dimensional complex vector space V such that ρ factors through the double cover $\mathrm{Mp}(2;\mathbb{Z}/N\mathbb{Z})$ of the finite group $\mathrm{SL}(2;\mathbb{Z}/N\mathbb{Z})$ for some positive integer N, i.e., $\ker \rho \subset \mathrm{Mp}(2;\mathbb{Z})[N]$ is a congruence subgroup of level N. Then*
a) $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho] = \{0\}$ *if $k < 0$,*
b) $[\mathrm{Mp}(2;\mathbb{Z}), 0, \rho] \cong \mathbb{C}^g$ *where g is the multiplicity of the trivial one-dimensional representation in ρ,*
c) $\dim[\mathrm{Mp}(2;\mathbb{Z}), k, \rho] < \infty$ *for all $k \in \mathbb{Z}$.*

PROOF All components f_v of f are elliptic modular forms with respect to the congruence subgroup $\ker \rho$. Thus a) and c) follow immediately from the well known facts about $[\ker \rho, k, 1]$. By considering the decomposition of ρ into irreducible representations $\rho_j : \mathrm{Mp}(2;\mathbb{Z}) \to \mathrm{GL}(V_j)$ b) follows from $[\ker \rho, 0, 1] \cong \mathbb{C}$, $[\mathrm{Mp}(2;\mathbb{Z}), 0, \chi] = \{0\}$ for all non-trivial Abelian characters $\chi : \mathrm{Mp}(2;\mathbb{Z}) \to \mathbb{C}$ and the fact that

$$\mathrm{span}\{\rho_j(\mathrm{Mp}(2;\mathbb{Z}))(f|_{V_j}(\tau));\ \tau \in \mathcal{H}\} = V_j$$

whenever $f|_{V_j} \neq 0$. ∎

For the construction of Borcherds products we need a certain type of non-holomorphic modular forms.

Definition 3.6 *A* nearly holomorphic modular form *of weight k with respect to ρ and $\mathrm{Mp}(2;\mathbb{Z})$ is a function $f : \mathcal{H} \to V$ satisfying*

(i) $f|_k g = \rho(g) f$ *for all* $g \in \mathrm{Mp}(2;\mathbb{Z})$,

(ii) f *is holomorphic on* $\mathcal{H}$,

(iii) f *has at most a pole in* ∞, *i.e., there exists an* $n_0 \in \mathbb{Z}$, $n_0 < 0$ *such that* f *has a Fourier expansion of the form*

$$f(\tau) = \sum_{v \in \mathcal{B}} \sum_{\substack{n \in \mathbb{Z} \\ n \geq n_0}} c_v(n/N) e^{2\pi i n \tau / N}\, v.$$

We denote the space of nearly holomorphic modular forms of weight k with respect to ρ and $\mathrm{Mp}(2;\mathbb{Z})$ by $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]_\infty$. The principal part *of f is given by*

$$\sum_{v \in \mathcal{B}} \sum_{\substack{n \in \mathbb{Z} \\ n < 0}} c_v(n/N) e^{2\pi i n \tau / N}\, v.$$

3.3. The Weil representation

In this section we introduce a special representation which plays an important role in the theory of Borcherds products.

Suppose that $S \in \mathrm{Sym}(l;\mathbb{R})$ is an even matrix of signature (b^+, b^-). Let $\Lambda = \mathbb{Z}^l$ be the associated lattice with bilinear form $(\cdot,\cdot) = (\cdot,\cdot)_S$ and the corresponding quadratic form $q = q_S$. Let $(e_\mu)_{\mu \in \Lambda^\sharp/\Lambda}$ be the standard basis of the group ring $\mathbb{C}[\Lambda^\sharp/\Lambda]$. Then there is a unitary representation ρ_S of $\mathrm{Mp}(2;\mathbb{Z})$ on $\mathbb{C}[\Lambda^\sharp/\Lambda]$ which is defined by

$$\begin{aligned} \rho_S(T) e_\mu &= e^{2\pi i q(\mu)} e_\mu, \\ \rho_S(J) e_\mu &= \frac{\sqrt{i}^{\,b^- - b^+}}{\sqrt{|\det S|}} \sum_{\nu \in \Lambda^\sharp/\Lambda} e^{-2\pi i (\mu,\nu)} e_\nu. \end{aligned}$$

Note that this implies

$$\rho_S(C) e_\mu = i^{b^- - b^+} e_{-\mu}. \tag{3.1}$$

This representation is essentially the Weil representation of the quadratic module $(\Lambda^\sharp/\Lambda, q)$. Let N be the level of Λ. Then the representation ρ_S factors through the finite group $\mathrm{SL}(2;\mathbb{Z}/N\mathbb{Z})$ if l is even, and through a double cover of $\mathrm{SL}(2;\mathbb{Z}/N\mathbb{Z})$ if l is odd. In particular, ρ_S is a finite representation.

We denote the dual representation of ρ_S by $\rho_S^\sharp$. Since ρ_S is a unitary representation the values $\rho_S^\sharp((M,\varphi))$, understood as element of $\mathrm{Mat}(l;\mathbb{C})$, are simply the complex conjugate of $\rho_S((M,\varphi))$. Note that $\rho_{-S} = \overline{\rho}_S$. Therefore all the results we state for ρ_S also hold for $\rho_S^\sharp$ if one replaces S by $-S$.

Now we consider modular forms with respect to those special representations. Assume that $f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$. In this case we denote the components of f by f_μ, so that $f = \sum_{\mu \in \Lambda^\sharp/\Lambda} f_\mu \; e_\mu$. Now f satisfying $f|_k(T) = \rho_S(T)f$ implies that $e^{-2\pi i q(\mu)\tau} f_\mu(\tau)$ is periodic with period 1 for all $\mu \in \Lambda^\sharp/\Lambda$. Therefore f has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\substack{n \in q(\mu)+\mathbb{Z} \\ n \geq 0}} c_\mu(n) q^n \; e_\mu, \tag{3.2}$$

where, as usual, $q = e^{2\pi i \tau}$ (not to be confused with the quadratic form $q = q_S$). Analogously, $f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S^\sharp]$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\substack{n \in -q(\mu)+\mathbb{Z} \\ n \geq 0}} c_\mu(n) q^n \; e_\mu. \tag{3.3}$$

Considering that C^2 acts trivially on $\tau \in \mathcal{H}$ we can deduce a first necessary condition on the weight for the existence of non-trivial modular forms.

Proposition 3.7 *If* $2k \not\equiv b^+ - b^- \pmod 2$ *then*

$$[\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S] = \{0\}.$$

PROOF Let $f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$. Then

$$(-1)^{-2k} f = f|_k(C^2) = \rho_S(C^2) f = (-1)^{b^- - b^+} f,$$

and thus $f = 0$ unless $2k \equiv b^+ - b^- \pmod 2$. ∎

The functional equation for modular forms with respect to ρ_S under $C \in \mathrm{Mp}(2;\mathbb{Z})$ implies

Proposition 3.8 *Let* $2k \equiv b^+ - b^- \pmod 2$ *and* $f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$ *with Fourier expansion (3.2). Then*

$$c_{-\mu}(n) = (-1)^{(2k+b^- - b^+)/2} c_\mu(n)$$

for all $\mu \in \Lambda^\sharp/\Lambda$ *and* $n \in \mathbb{Z} + q(\mu)$.

PROOF Let $f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$. Then

$$i^{-2k} f = f|_k C = \rho_S(C) f = i^{b^- - b^+} \sum_{\mu \in \Lambda^\sharp/\Lambda} f_\mu \; e_{-\mu}$$

yields $f_{-\mu} = i^{2k+b^- - b^+} f_\mu = (-1)^{(2k+b^- - b^+)/2} f_\mu$ for all $\mu \in \Lambda^\sharp/\Lambda$. ∎

If $\mu \equiv -\mu \pmod \Lambda$ for all $\mu \in \Lambda^\sharp$ then we get another necessary condition on the weight for the existence of non-trivial modular forms.

Corollary 3.9 *If* $2k + b^- - b^+ \equiv 2 \pmod 4$ *and* $\mu = -\mu$ *for all* $\mu \in \Lambda^\sharp/\Lambda$ *then*

$$[\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S] = \{0\}.$$

3.4. A dimension formula

If the representation ρ of $\mathrm{Mp}(2;\mathbb{Z})$ satisfies certain conditions then for $k \geq 2$ the dimension of $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$ can be calculated explicitly. In [Sk84] (see also [ES95]) Skoruppa determined a dimension formula using the Eichler-Selberg trace formula.

Theorem 3.10 *Let $k \in \frac{1}{2}\mathbb{Z}$, and let $\rho : \mathrm{Mp}(2;\mathbb{Z}) \to \mathrm{GL}(V)$ be a finite representation such that $\rho(C) = e^{-\pi i k}\,\mathrm{id}_V$. Then the dimension of $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$ is given by the following formula*

$$\dim[\mathrm{Mp}(2;\mathbb{Z}), k, \rho] - \dim[\mathrm{Mp}(2;\mathbb{Z}), 2-k, \overline{\rho}]_0 =$$
$$\frac{k+5}{12}n + \frac{1}{4}\,\mathrm{Re}(e^{\pi i k/2}\,\mathrm{trace}\,\rho(J)) + \frac{2}{3\sqrt{3}}\,\mathrm{Re}(e^{\pi i(2k+1)/6}\,\mathrm{trace}\,\rho(JT)) - \sum_{j=1}^{n}\lambda_j,$$

where $n = \dim V$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$, $0 \leq \lambda_j < 1$, such that the eigenvalues of $\rho(T)$ are $e^{2\pi i \lambda_j}$.

PROOF We show how this follows from the formula given in [ES95]. Since ρ is a finite representation we can find $\lambda_j \in \mathbb{Q}$ with $0 \leq \lambda_j < 1$ such that the eigenvalues of $\rho(T)$ are $e^{2\pi i\lambda_j}$. Then

$$\frac{1}{2}a(\rho) - \sum_{j=1}^{n}\mathbb{B}_1(\lambda_j) = \frac{1}{2}\sum_{\substack{1\leq j\leq n\\ \lambda_j\in\mathbb{Z}}} 1 - \sum_{\substack{1\leq j\leq n\\ \lambda_j\notin\mathbb{Z}}}(\lambda_j - \lfloor\lambda_j\rfloor - \frac{1}{2}) = -\sum_{j=1}^{n}(\lambda_j - \lfloor\lambda_j\rfloor - \frac{1}{2}) = \frac{n}{2} - \sum_{j=1}^{n}\lambda_j,$$

where $a(\rho)$ and $\mathbb{B}_1(\lambda_j)$ are defined as in [ES95] and where $\lfloor\cdot\rfloor$ is the greatest integer function. ∎

Remark 3.11 *a) If $k \geq 2$ then $\dim[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$ can be calculated directly using the above formula since the dimension of the spaces of cusp forms of non-positive weight is 0. In the cases $k = \frac{1}{2}$ and $k = \frac{3}{2}$ the dimension of $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$ can also be calculated explicitly (cf. [Sk84]). For the case $k = 1$ an explicit formula is not known to the author.*

b) In [Bo99, Sec. 4] Borcherds gives another dimension formula.

In general the dimension formula is not directly applicable to the Weil representation ρ_S because the condition on C acting as a scalar on $\mathbb{C}[\Lambda^\sharp/\Lambda]$ is usually not satisfied. But we can apply the formula to the induced representation of $\mathrm{Mp}(2;\mathbb{Z})$ on the subspace of $\mathbb{C}[\Lambda^\sharp/\Lambda]$ on which C acts as $e^{-\pi i k}$. According to (3.1) this space is spanned by $\{e_\mu + e_{-\mu};\ \mu \in \Lambda^\sharp/\Lambda\}$ whenever $2k + b^- - b^+ \equiv 0 \pmod 4$ and by $\{e_\mu - e_{-\mu};\ \mu \in \Lambda^\sharp/\Lambda\}$ whenever $2k + b^- - b^+ \equiv 2 \pmod 4$.

Luckily, according to Proposition 3.8, all $f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$ belong to the subspace spanned by $\{e_\mu + e_{-\mu};\ \mu \in \Lambda^\sharp/\Lambda\}$ if $2k + b^- - b^+ \equiv 0 \pmod 4$ and to the subspace spanned by $\{e_\mu - e_{-\mu};\ \mu \in \Lambda^\sharp/\Lambda\}$ if $2k + b^- - b^+ \equiv 2 \pmod 4$. So in those cases we can calculate the dimension of $[\mathrm{Mp}(2;\mathbb{Z}),k,\rho_S]$ by considering the induced representation of $\mathrm{Mp}(2;\mathbb{Z})$ on those subspaces of $\mathbb{C}[\Lambda^\sharp/\Lambda]$. We denote those induced representations by ρ_S^+ and ρ_S^-, respectively.

First we look at the case $S = A_3$. Then the Weil representation acts as follows.

$$\rho_{A_3}(T) = [1, e^{3\pi i/4}, -1, e^{3\pi i/4}],$$
$$\rho_{A_3}(J) = \frac{e^{-3\pi i/4}}{2}\begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{pmatrix}, \quad \rho_{A_3}(C) = i\cdot\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1\\ \cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot\end{pmatrix}.$$

Since C does not act as a scalar we have to consider the induced representations $\rho_{A_3}^+$ and $\rho_{A_3}^-$. We get

$$\rho_{A_3}^+(T) = [1, e^{3\pi i/4}, -1], \quad \rho_{A_3}^+(J) = \frac{e^{-3\pi i/4}}{2}\begin{pmatrix}1&1&1\\2&0&-2\\1&-1&1\end{pmatrix}, \quad \rho_{A_3}^+(C) = e^{-3\pi i/2}\cdot I_3$$

and

$$\rho_{A_3}^-(T) = e^{3\pi i/4}, \quad \rho_{A_3}^-(J) = e^{-\pi i/4}, \quad \rho_{A_3}^-(C) = e^{-\pi i/2}.$$

Lemma 3.12 *Suppose that $S = A_3$. Then for $k \in \frac{1}{2}\mathbb{Z}$, $k \geq 0$, we have*

$$\dim[\mathrm{Mp}(2;\mathbb{Z}),k,\rho_S] = \begin{cases} \left\lfloor \frac{k-\frac{3}{2}}{4} \right\rfloor + 1 & \text{if } k \in \frac{3}{2} + 2\mathbb{Z}, \\ \left\lfloor \frac{k-\frac{9}{2}}{12} \right\rfloor + 1 & \text{if } k \in \frac{1}{2} + 2\mathbb{Z}, k - \frac{13}{2} \not\equiv 0 \pmod{12}, \\ \left\lfloor \frac{k-\frac{9}{2}}{12} \right\rfloor & \text{if } k \in \frac{1}{2} + 2\mathbb{Z}, k - \frac{13}{2} \equiv 0 \pmod{12}, \\ 0 & \text{if } k \in \mathbb{Z}. \end{cases}$$

PROOF The assertion for $k \in \mathbb{Z}$ follows from Proposition 3.7. For $k \in \frac{1}{2} + \mathbb{Z}$, $k \geq 2$, we apply Theorem 3.10 on ρ_S^+ and ρ_S^-, respectively.

Since the eigenvalue $e^{3\pi i/4}$ of $\rho_S^-(T)$ is not of the form $e^{2\pi i \frac{n^2}{8}}$ with $n \in \mathbb{Z}$, the supplement to the dimension formula in [ES95, Sec. 4.2] yields $\dim[\mathrm{Mp}(2;\mathbb{Z}),\frac{1}{2},\rho_S] = \dim[\mathrm{Mp}(2;\mathbb{Z}),\frac{1}{2},\rho_S^-] = 0$. By the same argument we get $\dim[\mathrm{Mp}(2;\mathbb{Z}),\frac{1}{2},\overline{\rho_S^+}] = 0$, and thus $\dim[\mathrm{Mp}(2;\mathbb{Z}),\frac{1}{2},\overline{\rho_S^+}]_0 = 0$. Then Theorem 3.10 yields $\dim[\mathrm{Mp}(2;\mathbb{Z}),\frac{3}{2},\rho_S^+] = 1$. This completes the proof. ∎

Corollary 3.13 *Suppose that $S = A_3$. If $k \in \frac{1}{2} + 2\mathbb{Z}$, $k \geq \frac{9}{2}$ then $[\mathrm{Mp}(2;\mathbb{Z}),k,\rho_S]$ is isomorphic to the space of (elliptic) modular forms of (even) weight $k - \frac{9}{2}$ with respect to the full modular group $\mathrm{SL}(2;\mathbb{Z})$. The isomorphism is given by*

$$[\mathrm{SL}(2;\mathbb{Z}),k-\tfrac{9}{2},1] \to [\mathrm{Mp}(2;\mathbb{Z}),k,\rho_S], \quad f \mapsto \eta^9\cdot f\cdot(e_{(\frac{1}{4},\frac{1}{2},-\frac{1}{4})} - e_{(-\frac{1}{4},\frac{1}{2},\frac{1}{4})}).$$

PROOF Let $f \in [\mathrm{SL}(2;\mathbb{Z}), k-\frac{9}{2}, 1]$. According to Example 3.3, η^9 is a modular form of weight $\frac{9}{2}$ with respect to ρ_S^-. Thus Proposition 3.5 yields $\eta^9 \cdot f \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S^-]$ and by Proposition 3.8 we have

$$\eta^9 \cdot f \cdot (e_{(\frac{1}{4},\frac{1}{2},-\frac{1}{4})} - e_{(-\frac{1}{4},\frac{1}{2},\frac{1}{4})}) \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S].$$

A comparison of the dimension of the spaces completes the proof. ∎

Next we consider the case $S = A_1^{(3)}$. Then the Weil representation acts as follows.

$$\rho_{A_1^{(3)}}(T) = [1, i, i, i, -1, -1, -1, -i],$$

$$\rho_{A_1^{(3)}}(J) = \frac{e^{-3\pi i/4}}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix},$$

$$\rho_{A_1^{(3)}}(C) = e^{-3\pi i/2} \cdot I_8.$$

Since C acts as a scalar we can immediately apply the dimension formula on $\rho_{A_1^{(3)}}$.

Lemma 3.14 *Suppose that $S = A_1^{(3)}$. Then for $k \in \frac{1}{2}\mathbb{Z}$, $k \geq 0$ we have*

$$\dim[\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S] = \begin{cases} \lfloor \frac{2k+2}{3} \rfloor & \text{if } k \in \frac{3}{2} + 2\mathbb{Z}, \\ 0 & \text{if } k \notin \frac{3}{2} + 2\mathbb{Z}. \end{cases}$$

PROOF For $k \in \mathbb{Z}$ the assertion follows from Proposition 3.7, and for $k \in \frac{1}{2} + 2\mathbb{Z}$ the assertion follows from Corollary 3.9. A similar argument as in the proof of Lemma 3.12 yields $\dim[\mathrm{Mp}(2;\mathbb{Z}), \frac{1}{2}, \overline{\rho_S}]_0 = 0$. Then application of Theorem 3.10 completes the proof. ∎

Finally we consider the case $S = D_4$. Then the Weil representation acts as follows.

$$\rho_{D_4}(T) = [1, -1, -1, -1],$$

$$\rho_{D_4}(J) = -\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$\rho_{D_4}(C) = I_4.$$

Again C acts as a scalar so that we can immediately apply the dimension formula on ρ_{D_4}.

Lemma 3.15 *Suppose that $S = D_4$. Then for $k \in \mathbb{Z}$, $k \geq 0$ we have*

$$\dim[\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S] = \begin{cases} \left\lfloor \frac{k}{3} \right\rfloor & \text{if } k \equiv 0 \pmod 4, \\ \left\lfloor \frac{k}{3} \right\rfloor + 1 & \text{if } k \equiv 2 \pmod 4, \\ 0 & \text{if } k \text{ odd.} \end{cases}$$

PROOF The assertion for odd k follows from Corollary 3.9, the assertion for positive even k follows from Theorem 3.10, and the assertion for $k = 0$ follows from Proposition 3.5 b) and the fact that ρ_S decomposes into one irreducible two-dimensional representation and two non-trivial one-dimensional representations. ■

3.5. Examples of vector-valued modular forms

In this section we introduce two important examples of vector-valued modular forms, Eisenstein series and theta series.

3.5.1. Eisenstein series

Throughout this section we suppose that $S \in \mathrm{Sym}(l;\mathbb{R})$ is an even matrix of signature (b^+, b^-), and we set $\Lambda = \mathbb{Z}^l$.

Definition 3.16 *Let $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, such that $2k - b^+ + b^- \equiv 0 \pmod 4$. Moreover, let $v \in \mathbb{C}[\Lambda^\sharp/\Lambda]$ such that $\rho_S(T)v = v$. Then we define the Eisenstein series $E_k(\cdot;\ v, S) : \mathcal{H} \to \mathbb{C}[\Lambda^\sharp/\Lambda]$ by*

$$E_k(\tau;\ v, S) = \frac{1}{2} \sum_{g:\widetilde{\Gamma}_\infty\backslash \mathrm{Mp}(2;\mathbb{Z})} \rho_S(g)^{-1}\,(v|_k g)(\tau),$$

where the sum runs over a set of representatives of $\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z})$ and where v is considered as constant function $\mathcal{H} \to \mathbb{C}[\Lambda^\sharp/\Lambda]$.

Remark *a) Due to $\rho_S(T)v = v$ the definition is independent of the choice of representatives of $\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z})$. Moreover, just as in the scalar case, the series converges normally on $\mathcal{H}$ if (and only if) $k > 2$.*

b) The definition can be extended to allow arbitrary $v \in \mathbb{C}[\Lambda^\sharp/\Lambda]$ by taking the sum over a set of representatives of $(\widetilde{\Gamma}_\infty \cap \ker\rho_S)\backslash\mathrm{Mp}(2;\mathbb{Z})$ (cf. [De01, Sec. 3.2]).

Let $v = \sum_{\mu\in\Lambda^\sharp/\Lambda} a(\mu)e_\mu \in \mathbb{C}[\Lambda^\sharp/\Lambda]$. Then the condition $\rho_S(T)v = v$ is obviously equivalent to $a(\mu) = 0$ for all $\mu \in \Lambda^\sharp/\Lambda$ with $q(\mu) \notin \mathbb{Z}$. Moreover, $E_k(\cdot;\ v, S) = \sum_{\mu\in\Lambda^\sharp/\Lambda} a(\mu)E_k(\cdot;\ e_\mu, S)$ if $\rho_S(T)v = v$. Therefore it is sufficient to consider the Eisenstein series $E_k(\cdot;\ e_\beta, S)$ for $\beta \in \Lambda^\sharp/\Lambda$ with $q(\beta) \in \mathbb{Z}$.

Proposition 3.17 *Let $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, such that $2k - b^+ + b^- \equiv 0 \pmod 4$. Moreover, let $\beta \in \Lambda^\sharp/\Lambda$ with $q(\beta) \in \mathbb{Z}$. Then*

$$E_k(\cdot;\ e_\beta, S) \in [\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S].$$

PROOF $E_k(\cdot;\ e_\beta, S)$ converges normally on $\mathcal{H}$ and thus defines a holomorphic function on $\mathcal{H}$. Since $\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z}) \to \widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z})$, $\widetilde{\Gamma}_\infty g \mapsto \widetilde{\Gamma}_\infty gh$, is a bijection for all $h \in \mathrm{Mp}(2;\mathbb{Z})$ we have

$$\begin{aligned} E_k(\cdot;\ e_\beta, S)|_k h &= \frac{1}{2} \sum_{g:\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z})} \rho_S(g)^{-1}\,(e_\beta|_k g)|_k h \\ &= \frac{1}{2} \sum_{g:\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z})} \rho_S(ghh^{-1})^{-1}\, e_\beta|_k(gh) \\ &= \frac{1}{2} \sum_{g':\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}(2;\mathbb{Z})} \rho_S(h)\rho_S(g')^{-1}\, e_\beta|_k g' \\ &= \rho_S(h) E_k(\cdot;\ e_\beta, S) \end{aligned}$$

for all $h \in \mathrm{Mp}(2;\mathbb{Z})$. Finally, we have

$$\lim_{y\to\infty} E_k(iy;\ e_\beta, S) = \frac{1}{2} \sum_{g\in\langle C\rangle} \rho_S(g)^{-1}\, e_\beta|_k g = e_\beta + e_{-\beta},$$

i.e., $E_k(\cdot;\ e_\beta, S)$ is bounded on $\{\tau \in \mathbb{C};\ \mathrm{Im}(\tau) > y_0\}$ for all $y_0 > 0$. ∎

In [BK01] Bruinier and Kuss defined certain Eisenstein series E_β^{BK} and gave explicit formulas for the Fourier coefficients of E_0^{BK}. Their E_β^{BK} are defined via the dual representation $\rho_S^\sharp$ while our Eisenstein series are defined via ρ_S, but, according to the remarks in Section 3.3, $\rho_S^\sharp$ is essentially the same as ρ_{-S}, and thus we have $E_\beta^{\mathrm{BK}} = E_k(\cdot;\ e_\beta, -S)$. Due to this identification we can use their formulas to calculate the Fourier coefficients of $E_k(\cdot;\ e_0, S)$.

3.5.2. Theta series

In this section we introduce vector-valued theta series. Our definition is based on the one used in [Pf53] and [Sh73].

Definition 3.18 *Suppose that $S \in \mathrm{Sym}(l;\mathbb{R})$ is an even positive definite matrix. Let $\Lambda = \mathbb{Z}^l$.*

a) Let $r \in \mathbb{Z}$, $r \geq 0$, and additionally $r \leq 1$ if $l = 1$. A homogeneous spherical polynomial of degree r with respect to S *is a function $p : \mathbb{R}^l \to \mathbb{C}$ of the form*

$$p(x) = \sum_{v\in\mathbb{C}^l} \alpha_v\ ({}^t vSx)^r$$

with $\alpha_v \neq 0$ *for finitely many vectors* $v \in \mathbb{C}^l$ *satisfying* $S[v] = 0$ *if* $r > 1$.

b) *Let* p_r *be a homogeneous spherical polynomial of degree* r *with respect to* S. *Then we define the theta series* $\Theta(\cdot; S, p_r) : \mathcal{H} \to \mathbb{C}[\Lambda^\sharp/\Lambda]$ *by*

$$\Theta(\tau; S, p_r) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \theta_\mu(\tau; S, p_r)\; e_\mu$$

where

$$\theta_\mu(\tau; S, p_r) = \sum_{\lambda \in \mu + \Lambda} p_r(\lambda)\; e^{\pi i S[\lambda]\tau} \quad \textit{for } \tau \in \mathcal{H}.$$

According to Pfetzer ([Pf53]) and Shimura ([Sh73]), those theta series are holomorphic modular forms.

Theorem 3.19 *Suppose that* $S \in \mathrm{Sym}(l; \mathbb{R})$ *is an even positive definite matrix and that* p_r *is a homogeneous spherical polynomial of degree* r *with respect to* S. *Then* $\Theta(\cdot; S, p_r)$ *is a modular form of weight* $l/2 + r$ *with respect to the Weil representation* ρ_S, *i.e.,*

$$\Theta(\cdot; S, p_r) \in [\mathrm{Mp}(2; \mathbb{Z}), l/2 + r, \rho_S].$$

If $r > 0$ *then* $\Theta(\cdot; S, p_r)$ *is a cusp form.*

The Fourier expansion of the components θ_μ, $\mu \in \Lambda^\sharp/\Lambda$, *of* Θ *is given by*

$$\theta_\mu(\tau; S, p_r) = \sum_{\substack{n \in q(\mu) + \mathbb{Z} \\ n \geq 0}} c_\mu(n; p_r)\; e^{2\pi i n \tau}, \quad \textit{where} \quad c_\mu(n; p_r) = \sum_{\substack{\lambda \in \mu + \Lambda \\ q(\lambda) = n}} p_r(\lambda).$$

Concrete examples of theta series will be constructed in Section 5.2.

4. Borcherds Products

In this chapter we apply the results of [Bo98] and [Br02] to our special case.

Let $S \in \mathrm{Pos}(l;\mathbb{R})$ be an even positive definite matrix of degree $l \in \mathbb{N}$. Recall the definition of S_0 and S_1 as well as the definitions of the associated bilinear forms $(\cdot,\cdot)$, $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ and the corresponding quadratic forms q, q_0 and q_1 from Section 1.2. Moreover, let $\Lambda = \mathbb{Z}^l$, $\Lambda_0 = \mathbb{Z}^{l+2}$, $\Lambda_1 = \mathbb{Z}^{l+4}$ and $V = \mathbb{R}^l$, $V_0 = \mathbb{R}^{l+2}$, $V_1 = \mathbb{R}^{l+4}$. Note that Λ_1 together with $(\cdot,\cdot)_1$ is an even lattice of signature $(2, l+2)$. Furthermore, note that $q = q_S$ and thus $q_1((*,*,x,*,*)) + \mathbb{Z} = q_0((*,x,*)) + \mathbb{Z} = -q(x) + \mathbb{Z}$ for all $x \in V$.

Recall that the discriminant groups of Λ, Λ_0 and Λ_1 are canonically isomorphic. Therefore we will make no distinction between those groups or between the corresponding group algebras, i.e., we will often write $\mu(\in \Lambda^\sharp/\Lambda)$ and e_μ or $\mu_0(\in \Lambda_0^\sharp/\Lambda_0)$ and e_{μ_0} instead of the corresponding elements of $\Lambda_1^\sharp/\Lambda_1$ and $\mathbb{C}[\Lambda_1^\sharp/\Lambda_1]$. In particular, we will often denote the Fourier coefficients of a vector-valued modular form $f : \mathcal{H}_S \to \mathbb{C}[\Lambda_1^\sharp/\Lambda_1]$ by $c_\mu(n)$ or $c_{\mu_0}(n)$. Also, since the Weil representation ρ_{S_1} is essentially the same as the dual Weil representation $\rho_S^\sharp(\cong \rho_{-S})$, we will always use the latter even though using the former would be more correct. Moreover, by abuse of notation we will sometimes write an element μ of a discriminant group in place of an element λ of the corresponding dual lattice or vice versa. In this case we always mean an arbitrary element of the coset μ or the coset λ lies in, respectively. For example we often write $q(\mu) + \mathbb{Z}$ and $\mu + \Lambda$ for $\mu \in \Lambda^\sharp/\Lambda$, and we sometimes write $c_\lambda(n)$ instead of $c_{\lambda+\Lambda}(n)$ for $\lambda \in \Lambda^\sharp$. In any case it will always be clear from the context what is meant.

In order to apply Borcherds's theory we have to choose a primitive isotropic vector $z \in \Lambda_1$ and a second vector $z^\sharp \in \Lambda_1^\sharp$ such that $(z, z^\sharp)_1 = 1$. We choose and fix $z = (1,0,\ldots,0)$ and $z^\sharp = (0,\ldots,0,1)$. This choice allows us to identify $(\Lambda_1 \cap z^\perp)/\mathbb{Z}z$ with the Lorentzian lattice $\Lambda_0 \cong \{0\} \times \Lambda_0 \times \{0\} = \Lambda_1 \cap z^\perp \cap (z^\sharp)^\perp$.

4.1. Weyl chambers and the Weyl vector

We consider the Lorentzian lattice Λ_0, and set $z_0 = (1,0,\ldots,0)$ and $z_0^\sharp = (0,\ldots,0,1)$. Then z_0 is a primitive isotropic element of Λ_0 and $z_0 \in \Lambda_0^\sharp$ with $(z_0, z_0^\sharp)_0 = 1$. Therefore we can identify $(\Lambda_0 \cap z_0^\perp)/\mathbb{Z}z_0$ with the negative definite lattice $\Lambda \cong \{0\} \times \Lambda \times \{0\} = \Lambda_0 \cap z_0^\perp \cap (z_0^\sharp)^\perp$. Note that z_0 is in the closure of the cone $\mathcal{P}_S$ of positive norm vectors of $V_0 = \Lambda_0 \otimes \mathbb{R}$ we fixed in section 1.2.

Definition 4.1 *Let $\mathcal{P}_S^1 = \{v \in \mathcal{P}_S;\ q_0(v) = 1\}$ be the subset of norm 1 vectors in $\mathcal{P}_S$.*

a) For $\mu \in \Lambda^\sharp/\Lambda$ and $n \in -q(\mu) + \mathbb{Z}$, $n < 0$, we define the subset $H_0(\mu, n)$ of $\mathcal{P}_S^1$ by

$$H_0(\mu, n) = \bigcup_{\substack{\lambda_0 \in (0,\mu,0)+\Lambda_0 \\ q_0(\lambda_0)=n}} \lambda_0^\perp$$

where $\lambda_0^\perp$ is the orthogonal complement of λ_0 in $\mathcal{P}_S^1$. Then the connected components of $\mathcal{P}_S^1 - H_0(\mu, n)$ are called Weyl chambers of $\mathcal{P}_S^1$ of index (μ, n).

b) Let $f : \mathcal{H}_S \to \mathbb{C}[\Lambda_1^\sharp/\Lambda_1]$ be a nearly holomorphic modular form of weight $k = -l/2$ with respect to the dual Weil representation $\rho_S^\sharp$. Suppose that f has Fourier expansion

$$\sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{n \in -q(\mu)+\mathbb{Z}} c_\mu(n) q^n \, e_\mu.$$

Then the connected components of

$$\mathcal{P}_S^1 - \bigcup_{\mu \in \Lambda^\sharp/\Lambda} \bigcup_{\substack{n \in -q(\mu)+\mathbb{Z} \\ n<0,\, c_\mu(n)\neq 0}} H_0(\mu, n)$$

are called Weyl chambers of $\mathcal{P}_S^1$ with respect to f.

c) Let W be a Weyl chamber (of either type) and $\lambda_0 \in \Lambda_0^\sharp$. Then we write $(\lambda_0, W)_0 > 0$ if $(\lambda_0, w)_0 > 0$ for all $w \in W$.

The Weyl chambers are usually not explicitly given. Therefore a condition of the form $(\lambda_0, W)_0 > 0$ is hard to verify. Luckily, it often suffices to check the condition for a single element of a Weyl chamber.

Lemma 4.2 *Let W be a Weyl chamber of $\mathcal{P}_S^1$ with respect to a nearly holomorphic modular form f with Fourier expansion*

$$\sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{n \in -q(\mu)+\mathbb{Z}} c_\mu(n) q^n \, e_\mu.$$

If $\lambda_0 \in \Lambda_0^\sharp$ with $c_{\lambda_0}(q_0(\lambda_0)) \neq 0$ and $(\lambda_0, v)_0 > 0$ for one vector $v \in W$ then $(\lambda_0, W)_0 > 0$, i.e., $(\lambda_0, w)_0 > 0$ for all $w \in W$.

PROOF Obviously we have

$$W = \bigcap_{\mu \in \Lambda^\sharp/\Lambda} \bigcap_{\substack{n \in -q(\mu)+\mathbb{Z} \\ n<0,\, c_\mu(n)\neq 0}} W_{\mu,n}$$

for suitable Weyl chambers $W_{\mu,n}$ of index (μ, n) (cf. [Br02, p. 88]). Since $c_{\lambda_0}(q_0(\lambda_0)) \neq 0$ we either have $q_0(\lambda_0) \geq 0$ or $q_0(\lambda_0) = n < 0$ and $\lambda_0 \in (0, \mu, 0) + \Lambda_0$ for one of the Weyl chambers $W_{\mu,n}$ which occur in the above section. Therefore we can apply [Br02, La. 3.2] (where, in case of $q_0(\lambda_0) \geq 0$, we choose an arbitrary Weyl chamber $W_{\mu,n}$ occurring in the section) and get $(\lambda_0, w)_0 > 0$ for all $w \in W_{\mu,n} \supset W$. ∎

Definition 4.3 *Let f be a nearly holomorphic modular form with Fourier expansion*

$$\sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{n \in -q(\mu)+\mathbb{Z}} c_\mu(n) q^n \, e_\mu,$$

and let W be a Weyl chamber of $\mathcal{P}_S^1$ with respect to f such that z_0 lies in the closure of the positive cone generated by W. Then we define the Weyl vector $\varrho_f(W) \in V_0$ *of W by $\varrho_f(W) = (\varrho_{z_0}, \varrho, \varrho_{z_0^\sharp})$ where*

$$\begin{aligned}
\varrho_{z_0} &= \frac{1}{24} \sum_{\lambda \in \Lambda^\sharp} c_\lambda(-q(\lambda)), \\
\varrho &= -\frac{1}{2} \sum_{\substack{\lambda \in \Lambda^\sharp \\ ((0,\lambda,0),W)_0 > 0}} c_\lambda(-q(\lambda))\lambda, \qquad (4.1) \\
\varrho_{z_0^\sharp} &= \varrho_{z_0} - \sum_{n=1}^{\infty} \sigma_1(n) \sum_{\lambda \in \Lambda^\sharp} c_\lambda(-n - q(\lambda)),
\end{aligned}$$

and $\sigma_1(n) = \sum_{d|n} d$ is the sum of divisors of n.

Note that the sums which occur in the definition of the components of the Weyl vector are all finite since $q = q_S$ is positive definite. Therefore and due to Lemma 4.2 we can explicitly calculate the Weyl vector of the Weyl chamber W with respect to f if the Fourier coefficients of the principal part of f are known and if we find a suitable $v \in W$.

Proposition 4.4 *Our definition of the Weyl vector is compatible with Borcherds's definition in [Bo98, Sec. 10].*

PROOF According to [Bo98, Th. 10.4] and the correction in the introduction of [Bo00] the Weyl vector defined in [Bo98, Sec. 10] is equal to $(\varrho_{z_0}, \varrho, \varrho_{z_0^\sharp})$ with

$$\begin{aligned}
\varrho_{z_0} &= -q_0(z_0^\sharp)\varrho_{z_0^\sharp} + \frac{1}{4} \sum_{\lambda \in \Lambda^\sharp} \sum_{\substack{\delta \in \Lambda_0^\sharp/\Lambda_0 \\ \delta = (0,\lambda,0)+\Lambda_0}} c_\delta(-q(\lambda)) B_2((\delta, z_0^\sharp)_0), \\
\varrho &= -\frac{1}{2} \sum_{\substack{\lambda \in \Lambda^\sharp,\, (0,\lambda,0) \in \Lambda_0^\sharp \\ ((0,\lambda,0),W)_0 > 0}} c_{(0,\lambda,0)}(-q(\lambda))\lambda, \\
\varrho_{z_0^\sharp} &= \text{constant term of } \overline{\Theta}_\Lambda(\tau) f_\Lambda(\tau) E_2(\tau)/24,
\end{aligned}$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ for $0 \leq x \leq 1$ is a Bernoulli piecewise polynomial,

$$\overline{\Theta}_\Lambda(\tau) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\lambda \in \mu + \Lambda} e^{2\pi i q(\lambda)\tau} \, e_{-\mu}$$

is a certain vector-valued theta series, f_Λ (as defined in [Bo98, p. 512]) is in our situation equal to f,

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is the elliptic Eisenstein series of weight 2, and $\overline{\Theta}_\Lambda f_\Lambda$ is the inner product of $\overline{\Theta}_\Lambda$ and f_Λ with $(e_\mu, e_{\mu'}) = 1$ for $\mu, \mu' \in \Lambda^\sharp/\Lambda$ if $\mu + \mu' = 0$ and 0 otherwise.

Since $q_0(z_0^\sharp) = 0$ the first term in the formula for ϱ_{z_0} vanishes. Moreover, $(\delta, z_0^\sharp)_0 = 0$ for all $\delta \in \Lambda_0^\sharp/\Lambda_0$ with $\delta = (0, \lambda, 0) + \Lambda_0$. Thus the formula for ϱ_{z_0} can be simplified to

$$\varrho_{z_0} = \frac{1}{4} \sum_{\lambda \in \Lambda^\sharp} c_{(0,\lambda,0)}(-q(\lambda)) B_2(0) = \frac{1}{24} \sum_{\lambda \in \Lambda^\sharp} c_\lambda(-q(\lambda)).$$

Because of $\Lambda_0^\sharp = \mathbb{Z} \times \Lambda^\sharp \times \mathbb{Z}$ the additional condition $(0, \lambda, 0) \in \Lambda_0^\sharp$ in the formula for ϱ can be omitted.

Finally we calculate the constant term of $\overline{\Theta}_\Lambda(\tau) f(\tau) E_2(\tau)/24$. Let $\alpha(g; n)$ denote the n-th Fourier coefficient of g. Then

$$\varrho_{z_0^\sharp} = \alpha(\overline{\Theta}_\Lambda f E_2/24; 0) = \frac{1}{24} \alpha(\overline{\Theta}_\Lambda f; 0) - \sum_{n=1}^{\infty} \sigma_1(n) \alpha(\overline{\Theta}_\Lambda f; -n). \tag{4.2}$$

With the above Fourier expansions for f and Θ_Λ we get

$$\begin{aligned}
\overline{\Theta}_\Lambda(\tau) f(\tau) &= \left(\sum_{\mu \in \Lambda^\sharp/\Lambda} \left(\sum_{\lambda \in \mu + \Lambda} q^{q(\lambda)} \right) e_{-\mu} \right) \cdot \left(\sum_{\mu \in \Lambda^\sharp/\Lambda} \left(\sum_{n \in -q(\mu) + \mathbb{Z}} c_\mu(n) q^n \right) e_\mu \right) \\
&= \sum_{\mu \in \Lambda^\sharp/\Lambda} \left(\sum_{\lambda \in \mu + \Lambda} q^{q(\lambda)} \right) \cdot \left(\sum_{n \in -q(\mu) + \mathbb{Z}} c_\mu(n) q^n \right)
\end{aligned}$$

and thus

$$\alpha(\overline{\Theta}_\Lambda f; -n) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{\lambda \in \mu + \Lambda} c_\mu(-n - q(\lambda)) = \sum_{\lambda \in \Lambda^\sharp} c_\lambda(-n - q(\lambda)).$$

Inserting this into (4.2) yields

$$\begin{aligned}\varrho_{z_0^\sharp} &= \frac{1}{24}\sum_{\lambda\in\Lambda^\sharp} c_\lambda(-q(\lambda)) - \sum_{n=1}^{\infty}\sigma_1(n)\sum_{\lambda\in\Lambda^\sharp} c_\lambda(-n-q(\lambda)) \\ &= \varrho_{z_0} - \sum_{n=1}^{\infty}\sigma_1(n)\sum_{\lambda\in\Lambda^\sharp} c_\lambda(-n-q(\lambda)).\end{aligned}$$ ■

Remark *For the special case where Λ is the maximal order of an imaginary quadratic field with quadratic form $q(z) = |z|^2$ this result can already be found in [DK03].*

Next we will define a special Weyl chamber W_f which will allow us to replace the hard-to-check condition $(\lambda, W)_0 > 0$ appearing in the definition of the Weyl vector by a much nicer condition.

Proposition 4.5 *For $x \in \mathbb{R}$, $x > 0$, we define the vectors $\alpha(x) := (1, \dots, x^{l-1}) \in \mathbb{R}^l$ and $v(x) := (1, -x^2\alpha(x), x) \in \mathbb{R}^{l+2}$. Moreover, we set $v_1(x) := v(x)/\sqrt{q_0(v(x))}$ whenever $q_0(v(x)) > 0$.*
a) For small positive values of x we have $v(x) \in \mathcal{P}_S$ and $v_1(x) \in \mathcal{P}_S^1$.
b) Suppose that f is a nearly holomorphic modular form with Fourier expansion

$$\sum_{\mu\in\Lambda^\sharp/\Lambda}\ \sum_{n\in -q(\mu)+\mathbb{Z}} c_\mu(n)q^n\, e_\mu.$$

Then there exists a Weyl chamber W of $\mathcal{P}_S^1$ with respect to f such that $v_1(x) \in W$ for small values of $x > 0$, i.e., there exists an $x_0 \in \mathbb{R}$, $x_0 > 0$, such that $\{v_1(x);\ 0 < x < x_0\} \cap \lambda_0^\perp = \emptyset$ for all $\lambda_0 \in \Lambda_0^\sharp$ with $q_0(\lambda_0) < 0$ and $c_{\lambda_0}(q_0(\lambda_0)) \neq 0$.

PROOF a) For $x \to 0$ we have $q_0(v(x)) = x - x^4 q(\alpha(x)) = x + O(x^4)$. Thus the definition of $\mathcal{P}_S$ implies that $v(x) \in \mathcal{P}_S$ for small positive values of x. For those x we obviously have $v_1(x) \in \mathcal{P}_S^1$.

b) By virtue of a) we have $v_1(x) \in \mathcal{P}_S^1$ for small values of $x > 0$. Suppose that for arbitrary small values of $x_0 > 0$ there is a $\lambda_0 = (m, \lambda, n) \in \Lambda_0^\sharp$ with $q_0(\lambda_0) < 0$ and $c_{\lambda_0}(q_0(\lambda_0)) \neq 0$ such that $v_1(x) \in \lambda_0^\perp$ for some $0 < x < x_0$. Then

$$0 = (v_1(x), \lambda_0) = (v(x), \lambda_0) = mx + n + x^2(\alpha(x), \lambda).$$

Case 1: $(m, n) \neq (0, 0)$. Due to the Cauchy-Schwarz inequality we have

$$(mx+n)^2 = x^4(\alpha(x), \lambda)^2 \le x^4(\alpha(x), \alpha(x)) \cdot (\lambda, \lambda) = 4x^4 q(\alpha(x))q(\lambda),$$

and thus

$$q_0(\lambda_0) = mn - q(\lambda) \le mn - \frac{(mx+n)^2}{4x^4 q(\alpha(x))}.$$

This implies that $q_0(\lambda_0)$ tends to $-\infty$ for small values of $x > 0$ which is impossible because $c_{\lambda_0}(m) = 0$ for $m \ll 0$.
Case 2: $(m, n) = (0, 0)$. Since $\Lambda^\sharp = S^{-1}\Lambda$ there is a $t = {}^t(t_1, \ldots, t_l) \in \Lambda = \mathbb{Z}^l$ such that $\lambda = S^{-1}t$. Note that $t \neq 0$ because $q_0(\lambda_0) = -q(\lambda) < 0$. Because of $m = n = 0$ we have

$$0 = (\alpha(x), \lambda) = (\alpha(x), S^{-1}t) = \sum_{j=1}^{l} t_j x^{j-1}. \tag{4.3}$$

Let $r := \min_{1 \leq j \leq l}\{j;\ t_j \neq 0\}$ and $s := \max_{1 \leq j \leq l}\{j;\ t_j \neq 0\}$. Then (4.3) and $x \neq 0$ imply $r \neq s$. We get

$$t_s = -\sum_{j=r}^{s-1} t_j x^{j-s} = -\frac{t_r}{x^{s-r}} + O\left(\frac{1}{x^{s-r-1}}\right)$$

for $x \downarrow 0$. Now we consider

$$\begin{aligned}
-q_0(\lambda_0) = q(\lambda) = q_{S^{-1}}(t) &= \sum_{r \leq j,k \leq s} (S^{-1})_{j,k} t_j t_k \\
&= \sum_{r \leq j,k \leq s-1} (S^{-1})_{j,k} t_j t_k + 2\sum_{j=r}^{s-1} (S^{-1})_{j,s} t_j t_s + (S^{-1})_{s,s} t_s^2 \\
&= (S^{-1})_{s,s} \frac{t_r^2}{x^{2s-2r}} + O\left(\frac{1}{x^{2s-2r-1}}\right). \qquad \blacksquare
\end{aligned}$$

Since S is positive definite we have $(S^{-1})_{s,s} > 0$. Thus, just as in the first case, we get a contradiction because $q_0(\lambda_0)$ tends to $-\infty$ for small values of $x > 0$.

Definition 4.6 *For $x \in \mathbb{R}$, $0 < x \ll 1$, let $v_1(x)$ be defined as in Proposition 4.5. Let f be a nearly holomorphic modular form. Then we denote the uniquely determined Weyl chamber of $\mathcal{P}_S^1$ with respect to f that contains $v_1(x)$ for small values of x by W_f and call it* the Weyl chamber of f. *Moreover, we denote the corresponding Weyl vector $\varrho_f(W_f)$ simply by ϱ_f and call it* the Weyl vector of f.

Note that $z_0 = (1, 0 \ldots, 0)$ is contained in the closure of the positive cone of the Weyl chamber W_f since $v(x)$ (as defined in Proposition 4.5) is contained in the positive cone for small values of x and converges to z_0 for $x \to 0$. Thus the Weyl vector ϱ_f is well-defined.

Next we define a certain type of positive vectors. As we will show this positiveness coincides with positiveness with respect to the Weyl chamber W_f. The definition differs from the definition of positive vectors introduced in Section 2.1, but this should not lead to confusion.

Definition 4.7 *Let $t = {}^t(t_1, \ldots, t_l) \in \Lambda = \mathbb{Z}^l$. We write $t > 0$ if there is a $j \in \mathbb{N}$, $1 \leq j \leq l$, such that $t_1 = \ldots = t_{j-1} = 0$ and $t_j > 0$. For $\lambda = S^{-1}t \in \Lambda^\sharp$ we write $\lambda > 0$*

if $t > 0$*, and for* $\lambda_0 = (m, \lambda, n) \in \Lambda_0^\sharp$ *we write* $\lambda_0 > 0$ *if* $n > 0$ *or* $n = 0$ *and* $m > 0$ *or* $m = n = 0$ *and* $\lambda > 0$*. Additionally, we write* $t < 0$*,* $\lambda < 0$ *and* $\lambda_0 < 0$ *if* $-t > 0$*,* $-\lambda > 0$ *and* $-\lambda_0 > 0$*, respectively.*

Note that for each $t \in \mathbb{Z}^l$ we have either $t > 0$ or $t < 0$ or $t = 0$. Analogous assertions hold for $\lambda \in \Lambda^\sharp$ and $\lambda_0 \in \Lambda_0^\sharp$.

Proposition 4.8 *Suppose that* f *is a nearly holomorphic modular form with Fourier expansion*

$$\sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{n \in -q(\mu)+\mathbb{Z}} c_\mu(n) q^n \, e_\mu.$$

Let W_f *be the corresponding Weyl chamber of* $\mathcal{P}_S^1$*. Then for all* $\lambda_0 \in \Lambda_0^\sharp$ *with* $c_{\lambda_0}(q_0(\lambda_0)) \neq 0$ *we have* $(\lambda_0, W_f) > 0$ *if and only if* $\lambda_0 > 0$*.*

PROOF Let $\lambda_0 = (m, \lambda, n) \in \Lambda_0^\sharp$ with $c_{\lambda_0}(q_0(\lambda_0)) \neq 0$, and let $t = {}^t(t_1, \ldots, t_l) \in \Lambda = \mathbb{Z}^l$ such that $\lambda = S^{-1}t$. For $x \in \mathbb{R}$, $0 < x \ll 1$ let $v(x)$ and $v_1(x)$ be defined as in Proposition 4.5. By virtue of Lemma 4.2 we have $(\lambda_0, W_f) > 0$ if and only if $(\lambda_0, v_1(x)) > 0$ (whenever $x > 0$ such that $v_1(x) \in W_f$). Since $v_1(x)$ is a positive multiple of $v(x)$ we have $(\lambda_0, v_1(x)) > 0$ if and only if $(\lambda_0, v(x)) > 0$. The claim now follows from the fact that the inequality

$$(\lambda_0, v(x)) = n + mx + x^2(t_1 + \ldots + t_l x^{l-1}) > 0$$

is satisfied for arbitrary small values of x if and only if $\lambda_0 > 0$. ∎

4.2. Quadratic divisors

For the purpose of this chapter we introduce a different realization of $\mathcal{H}_S$ as subvariety of the projective space $P(V_1(\mathbb{C})) := \{[Z];\ Z \in V_1(\mathbb{C})\}$ associated to the complexification $V_1(\mathbb{C}) = V_1 \otimes \mathbb{C}$ of V_1. We extend the bilinear form $(\cdot,\cdot)_1 : V_1 \times V_1 \to \mathbb{R}$ to a $\mathbb{C}$-bilinear form on $V_1(\mathbb{C})$. Let

$$\mathcal{N} := \{[Z] \in P(V_1(\mathbb{C}));\ q_1(Z) = 0\}$$

be the zero-quadric in $P(V_1(\mathbb{C}))$ and

$$\mathcal{K} := \{[Z] \in \mathcal{N};\ (Z, \overline{Z})_1 > 0\}.$$

If $Z = X + iY \in V_1(\mathbb{C})$ then $[Z] \in \mathcal{K}$ if and only if $q_1(X) = q_1(Y) > 0$ and $(X, Y)_1 = 0$. We define a map $\iota : \mathcal{H}_S \cup (-\mathcal{H}_S) \to P(V_1(\mathbb{C}))$ by

$$\iota(w) = [(-q_0(w), w, 1)] \qquad \text{for all } w \in \mathcal{H}_S \cup (-\mathcal{H}_S).$$

Let $w = u + iv \in \mathcal{H}_S \cup (-\mathcal{H}_S)$. Then $\iota(w) = [X + iY]$ with $X = (q_0(v) - q_0(u), u, 1)$ and $Y = (-(u, v)_0, v, 0)$. Because of $q_1(X) = q_1(Y) = q_0(v) > 0$ and $(X, Y)_1 = (u, v)_0 - (u, v)_0 = 0$ for all $w \in \mathcal{H}_S \cup (-\mathcal{H}_S)$ we conclude $\iota(w) \in \mathcal{K}$ for all $w \in \mathcal{H}_S \cup (-\mathcal{H}_S)$.

Conversely, let $[Z]=[X+iY]\in\mathcal{K}$. Since X and Y span a two-dimensional (and thus maximal) positive definite subspace of V_1 we have $(Z,z)_1\neq 0$, where $z=(1,0,\ldots,0)$ is the isotropic vector we fixed at the begin of this chapter. Therefore $[Z]$ has a unique representation of the form $[(-q_0(Z_0),Z_0,1)]$, $Z_0\in V_0(\mathbb{C})\cong\mathbb{C}^{l+2}$. Now as above $[Z]\in\mathcal{K}$ implies $q_0(\mathrm{Im}(Z_0))>0$ and thus $Z_0\in\mathcal{H}_S\cup(-\mathcal{H}_S)$.

We conclude that ι biholomorphically maps $\mathcal{H}_S\cup(-\mathcal{H}_S)$ to $\mathcal{K}$, and we denote the image of $\mathcal{H}_S$ under ι by $\mathcal{K}^+$.

On $\mathcal{K}$ the orthogonal group $\mathrm{O}(S_1;\mathbb{R})$ acts in a natural way (induced by the action on V_1). This action is (of course) exactly the same as the action of $\mathrm{O}(S_1;\mathbb{R})$ on $\mathcal{H}_S\cup(-\mathcal{H}_S)$ we introduced in Section 1.2. The subgroup $\mathrm{O}^+(S_1;\mathbb{R})$ of $\mathrm{O}(S_1;\mathbb{R})$ maps $\mathcal{K}^+$ onto itself.

Definition 4.9 *Suppose* $0\neq\lambda=(l_{-1},\lambda_0,l_{l+2})\in\Lambda_1^\sharp$. *We define the* rational quadratic divisor $\lambda^\perp$ *for* λ *by*

$$\lambda^\perp=\{w\in\mathcal{H}_S;\ l_{-1}+(\lambda_0,w)_0-l_{l+2}q_0(w)=0\}.$$

Let $\lambda_p\in\mathbb{Q}\lambda\cap\Lambda_1^\sharp$ *be primitive. Then the* discriminant $\delta(\lambda^\perp)$ *of* $\lambda^\perp$ *is defined by*

$$\delta(\lambda^\perp)=-Nq_1(\lambda_p)$$

where N *is the level of* Λ_1.

Remark 4.10 *The discriminant is well-defined since the primitive vector* λ_p *corresponding to* λ *is uniquely determined up to the sign.*

We have $w\in\lambda^\perp$ if and only if $(\lambda,(-q_0(w),w,1))_1=0$. Thus $\mathrm{O}^+(S_1;\mathbb{R})$ acts on the set of all rational quadratic divisors via

$$M\lambda^\perp:=(M\lambda)^\perp=\{M\langle w\rangle;\ w\in\lambda^\perp\}.$$

Obviously, the discriminant is invariant under this action.

Proposition 4.11 *Let* S *be one of the matrices listed in (1.2). Then* Γ_S *acts transitively on the set of rational quadratic divisors of fixed discriminant, i.e., if* $\lambda_1,\lambda_2\in\Lambda_1^\sharp$ *such that* $\delta(\lambda_1^\perp)=\delta(\lambda_2^\perp)$ *then there exists an* $M\in\Gamma_S$ *such that* $\lambda_1^\perp=M\lambda_2^\perp$.

PROOF Each rational quadratic divisor is generated by a uniquely (up to the sign) determined primitive vector in $\Lambda_1^\sharp$. According to [FH00, La. 4.6] the group Γ_S acts transitively on the set of primitive vectors in $\Lambda_1^\sharp$ of the same norm. Thus Γ_S also acts transitively on the set of rational quadratic divisors of the same discriminant. ∎

4.3. Borcherds products

Now we can state the main result of [Bo98] adapted to our situation.

Theorem 4.12 *Suppose that S is an even positive definite matrix of degree l. Given a nearly holomorphic modular form $f \in [\mathrm{Mp}(2;\mathbb{Z}), -l/2, \rho_S^\sharp]_\infty$ of weight $-l/2$ with respect to the dual Weil representation $\rho_S^\sharp$ with Fourier expansion*

$$f(\tau) = \sum_{\mu \in \Lambda^\sharp/\Lambda} \sum_{n \in -q(\mu)+\mathbb{Z}} c_\mu(n) q^n \, e_\mu$$

such that $c_0(0) \in 2\mathbb{Z}$ and $c_\mu(n) \in \mathbb{Z}$ whenever $n < 0$, there exists a Borcherds product $\psi_k : \mathcal{H}_S \to \mathbb{C}$ with the following properties:

a) ψ_k is a meromorphic modular form of weight $k = c_0(0)/2$ with respect to $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$ and some Abelian character χ of finite order.

b) The only zeros and poles of ψ_k lie on rational quadratic divisors. If $\lambda \in \Lambda_1^\sharp$ is primitive with $q_1(\lambda) < 0$ then the order of ψ_k along $\lambda^\perp$ is given by

$$\sum_{r=1}^{\infty} c_{r\lambda}(r^2 q_1(\lambda)).$$

c) Let ϱ_f be the Weyl vector of f. Moreover, let $n_0 := \min\{n \in \mathbb{Q};\ c_\gamma(n) \neq 0\}$, and let $\mathcal{S}$ be the set of poles of ψ_k. Then on $\{w = u + iv \in \mathcal{H}_S;\ q_0(v) > |n_0|\} - \mathcal{S}$ the function ψ_k is given by the normally convergent product expansion

$$\psi_k(w) = e^{2\pi i (\varrho_f, w)_0} \prod_{\substack{\lambda_0 \in \Lambda_0^\sharp \\ \lambda_0 > 0}} \left(1 - e^{2\pi i (\lambda_0, w)_0}\right)^{c_{\lambda_0}(q_0(\lambda_0))}. \tag{4.4}$$

PROOF Apply [Bo98, Thm. 13.3] and [Br02, Thm. 3.22] to our special case and take the other results from this chapter into account. ∎

Note that the theorem only gives us modular forms with respect to the subgroup $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$ of the full modular group Γ_S. But due to the explicitly given product expansion (4.4) we can explicitly check how the Borcherds product ψ_k transforms under the additional generators of Γ_S which do not fix the discriminant group, and thus we can show that the Borcherds products are in fact modular forms with respect to the full modular group. Moreover, we get explicit formulas for the values of the characters of the Borcherds products. By virtue of Proposition 1.15 we only have to consider how ψ_k transforms under matrices of the form R_A, $A \in \mathrm{O}(\Lambda)$.

Proposition 4.13 *Let $A \in \mathrm{O}(\Lambda)$, and let ψ be a Borcherds product with product expansion*

(4.4). Then

$$\begin{aligned}\frac{\psi(R_A\langle w\rangle)}{\psi(w)} = &\prod_{\substack{t\in\mathbb{Z}^l\\ t>0,\ {}^tAt<0\\ \lambda=S^{-1}t}} \left(e^{\pi i\,{}^t({}^tAt-t)z}\,\frac{1-e^{-2\pi i\,{}^t({}^tAt)z}}{1-e^{-2\pi i\,{}^ttz}}\right)^{c_\lambda(-q(\lambda))} \times \\ &\times \prod_{\substack{t\in\mathbb{Z}^l\\ t>0,\ {}^tAt>0\\ \lambda=S^{-1}t}} \left(e^{\pi i\,{}^t({}^tAt-t)z}\right)^{c_\lambda(-q(\lambda))}\end{aligned} \tag{4.5}$$

for all $w=(\tau_1,z,\tau_2)$ *in the domain of convergence.*

PROOF First of all note that for all w in the domain of convergence $R_A\langle w\rangle = \widetilde{R}_A w$, where $\widetilde{R}_A = I_1\times A\times I_1$, also lies in the domain of convergence. Thus we can insert $R_A\langle w\rangle$ in the product expansion of ψ and get

$$\frac{\psi(R_A\langle w\rangle)}{\psi(w)} = e^{2\pi i(\varrho_f,R_A\langle w\rangle-w)_0}\prod_{\substack{\lambda_0\in\Lambda_0^\sharp\\ \lambda_0>0}}\left(\frac{1-e^{2\pi i(\lambda_0,R_A\langle w\rangle)_0}}{1-e^{2\pi i(\lambda_0,w)_0}}\right)^{c_{\lambda_0}(q_0(\lambda_0))}.$$

First we look at $(\lambda_0,R_A\langle w\rangle)_0$. Let $\lambda_0=(m,S^{-1}t,n)$ and $w=(\tau_1,z,\tau_2)$. We have

$$(S^{-1}t,Az) = {}^t(S^{-1}t)SAz = {}^t({}^tAt)z$$

and

$$S^{-1}\,{}^tA = A^{-1}S^{-1}$$

since $A\in \mathrm{O}(\Lambda)$. Therefore

$$\begin{aligned}(\lambda_0,R_A\langle w\rangle)_0 &= \big((m,S^{-1}t,n),(\tau_1,Az,\tau_2)\big)_0 = m\tau_2+n\tau_1-(S^{-1}t,Az)\\ &= m\tau_2+n\tau_1-{}^t({}^tAt)z = \big((m,S^{-1}({}^tAt),n),w\big)_0 = (\widetilde{R}_{A^{-1}}\lambda_0,w)_0.\end{aligned}$$

Because of $\widetilde{R}_{A^{-1}}\Lambda_0^\sharp=\Lambda_0^\sharp$ all terms for which $\widetilde{R}_{A^{-1}}\lambda_0=(m,S^{-1}({}^tAt),n)>0$ cancel out, and thus we get

$$\begin{aligned}\frac{\psi(R_A\langle w\rangle)}{\psi(w)} &= e^{2\pi i(\varrho_f,R_A\langle w\rangle-w)_0}\prod_{\substack{t\in\mathbb{Z}^l\\ t>0,\ {}^tAt<0\\ \lambda_0=(0,S^{-1}t,0)}}\left(\frac{1-e^{-2\pi i\,{}^t({}^tAt)z}}{1-e^{-2\pi i\,{}^ttz}}\right)^{c_{\lambda_0}(q_0(\lambda_0))}\\ &= e^{2\pi i(\varrho_f,R_A\langle w\rangle-w)_0}\prod_{\substack{t\in\mathbb{Z}^l\\ t>0,\ {}^tAt<0\\ \lambda=S^{-1}t}}\left(\frac{1-e^{-2\pi i\,{}^t({}^tAt)z}}{1-e^{-2\pi i\,{}^ttz}}\right)^{c_\lambda(-q(\lambda))}.\end{aligned}$$

Next we consider $(\varrho_f, R_A\langle w\rangle - w)_0 = (\varrho_f, (0, Az - z, 0))_0 = -(\varrho, Az - z)$. Inserting the explicit formula (4.1) for ϱ yields

$$\begin{aligned}(\varrho_f, R_A\langle w\rangle - w)_0 &= \frac{1}{2} \sum_{\substack{t\in\mathbb{Z}^l,\, t>0 \\ \lambda=S^{-1}t}} c_\lambda(-q(\lambda))\, {}^t\lambda S(Az - z) \\ &= \frac{1}{2} \sum_{\substack{t\in\mathbb{Z}^l,\, t>0 \\ \lambda=S^{-1}t}} c_\lambda(-q(\lambda))\, {}^t({}^tAt - t)z.\end{aligned}$$

This completes the proof. ■

In order to construct concrete Borcherds products with known weight and known zeros and poles we need nearly holomorphic modular forms of weight $-l/2$ with respect to the dual Weil representation $\rho_S^\sharp$ with known principal part and constant term. In [Bo99] Borcherds gives a necessary and sufficient condition for the existence of nearly holomorphic modular forms with prescribed principal part and constant term. Note that, according to [Br02, Prop. 1.12], nearly holomorphic modular forms are uniquely determined by their principal part.

Theorem 4.14 *Suppose that S is an even positive definite matrix of degree l. There exists a nearly holomorphic modular form $f \in [\mathrm{Mp}(2;\mathbb{Z}), -l/2, \rho_S^\sharp]_\infty$ of weight $-l/2$ with respect to the dual Weil representation $\rho_S^\sharp$ with principal part and constant term*

$$\sum_{\mu\in\Lambda^\sharp/\Lambda} \sum_{\substack{n\in-q(\mu)+\mathbb{Z} \\ n\leq 0}} c_\mu(n) q^n\, e_\mu,$$

if and only if

$$\sum_{\mu\in\Lambda^\sharp/\Lambda} \sum_{\substack{n\in-q(\mu)+\mathbb{Z} \\ n\leq 0}} c_\mu(n)\alpha_\mu(-n) = 0$$

for all holomorphic modular forms $g \in [\mathrm{Mp}(2;\mathbb{Z}), 2 + l/2, \rho_S]$ (the so-called obstruction space*) with Fourier expansion*

$$g(\tau) = \sum_{\mu\in\Lambda^\sharp/\Lambda} \sum_{\substack{n\in q(\mu)+\mathbb{Z} \\ n\geq 0}} \alpha_\mu(n) q^n\, e_\mu.$$

PROOF [Bo99, Thm. 3.1] ■

4.3.1. Borcherds products for $S = A_3$

In this case we only have to check how the Borcherds products transform under $M_{\mathrm{tr}} = R_A$, $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, because, according to Corollary 1.23 and Proposition 1.24, we have $\Gamma_S = \langle \mathrm{O_d}(\Lambda_1) \cap \Gamma_S,\ M_{\mathrm{tr}}\rangle$.

Proposition 4.15 *Let ψ be a Borcherds product with product expansion (4.4). Then*

$$\psi(M_{\mathrm{tr}}\langle w\rangle) = \left(\prod_{\substack{t_2,t_3\in\mathbb{Z}\\(t_2,t_3)>0}} (-1)^{c_\lambda(-q(\lambda))}\right)\psi(w)$$

for all w in the domain of convergence where $\lambda = S^{-1}\,{}^t(0,t_2,t_3)$. In particular, all Borcherds products are modular forms with respect to the full modular group.

PROOF We apply Proposition 4.13. Let $t = (t_1,t_2,t_3) \in \mathbb{Z}^3$, $t > 0$. Then

$${}^tAt = \begin{pmatrix} t_1 \\ t_1 - t_2 \\ -t_3 \end{pmatrix} \quad \text{and} \quad {}^tAt - t = \begin{pmatrix} 0 \\ t_1 - 2t_2 \\ -2t_3 \end{pmatrix}.$$

Thus

$$\begin{aligned} t > 0 \text{ and } {}^tAt > 0 \quad &\Longleftrightarrow \quad t_1 > 0,\ t_2,t_3 \in \mathbb{Z}, \\ t > 0 \text{ and } {}^tAt < 0 \quad &\Longleftrightarrow \quad t_1 = 0,\ (t_2,t_3) > 0. \end{aligned}$$

First we consider the case $t > 0$ and ${}^tAt < 0$, i.e., $t = (0,t_2,t_3) > 0$. In this case

$$e^{\pi i\,{}^t({}^tAt-t)z}\,\frac{1-e^{-2\pi i\,{}^t({}^tAt)z}}{1-e^{-2\pi i\,{}^ttz}} = e^{\pi i(-2t_2z_2-2t_3z_3)}\,\frac{1-e^{\pi i(2t_2z_2+2t_3z_3)}}{1-e^{\pi i(-2t_2z_2-2t_3z_3)}} = -1.$$

Therefore the first product in (4.5) becomes

$$\prod_{\substack{t_2,t_3\in\mathbb{Z}\\(t_2,t_3)>0}} (-1)^{c_\lambda(-q(\lambda))}$$

where $\lambda = S^{-1}\,{}^t(0,t_2,t_3)$.

Next we consider the case $t > 0$ and ${}^tAt > 0$, i.e., $t = (t_1,t_2,t_3)$, $t_1 > 0$. We will show that the second product in (4.5) equals 1. The set $\{t \in \mathbb{Z}^3;\ t_1 > 0\}$ splits into the disjoint sets

$$\begin{gathered} \{t\in\mathbb{Z}^3;\ t_1>0,\ 2t_2>t_1\}, \quad \{t\in\mathbb{Z}^3;\ t_1>0,\ 2t_2<t_1\}, \\ \{t\in\mathbb{Z}^3;\ t_1>0,\ 2t_2=t_1,\ t_3>0\}, \quad \{t\in\mathbb{Z}^3;\ t_1>0,\ 2t_2=t_1,\ t_3<0\}, \\ \{t\in\mathbb{Z}^3;\ t_1>0,\ 2t_2=t_1,\ t_3=0\}. \end{gathered}$$

For each $t = (t_1,t_2,t_3)$ in the first or third set $t' = (t_1',t_2',t_3') = (t_1, t_1-t_2, -t_3) = At$ is in the second or fourth set, respectively. We have

$$e^{\pi i\,{}^t({}^tAt-t)z} = e^{\pi i((t_1-2t_2)z_2+(-2t_3z_3))}$$

and

$$e^{\pi i\, {}^t({}^tAt'-t')z} = e^{-\pi i((t_1-2t_2)z_2+(-2t_3z_3))}.$$

Moreover, for $\lambda = S^{-1}t$ and $\lambda' = S^{-1}t'$ one easily verifies that $\lambda + \Lambda = -\lambda' + \Lambda$ and $q(\lambda) = q(\lambda')$. Thus $c_\lambda(-q(\lambda)) = c_{\lambda'}(-q(\lambda'))$, and consequently the terms for t in the first and third set and the terms for the corresponding t' in the second and fourth set cancel each other out in the second product in (4.5). The remaining terms for t in the fifth set are all equal to 1. This completes the proof. ∎

4.3.2. Borcherds products for $S = A_1^{(3)}$

According to Corollary 1.23 and Proposition 1.24, we have $\Gamma_S = \langle \mathrm{O_d}(\Lambda_1) \cap \Gamma_S,\ R_A,\ R_B\rangle$, where $A = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. So in order to show that the Borcherds products are modular forms with respect to the full modular group Γ_S we have to consider how the Borcherds products transform under R_A and R_B (or some alternative generators). This will also help us to determine the Abelian characters of the Borcherds products.

Proposition 4.16 *Suppose that ψ is a Borcherds product for $S = A_1^{(3)}$ with product expansion (4.4). Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then

$$\psi(R_A\langle w\rangle) = \left(\prod_{t=1}^{\infty} (-1)^{c_{(0,t/2,t/2)}\left(-t^2/2\right)}\right) \psi(w)$$

and

$$\psi(R_B\langle w\rangle) = \left(\prod_{t=1}^{\infty} (-1)^{c_{(t/2,t/2,0)}\left(-t^2/2\right)+c_{(0,0,t/2)}\left(-t^2/4\right)}\right) \psi(w)$$

for all w in the domain of convergence. In particular, all Borcherds products are modular forms with respect to the full modular group.

PROOF This can be proved analogously to Proposition 4.15. ∎

5. Graded Rings of Orthogonal Modular Forms

5.1. The graded ring for $S = A_3$

In this section we will determine generators and algebraic structure of the graded ring of orthogonal modular forms in the case $S = A_3$, i.e.,

$$\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma'_S, k, 1].$$

First we construct some suitable Borcherds products. As input we need nearly holomorphic modular forms $f \in [\mathrm{Mp}(2;\mathbb{Z}), -3/2, \rho_S^\sharp]_\infty$ of small pole order. According to Theorem 4.14, the Fourier coefficients of those forms have to satisfy a certain condition for all elements of the obstruction space $[\mathrm{Mp}(2;\mathbb{Z}), 7/2, \rho_S]$. By virtue of Lemma 3.12 the obstruction space has dimension 1. It is spanned by the Eisenstein series $E_{7/2} = E_{7/2}(\cdot;\ e_0, A_3)$. Using the formulas in [BK01] we can calculate the Fourier expansion of this Eisenstein series. (We used the program *eis* which is available for download on Bruinier's homepage and verified the results with independent calculations.) We get

$$\begin{aligned}
E_{7/2,(0,0,0)}(\tau) &= 1 - 108q - 450q^2 - 1656q^3 + O(q^4),\\
E_{7/2,\pm(\frac14,\frac12,-\frac14)}(\tau) &= -8q^{3/8} - 216q^{11/8} - 792q^{19/8} + O(q^{27/8}),\\
E_{7/2,(\frac12,0,\frac12)}(\tau) &= -18q^{1/2} - 232q^{3/2} - 1080q^{5/2} + O(q^{7/2}),
\end{aligned}$$

where $q = e^{2\pi i\tau}$. Using Theorem 4.14 we deduce the following condition for principal part and constant term of elements of $[\mathrm{Mp}(2;\mathbb{Z}), -3/2, \rho_S^\sharp]_\infty$:

$$c_0(0) = 8\ \Big(c_{(\frac14,\frac12,-\frac14)}(-\tfrac38) + c_{(-\frac14,\frac12,\frac14)}(-\tfrac38)\Big) + 18\ c_{(\frac12,0,\frac12)}(-\tfrac12) + 108\ c_0(-1) + \cdots .$$

Thus possible principal parts and constant terms of nearly holomorphic modular forms are given by

$$\begin{array}{lllr}
 & & q^{-3/8}\ \big(e_{1/4} + e_{-1/4}\big) & +\,16\ e_0,\\
 & q^{-1/2}\ e_{1/2} & & +\,18\ e_0,\\
q^{-1}\ e_0 & & & +\,108\ e_0,
\end{array}$$

where we use the following abbreviations for the basis elements of $\mathbb{C}[\Lambda^\sharp/\Lambda]$: $e_0 = e_{0+\Lambda}$, $e_{\pm 1/4} = e_{\pm(\frac{1}{4},\frac{1}{2},-\frac{1}{4})+\Lambda}$, $e_{1/2} = e_{(\frac{1}{2},0,\frac{1}{2})+\Lambda}$. By applying Theorem 4.12 we obtain Borcherds products ψ_k which have zeros along rational quadratic divisors with discriminant ≤ 8. According to Proposition 4.11 the modular group Γ_S acts transitively on the set of rational quadratic divisors of fixed discriminant. Therefore it suffices to consider the following representatives $\lambda_\delta^\perp$ of discriminant δ:

$$\begin{aligned}
\lambda_3^\perp &= \{w \in \mathcal{H}_S;\ z_3 = 0\} \cong \mathcal{H}_{A_2},\\
\lambda_4^\perp &= \{w \in \mathcal{H}_S;\ z_2 = 0\} \cong \mathcal{H}_{A_1^{(2)}},\\
\lambda_8^\perp &= \{w \in \mathcal{H}_S;\ z_3 = -z_1\} \cong \mathcal{H}_{S_2},
\end{aligned}$$

where $w = (\tau_1, z_1, z_2, z_3, \tau_2)$.

Theorem 5.1 *Let $S = A_3$. Then there exist Borcherds products*

$$\psi_8 \in [\Gamma_S, 8, 1]_0, \qquad \psi_9 \in [\Gamma_S, 9, \nu_\pi]_0 \qquad \textit{and} \qquad \psi_{54} \in [\Gamma_S, 54, \nu_\pi \det]_0.$$

The zeros of the products are all of first order and are given by

$$\bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_{A_2}\rangle, \qquad \bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_{A_1^{(2)}}\rangle \qquad \textit{and} \qquad \bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_{S_2}\rangle,$$

respectively.

PROOF Theorem 4.12 yields the existence of holomorphic modular forms of the given weights with respect to $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$ and some Abelian character χ and with the given zeros (and no poles). By virtue of Proposition 4.15 the ψ_k are in fact modular forms with respect to the full modular group Γ_S, and thus $\chi \in \Gamma_S^{\mathrm{ab}}$. Moreover, the proposition allows us to calculate the value of $\chi(M_{\mathrm{tr}})$ explicitly. In view of Corollary 2.3 the character is uniquely determined by this value. Finally, the Borcherds products ψ_k obviously vanish on $\mathcal{H} \times \{0\}^3 \times \mathcal{H} \subset \lambda_\delta^\perp$ which yields $\psi_k|\Phi = 0$. This completes the proof. ■

Remark 5.2 *a) The Borcherds products ψ_8 and ψ_9 occurred already in [FH00, 13.11, 13.12]. Using Theorem 2.31 and Theorem 2.36 we can identify the restrictions of the Borcherds products to the submanifolds $\mathcal{H}_{A_1^{(2)}}$ and $\mathcal{H}_{A_2}$. We get*

$$\psi_8|\mathcal{H}_{A_1^{(2)}} = (\phi_4^{A_1^{(2)}})^2 \quad \textit{and} \quad \psi_{54}|\mathcal{H}_{A_1^{(2)}} \in \phi_4^{A_1^{(2)}} \phi_{30}^{A_1^{(2)}} \cdot [\Gamma_{A_1^{(2)}}, 20, 1],$$

$$\psi_9|\mathcal{H}_{A_2} = \phi_9^{A_2} \quad \textit{and} \quad \psi_{54}|\mathcal{H}_{A_2} = \phi_9^{A_2}\phi_{45}^{A_2}.$$

b) In [Kra] Krieg constructed lifts of ψ_8 and ψ_9^2 to quaternionic modular forms. $\psi_8^{\mathit{Krieg}} = 2\psi_8$ is given as restriction of the sum of a certain Maaß form and a twisted version of the same Maaß form. This allows us to calculate the Fourier expansion of ψ_8^{Krieg}. Moreover, *he writes that there are $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that*

$$\alpha\psi_8^{\text{Krieg}}(E_{10} - E_4E_6) + \beta\psi_9^2 = F|\mathcal{H}_{A_3},$$

for some cusp form $F \in [\Gamma_{D_4}, 18, 1]$ which is explicitly given as polynomial of Eisenstein series. ψ_9 vanishes on $\mathcal{H}_{A_1^{(2)}}$, but F does not vanish on $\mathcal{H}_{A_1^{(2)}}$. Therefore we can determine α by restricting the above equation to $\mathcal{H}_{A_1^{(2)}}$. We get $\alpha = \frac{17}{161280}$ and $\beta = 9$ (where we choose β such that the Fourier coefficients of ψ_9^2 are minimal but still integral). In particular, we can explicitly calculate the Fourier coefficients of ψ_8 and ψ_9^2.

The Borcherds products vanish on quadratic divisors of first order. Therefore, if a modular form vanishes on a quadratic divisor one of the Borcherds products vanishes on, then we can divide this modular form by the Borcherds product. Luckily, for some of the quadratic divisors the Borcherds products vanish on there exist non-trivial elements of Γ_S stabilizing those quadratic divisors pointwise. Now, if a modular form is not stabilized by such a non-trivial element $M \in \Gamma_S$, then this modular form must vanish on the quadratic divisor which is stabilized by M. This way we can show that modular forms with respect to certain Abelian characters must be divisible by certain Borcherds products. The result is summarized in the following

Lemma 5.3 *Let $S = A_3$ and $k \in \mathbb{Z}$.*

a) If k is odd, $m \in \{0, 1\}$, and $f \in [\Gamma_S, k, \nu_\pi^{m+1}\det^m]$ then f vanishes on $\mathcal{H}_{A_1^{(2)}}$ and we have $f/\psi_9 \in [\Gamma_S, k-9, \nu_\pi^m \det^m]$.

b) If $f \in [\Gamma_S, k, \nu_\pi^{k+1}\det]$ then f vanishes on $\mathcal{H}_{S_2}$ and $f/\psi_{54} \in [\Gamma_S, k-54, \nu_\pi^k]$.

PROOF a) Let $k \in \mathbb{Z}$ be odd and $f \in [\Gamma_S, k, \nu_\pi^{m+1}\det^m]$. Then f vanishes on $\mathcal{H}_{A_1^{(2)}}$ according to Corollary 2.29. Therefore Theorem 5.1 yields $f/\psi_9 \in [\Gamma_S, k-9, \nu_\pi^m \det^m]$.

b) Let $f \in [\Gamma_S, k, \nu_\pi^{k+1}\det]$. By virtue of Corollary 2.29 f vanishes on $\mathcal{H}_{S_2}$. Thus Theorem 5.1 yields $f/\psi_{54} \in [\Gamma_S, k-54, \nu_\pi^k]$. ∎

The preceding result allows us to give some more information about ψ_9.

Corollary 5.4 *ψ_9 is a Maaß form.*

PROOF According to Corollary 2.24 there is, up to a scalar factor, exactly one Maaß form f_9 of weight 9. By virtue of the preceding lemma we have $f_9 = \psi_9 \cdot f_0$ for some $f_0 \in [\Gamma'_S, 0] = \mathbb{C}$ which yields the assertion. ∎

Due to the above lemma we can reduce any modular form of odd weight and any modular form with respect to a non-trivial Abelian character to a modular form of even weight with respect to the trivial character by dividing the modular form by a suitable produce of ψ_9 and ψ_{54}. This way we have reduced the problem of determining the graded ring

$$\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma'_S, k, 1]$$

of modular forms with respect to Γ'_S to the problem of determining the graded ring

$$\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_S, k, 1] = \bigoplus_{k\in\mathbb{Z}}[\Gamma_S, 2k, 1],$$

of modular forms of even weight with respect to the full modular group Γ_S (and trivial character). Elements of this graded ring are given by the Eisenstein series $E_k = E_k^{A_3}$, $k \geq 4$, we defined in Section 2.5.2 and, of course, also by ψ_8, ψ_9^2 and ψ_{54}^2. Using our knowledge about the graded ring of modular forms on $\mathcal{H}_{A_2}$ we will now show that for each $f \in \mathcal{A}(\Gamma_S)$ we can find a polynomial in E_4, E_6, E_{10}, E_{12} and ψ_9^2 such that the restriction of f to $\mathcal{H}_{A_2}$ coincides with the restriction of this polynomial.

Lemma 5.5 *Let $S = A_3$, $k \in 2\mathbb{Z}$, and $f \in [\Gamma_S, k, 1]$. Then there exists a polynomial p such that*

$$f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$$

vanishes on $\mathcal{H}_{A_2}$.

PROOF Let $k \in \mathbb{Z}$, k even, and $f \in [\Gamma_{A_3}, k, 1]$. Then due to Theorem 2.31 $f|\mathcal{H}_{A_2} \in [\Gamma_{A_2}, k, 1]$. By virtue of Theorem 2.36 b) $f|\mathcal{H}_{A_2}$ is a polynomial in $E_4^{A_2}$, $E_6^{A_2}$, $E_{10}^{A_2}$, $E_{12}^{A_2}$ and ϕ_9^2. Since $\dim \mathcal{M}(\Gamma_{A_2}, k) = 1$ for $k \in \{4, 6\}$ and $E_k|\mathcal{H}_{A_2} \in \mathcal{M}(\Gamma_{A_2}, k)$ we have $E_4|\mathcal{H}_{A_2} = E_4^{A_2}$ and $E_6|\mathcal{H}_{A_2} = E_6^{A_2}$. Moreover, we have $\psi_9|\mathcal{H}_{A_2} = \phi_9$. It remains to be shown that $E_{10}^{A_2}$ and $E_{12}^{A_2}$ can be expressed as polynomials in $E_4^{A_2}$, $E_6^{A_2}$, $E_{10}|\mathcal{H}_{A_2}$ and $E_{12}|\mathcal{H}_{A_2}$. This can easily be verified by comparing some Fourier coefficients. ∎

The Eisenstein series E_{10} and E_{12} can be replaced by the cusp forms

$$f_{10} := E_{10} - E_4 \cdot E_6 \quad \text{and} \quad f_{12} := E_{12} - \frac{441}{691}E_4^3 - \frac{250}{691}E_6^2.$$

If we denote the normalized elliptic Eisenstein series of weight k by G_k, then we obtain

$$f_{10}|\Phi = G_{10} - G_4 \cdot G_6 = 0 \quad \text{and} \quad f_{12}|\Phi = G_{12} - \frac{441}{691}G_4^3 - \frac{250}{691}G_6^2 = 0.$$

Thus f_{10} and f_{12} are indeed cusp forms according to Proposition 2.10. By explicitly calculating the first Fourier coefficients of f_{10} and f_{12} we can verify that f_{10} and f_{12} do not vanish identically on $\mathcal{H}_{A_2}$.

Now we can prove our main result in the case $S = A_3$.

Theorem 5.6 *Let $S = A_3$.*
a) The graded ring $\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_S, 2k, 1]$ is generated by

$$E_4,\ E_6,\ \psi_8,\ E_{10},\ E_{12}\ \textit{and}\ \psi_9^2.$$

b) The graded ring $\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_S, k, 1]$ is generated by

$$E_4,\ E_6,\ \psi_8,\ \psi_9,\ E_{10},\ E_{12}\ \textit{and}\ \psi_{54}.$$

c) *The ideal of cusp forms in $\mathcal{A}(\Gamma'_S)$ is generated by*

$$\psi_8,\ \psi_9,\ f_{10},\ f_{12}\ \textit{and}\ \psi_{54}.$$

PROOF a) Let $k \in \mathbb{Z}$ be even, and let $f \in [\Gamma_S, k, 1]$. According to Lemma 5.5, there exists a polynomial p such that

$$\widetilde{f} := f - p(E_4, E_6, E_{10}, E_{12}, \psi_9^2)$$

vanishes on $\mathcal{H}_{A_2}$. Then Theorem 5.1 leads to

$$\widetilde{f}/\psi_8 \in [\Gamma_S, k-8, 1],$$

and an induction yields the assertion.

b) Let $f \in [\Gamma'_S, k, 1]$. If k is odd then, according to Lemma 5.3, the function f vanishes on $\mathcal{H}_{A_1^{(2)}}$ and we have $f/\psi_9 \in [\Gamma'_S, k-9, 1]$. So we can assume that k is even. Due to Corollary 2.3 we know that $[\Gamma'_S, k, 1] = [\Gamma_S, k, 1] \oplus [\Gamma_S, k, \nu_\pi \det]$ for even k. Thus $f = f_1 + f_{\nu_\pi \det}$ with $f_\chi \in [\Gamma_S, k, \chi]$. The function $f_{\nu_\pi \det}$ vanishes on $\mathcal{H}_{S_2}$, and we get $f_{\nu_\pi \det}/\psi_{54} \in [\Gamma_S, k-54, 1]$. Applying part a) on f_1 and $f_{\nu_\pi \det}/\psi_{54}$ completes the proof.

c) Let $\mathcal{I}$ be the ideal generated by the cusp forms ψ_8, ψ_9, f_{10}, f_{12} and ψ_{54}, and let $f \in [\Gamma'_S, k]_0$. According to part b) we can write f as a polynomial in E_4, E_6, ψ_8, ψ_9, E_{10}, E_{12} and ψ_{54}. In view of the above comments about f_{10} and f_{12} we can also write f as a polynomial in E_4, E_6, ψ_8, ψ_9, f_{10}, f_{12} and ψ_{54}. Therefore there exists a polynomial $p \in \mathbb{C}[X_1, X_2]$ such that

$$f - p(E_4, E_6) \in \mathcal{I}.$$

Application of Siegel's Φ-operator yields

$$0 = (f - p(E_4, E_6))|\Phi = p(E_4|\Phi, E_6|\Phi) = p(G_4, G_6)$$

where G_4 and G_6 are the normalized elliptic Eisenstein series of the indicated weight. Since G_4 and G_6 are algebraically independent, we have $p = 0$, and thus $f \in \mathcal{I}$. ■

Some more results are given in the following

Theorem 5.7 *Let $S = A_3$.*

a) *The orthogonal modular forms E_4, E_6, ψ_8, ψ_9, E_{10} and E_{12} are algebraically independent.*

b) *There is a unique polynomial $p \in \mathbb{C}[X_1, \dots, X_6]$ such that*

$$\psi_{54}^2 = p(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}).$$

c) *We have*

$$\mathcal{A}(\Gamma'_S) \cong \mathbb{C}[X_1, \dots, X_7]/(X_7^2 - p(X_1, \dots, X_6))$$

and

$$\sum_{k=0}^{\infty} \dim[\Gamma'_S, k] t^k = \frac{1+t^{54}}{(1-t^4)(1-t^6)(1-t^8)(1-t^9)(1-t^{10})(1-t^{12})}.$$

PROOF a) The restrictions of E_4, E_6, ψ_9, E_{10} and E_{12} to $\mathcal{H}_{A_2}$ are algebraically independent due to Theorem 2.36. Moreover, ψ_8 vanishes on $\mathcal{H}_{A_2}$ according to Theorem 5.1. This yields the assertion.

b) Because of $\psi_{54}^2 \in [\Gamma_S, 108, 1]$ the existence of p follows from Theorem 5.6. The uniqueness of p is a consequence of part a).

c) Let $Q \in \mathbb{C}[X_1, \ldots, X_7]$ such that $Q(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}, \psi_{54}) = 0$. There exist polynomials $Q_0, Q_1 \in \mathbb{C}[X_1, \ldots, X_6]$ such that $Q - Q_0 - X_7 Q_1 \in (X_7^2 - p(X_1, \ldots, X_6))$, hence

$$Q_0(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) + \psi_{54} \cdot Q_1(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) = 0. \tag{5.1}$$

Let $M = R_{(-I_3)} M_{\mathrm{tr}}$. Then the modular substitution $w \mapsto M\langle w \rangle$ maps ψ_{54} to $-\psi_{54}$ and leaves E_4, E_6, ψ_8, ψ_9, E_{10} and E_{12} invariant. Therefore, by applying this substitution on (5.1) we get

$$Q_0(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) - \psi_{54} \cdot Q_1(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}) = 0.$$

Since E_4, E_6, ψ_8, ψ_9, E_{10} and E_{12} are algebraically independent Q_0 and Q_1 both have to vanish identically. Thus we have $Q \in (X_7^2 - p(X_1, \ldots, X_6))$. The dimension formula is a direct consequence of the algebraic structure of $\mathcal{A}(\Gamma'_S)$. ∎

The dimension formula for the Maaß space in Corollary 2.24 and Theorem 5.7 imply that all modular forms of weight $k \leq 10$ are Maaß forms, i.e., we get the following

Corollary 5.8 *For $k \leq 10$ we have*

$$[\Gamma'_S, k, 1] = \mathcal{M}(\Gamma'_S, k).$$

In particular, the Borcherds products ψ_8 and ψ_9 are Maaß forms.

Similarly to Aoki-Ibukiyama [AI05] and Krieg [Kra] we can construct the Borcherds product ψ_{54} from the algebraically independent primary generators of $\mathcal{A}(\Gamma'_{A_3})$ via the Rankin-Cohen type differential operator we introduced in Section 2.2.

Corollary 5.9 *There exists a constant $c \in \mathbb{C}$, $c \neq 0$, such that*

$$\{E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}\} = c\psi_{54}.$$

PROOF Since E_4, E_6, ψ_8, ψ_9, E_{10} and E_{12} are algebraically independent, we have $0 \neq g := \{E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12}\} \in [\Gamma_{A_3}, 54, \nu_\pi \det]$ according to Proposition 2.14. Due to the character Lemma 5.3 yields $g/\psi_{54} \in [\Gamma_{A_3}, 0, 1] = \mathbb{C}$. ∎

In [Kra] Krieg determines the graded rings $\mathcal{A}(\Gamma_{D_4})$ and $\mathcal{A}(\Gamma'_{D_4})$ of quaternionic modular forms of degree 2. He shows that $\mathcal{A}(\Gamma_{D_4})$ is generated by the Eisenstein series $E_6^{D_4}$ and six modular forms f_j, $j \in \{2, 5, 6, 8, 9, 12\}$, of weight $2j$ (not to be confused with the cusp forms f_{10} and f_{12}) given as polynomials in six theta series. We examine the restrictions of those generators (where we denote the restrictions of the f_j again by f_j) to $\mathcal{H}_{A_3}$. Computing the Fourier expansions we get

$$\begin{aligned}
E_4 &= f_2,\\
51E_{10} &= 35f_5 + 16f_2E_6,\\
21421E_{12} &= 22050f_6 + 400E_6^2 - 1029f_2^3,\\
382205952\psi_8^2 &= 27f_8 - 30f_2f_6 - 4E_6f_5 + 2f_2E_6^2 + 5f_2^4,\\
2779890176\psi_9^2 &= -54f_9 - 9E_6f_6 - 41472\psi_8(f_5 - f_2E_6) + 2f_2^2f_5 + E_6^3 + 6f_2^3E_6.
\end{aligned}$$

So obviously we can replace some of the generators of the graded ring $\mathcal{A}(\Gamma_{A_3})$ by some of the restrictions of the f_j.

Corollary 5.10 *The graded ring $\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma_S, 2k, 1]$ is generated by*

$$f_2|\mathcal{H}_{A_3},\ E_6,\ \psi_8,\ f_5|\mathcal{H}_{A_3},\ f_6|\mathcal{H}_{A_3}\ \textit{ and }\ f_9|\mathcal{H}_{A_3}.$$

Due to Baily-Borel's theory of compactification of arithmetic quotients of bounded symmetric domains ([BB66]) each orthogonal modular function, i.e., each meromorphic modular form of weight 0, is a quotient of two orthogonal modular forms of the same weight. Therefore the above results allow us to determine the algebraic structure of the field of orthogonal modular functions. We denote this field by $\mathcal{K}(\Gamma_S)$. Moreover, we denote the space of meromorphic modular forms with respect to an Abelian character χ by $[\Gamma_S, k, \chi]_{\mathrm{mer}}$.

Theorem 5.11 *Let $S = A_3$.*

a) The field $\mathcal{K}(\Gamma_S)$ of orthogonal modular functions with respect to Γ_S and the trivial character is a rational function field in the generators

$$\frac{E_6^2}{E_4^3},\quad \frac{\psi_8}{E_4^2},\quad \frac{E_{10}}{E_4E_6},\quad \frac{E_{12}}{E_4^3}\quad \textit{and}\quad \frac{\psi_9^2}{E_6^3}.$$

b) The field $\mathcal{K}(\Gamma'_S)$ of all orthogonal modular functions with respect to Γ'_S is an extension of degree 2 over $\mathcal{K}(\Gamma_S)$ generated by ψ_{54}/ψ_9^6.

PROOF a) Let $f \in \mathcal{K}(\Gamma_S)$. Due to Baily-Borel ([BB66, Cor. 10.12]) there exist $g, h \in [\Gamma'_S, k]$ such that $f = g/h$. Since f is a modular function with respect to the trivial character g and h have to be modular forms with respect to the same character χ. Because of Lemma 5.3 we can assume $\chi = 1$ and k even. Then, due to Theorem 5.6, f is a quotient of polynomials in E_4, E_6, ψ_8, E_{10}, E_{12} and ψ_9^2. After dividing the polynomials by a suitable modular form $E_4^{l_4}E_6^{l_6}$ of weight $4l_4 + 6l_6 = k$ it remains to be shown that all monomials $E_4^{k_4}E_6^{k_6}\psi_8^{k_8}E_{10}^{k_{10}}E_{12}^{k_{12}}\psi_9^{2k_{18}}$ with $k_j \in \mathbb{Z}$ and $\sum_j j \cdot k_j = 0$ can be written in the above generators. This follows from

$$E_4^{k_4}E_6^{k_6}\psi_8^{k_8}E_{10}^{k_{10}}E_{12}^{k_{12}}\psi_9^{2k_{18}} = \left(\frac{E_6^2}{E_4^3}\right)^{-k_4-k_6-2k_8-2k_{10}-3k_{12}-3k_{18}} \left(\frac{\psi_8}{E_4^2}\right)^{k_8} \left(\frac{E_{10}}{E_4E_6}\right)^{k_{10}} \left(\frac{E_{12}}{E_4^3}\right)^{k_{12}} \left(\frac{\psi_9^2}{E_6^3}\right)^{k_{18}}.$$

Hence $\mathcal{K}(\Gamma_S)$ is a function field in the above generators which are algebraically independent according to Theorem 5.7.

b) The function $g = \psi_{54}/\psi_9^6$ is obviously a modular function with respect to the character $\chi = \nu_\pi \det$. If f is another modular function with respect to χ then $f/g \in [\Gamma_S, 0, 1]_{\mathrm{mer}} = \mathcal{K}(\Gamma_S)$. Therefore

$$\begin{aligned}\mathcal{K}(\Gamma'_S) &= [\Gamma'_S, 0, 1]_{\mathrm{mer}} = [\Gamma_S, 0, 1]_{\mathrm{mer}} \oplus [\Gamma_S, 0, \nu_\pi \det]_{\mathrm{mer}} \\ &= \mathcal{K}(\Gamma_S) \oplus g \cdot \mathcal{K}(\Gamma_S) = \mathcal{K}(\Gamma_S)[g].\end{aligned}$$

Due to Theorem 5.7 we have $g^2 \in \mathcal{K}(\Gamma_S)$. Thus $\mathcal{K}(\Gamma'_S)$ is an extension of degree 2 over $\mathcal{K}(\Gamma_S)$. ∎

Remark *There are no non-trivial modular functions with respect to* Γ_S *and the Abelian characters* $\det$ *or* ν_π.

5.2. The graded ring for $S = A_1^{(3)}$

In this section we will determine the algebraic structure of the graded ring of orthogonal modular forms in the case $S = A_1^{(3)}$, i.e.,

$$\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma'_S, k, 1].$$

Just as in the case $S = A_3$ we will construct suitable Borcherds products in order to reduce this problem to the problem of determining the structure of

$$\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma_S, 2k, 1].$$

The structure of this algebra can be easily derived from the structure of

$$\mathcal{A}(\Gamma_{A_1^{(2)}}) = \bigoplus_{k\in\mathbb{Z}} [\Gamma_{A_1^{(2)}}, 2k, 1].$$

First we will again construct some suitable Borcherds products. In this case, by virtue of Lemma 3.14, the obstruction space $[\mathrm{Mp}(2;\mathbb{Z}), 7/2, \rho_S]$ has dimension 3. So in addition to the Eisenstein series $E_{7/2} = E_{7/2}(\cdot;\ e_0, A_1^{(3)})$ we need two more generators. They are given by theta series. According to Theorem 3.19 we have to find homogeneous spherical polynomials p of degree 2 with respect to S in order to get suitable theta series $\Theta(\cdot; S, p)$. We can choose $p_1(x) = x_1^2 - x_2^2$ and $p_2(x) = x_2^2 - x_3^2$.

Using the formulas in [BK01] we can calculate the Fourier expansion of the Eisenstein series. (Again we used the program *eis* and verified the results with independent calculations.) We get

$$\begin{aligned}
E_{7/2,(0,0,0)}(\tau) &= 1 - 66q - 396q^2 + O(q^3),\\
E_{7/2,(\frac12,0,0)}(\tau) = E_{7/2,(0,\frac12,0)}(\tau) = E_{7/2,(0,0,\frac12)}(\tau) &= -2q^{1/4} - 120q^{5/4} + O(q^{9/4}),\\
E_{7/2,(0,\frac12,\frac12)}(\tau) = E_{7/2,(\frac12,0,\frac12)}(\tau) = E_{7/2,(\frac12,\frac12,0)}(\tau) &= -12q^{1/2} - 184q^{3/2} + O(q^{7/2}),\\
E_{7/2,(\frac12,\frac12,\frac12)}(\tau) &= -40q^{3/4} - 192q^{7/4} + O(q^{11/4}),
\end{aligned}$$

where $q = e^{2\pi i\tau}$. According to Theorem 3.19, for the components of the two theta series we get the Fourier expansions

$$\begin{aligned}
\theta_{(0,0,0)}(\tau; S, p_1) &= 0,\\
\theta_{(\frac12,0,0)}(\tau; S, p_1) &= \frac12 q^{1/4} - 2q^{5/4} - \frac32 q^{9/4} + O(q^{13/4}),\\
\theta_{(0,\frac12,0)}(\tau; S, p_1) &= -\theta_{(\frac12,0,0)}(\tau; S, p_1),\\
\theta_{(0,0,\frac12)}(\tau; S, p_1) &= 0,\\
\theta_{(0,\frac12,\frac12)}(\tau; S, p_1) &= -q^{1/2} + 6q^{3/2} - 10q^{5/2} + O(q^{7/2}),\\
\theta_{(\frac12,0,\frac12)}(\tau; S, p_1) &= -\theta_{(0,\frac12,\frac12)}(\tau; S, p_1),\\
\theta_{(\frac12,\frac12,0)}(\tau; S, p_1) &= 0,\\
\theta_{(\frac12,\frac12,\frac12)}(\tau; S, p_1) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\theta_{(0,0,0)}(\tau; S, p_2) &= 0,\\
\theta_{(\frac12,0,0)}(\tau; S, p_2) &= 0,\\
\theta_{(0,\frac12,0)}(\tau; S, p_2) &= \theta_{(\frac12,0,0)}(\tau; S, p_1),\\
\theta_{(0,0,\frac12)}(\tau; S, p_2) &= \theta_{(0,\frac12,0)}(\tau; S, p_1),\\
\theta_{(0,\frac12,\frac12)}(\tau; S, p_2) &= 0,\\
\theta_{(\frac12,0,\frac12)}(\tau; S, p_2) &= \theta_{(0,\frac12,\frac12)}(\tau; S, p_1)\\
\theta_{(\frac12,\frac12,0)}(\tau; S, p_2) &= \theta_{(\frac12,0,\frac12)}(\tau; S, p_1),\\
\theta_{(\frac12,\frac12,\frac12)}(\tau; S, p_2) &= 0.
\end{aligned}$$

Inserting the theta series into the obstruction condition (cf. Theorem 4.14) yields

$$h_{(\frac{1}{2},0,0)} = h_{(0,\frac{1}{2},0)} = h_{(0,0,\frac{1}{2})} \quad \text{and} \quad h_{(0,\frac{1}{2},\frac{1}{2})} = h_{(\frac{1}{2},0,\frac{1}{2})} = h_{(\frac{1}{2},\frac{1}{2},0)}$$

for all $h \in [\mathrm{Mp}(2;\mathbb{Z}), -3/2, \rho_S^\sharp]_\infty$. Thus, using the Fourier expansion of the Eisenstein series $E_{7/2}$, we see that the terms

$$\begin{array}{rrrr} & & 3\cdot q^{-1/4} & +6, \\ & 3\cdot q^{-1/2} & & +36, \\ q^{-3/4} & & & +40, \\ q^{-1} & & -3\cdot q^{-1/4} & +60, \end{array}$$

where the Fourier expansion of the components can be easily reconstructed from, are valid principal parts and constant terms of nearly holomorphic modular forms of weight $-3/2$ with respect to $\rho_S^\sharp$. By applying Theorem 4.12 we obtain Borcherds products ψ_k with zeros along rational quadratic divisors of discriminant ≤ 8. Just as in the case $S = A_3$ it suffices to consider the following representatives $\lambda_\delta^\perp$ of discriminant δ:

$$\begin{aligned} \lambda_2^\perp &= \{w \in \mathcal{H}_S;\ z_3 = 0\} \cong \mathcal{H}_{A_1^{(2)}}, \\ \lambda_4^\perp &= \{w \in \mathcal{H}_S;\ z_2 = z_3\} \cong \mathcal{H}_{S_2}, \\ \lambda_6^\perp &= \{w \in \mathcal{H}_S;\ z_3 = z_1 + z_2\} \cong \mathcal{H}_{2A_2}, \\ \lambda_8^\perp &= \{w \in \mathcal{H}_S;\ z_3 = \tfrac{1}{2}\} =: \mathcal{H}_8, \end{aligned}$$

where $w = (\tau_1, z_1, z_2, z_3, \tau_2)$.

Theorem 5.12 *Let $S = A_1^{(3)}$. Then there exist Borcherds products*

$$\psi_3 \in [\Gamma_S, 3, \nu_2\nu_\pi \det]_0,\ \psi_{18} \in [\Gamma_S, 18, \nu_\pi]_0,\ \psi_{20} \in [\Gamma_S, 20, 1]_0 \ \textit{and}\ \psi_{30} \in [\Gamma_S, 30, \nu_2]_0.$$

The zeros of the products are all of first order and are given by

$$\bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_{A_1^{(2)}}\rangle, \quad \bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_{S_2}\rangle, \quad \bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_{2A_2}\rangle \quad \textit{and} \quad \bigcup_{M\in\Gamma_S} M\langle\mathcal{H}_8\rangle,$$

respectively.

PROOF Theorem 4.12 yields the existence of holomorphic modular forms of the given weights with respect to $\mathrm{O_d}(\Lambda_1) \cap \Gamma_S$ and some Abelian character χ and with the given zeros. By virtue of Proposition 4.16 the ψ_k are in fact modular forms with respect to the full modular group Γ_S and thus $\chi \in \Gamma_S^{\mathrm{ab}}$. Moreover, the proposition allows us to explicitly calculate the value of χ for two elements of Γ_S. In view of Corollary 2.3 the character is uniquely determined by those values. The Borcherds products ψ_3, ψ_{18} and ψ_{30} are cusp forms since they are modular forms with respect to a non-trivial character. Moreover, ψ_{20}

obviously vanishes on $\mathcal{H} \times \{0\}^3 \times \mathcal{H} \subset \lambda_6^\perp$ which implies $\psi_{20}|\Phi = 0$. This completes the proof. ■

Remark 5.13 *Using Theorems 2.34 and 2.36 and comparing the divisors of the Borcherds products we can identify the restrictions of the Borcherds products to the submanifold $\mathcal{H}_{A_1^{(2)}}$. For example ψ_{18} vanishes on $\mathcal{H}_{S_2}$, and thus, in particular, on $H(2;\mathbb{R}) \cong \mathcal{H}_{A_1} \subset \mathcal{H}_{S_2}$. This implies that its restriction to $\mathcal{H}_{A_1^{(2)}}$ is divisible by $\phi_4^{A_1^{(2)}}$. Due to the character we then conclude that $\psi_{18}|\mathcal{H}_{A_1^{(2)}} = (\phi_4^{A_1^{(2)}})^2\phi_{10}^{A_1^{(2)}}$. For the other two Borcherds products we get*

$$\psi_{20}|\mathcal{H}_{A_1^{(2)}} \in \phi_{10}^{A_1^{(2)}} \cdot [\Gamma_{A_1^{(2)}}, 10, 1] \quad \textit{and} \quad \psi_{30}|\mathcal{H}_{A_1^{(2)}} = \phi_{30}^{A_1^{(2)}}.$$

The restriction of ψ_3 to $\mathcal{H}_{S_2}$ is equal to the Borcherds product ϕ_3 occurring in [DK04].

Let $f = X_1 \cdot \ldots \cdot X_{10}$ be the product of the ten theta series in [FH00, Def. 10.3]. According to [FH00, Prop. 11.9] this product is a non-trivial modular form of weight 20 with respect to $\Gamma_{A_1^{(3)}}$ vanishing on $\mathcal{H}_{2A_2}$. Hence f/ψ_{20} is a holomorphic modular form of weight 0, and thus

$$f = c\psi_{20}$$

for some $c \in \mathbb{C} \setminus \{0\}$.

Just as in the case $S = A_3$ the fact that the Borcherds products vanish on quadratic divisors of first order allows us to conclude that modular forms with respect to certain Abelian characters must be divisible by certain Borcherds products. The result is summarized in the following

Lemma 5.14 *Let $S = A_1^{(3)}$, $k \in \mathbb{Z}$, and $m \in \{0, 1\}$.*

a) If k is odd and $f \in [\Gamma'_S, k, 1]$ then f vanishes on $\mathcal{H}_{A_1^{(2)}}$ and $f/\psi_3 \in [\Gamma'_S, k-3, 1]$.

b) If $f \in [\Gamma_S, k, \nu_2^m\nu_\pi^{k+1}\det^k]$ then f vanishes on $\mathcal{H}_{S_2}$ and $f/\psi_{18} \in [\Gamma_S, k-18, \nu_2^m\nu_\pi^k\det^k]$.

c) If $f \in [\Gamma_S, k, \nu_2^{k+1}\nu_\pi^m\det^k]$ then f vanishes on $\mathcal{H}_8$ and $f/\psi_{30} \in [\Gamma_S, k-30, \nu_2^k\nu_\pi^m\det^k]$.

PROOF a) Let $k \in \mathbb{Z}$ be odd and $f \in [\Gamma'_S, k, 1]$. Then f vanishes on $\mathcal{H}_{A_1^{(2)}}$ according to Corollary 2.32. Therefore Theorem 5.12 yields $f/\psi_3 \in [\Gamma'_S, k-3, 1]$.

b) Let $f \in [\Gamma_S, k, \nu_2^m\nu_\pi^{k+1}\det^k]$. By virtue of Corollary 2.32 f vanishes on $\mathcal{H}_{S_2}$. Thus Theorem 5.12 yields $f/\psi_{18} \in [\Gamma_S, k-18, \nu_2^m\nu_\pi^k\det^k]$.

c) We have

$$\mathcal{H}_8 = \left\{ w \in \mathcal{H}_S;\ w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w + {}^t(0,0,0,1,0) = \left(T_{e_4} R_{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)}\right)\langle w \rangle \right\}.$$

Let $\chi = \nu_2^{k+1}\nu_\pi^m\det^k$ and $f \in [\Gamma_S, k, \chi]$. Then for all $w \in \mathcal{H}_8$ we have

$$f(w) = (f|_k M)(w) = \chi(M)f(w) = -f(w)$$

if $M = T_{e_4} R_{\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right)}$. Hence f vanishes on $\mathcal{H}_8$, and by virtue of Theorem 5.12 we conclude $f/\psi_{30} \in [\Gamma_S, k-30, \nu_2^k \nu_\pi^m \det^k]$. ∎

Due to the above lemma we can reduce any modular form of odd weight and any modular form with respect to a non-trivial Abelian character of Γ_S to a modular form of even weight with respect to the trivial character by dividing the modular form by suitable powers of ψ_3, ψ_{18} and ψ_{30}. This way we have reduced the problem of determining the graded ring

$$\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma'_S, k, 1]$$

of modular forms with respect to Γ'_S to the problem of determining the graded ring

$$\mathcal{A}(\Gamma_S) = \bigoplus_{k\in\mathbb{Z}} [\Gamma_S, k, 1] = \bigoplus_{k\in\mathbb{Z}} [\Gamma_S, 2k, 1],$$

of modular forms of even weight with respect to the full modular group Γ_S (and trivial character). Elements of this ring are given by the Eisenstein series $E_k = E_k^{A_1^{(3)}}$, $k \geq 4$, we defined in Section 2.5.2, by the invariants h_k, we determined in Section 2.8, and, of course, also by ψ_3^2, ψ_{20}, ψ_{18}^2 and ψ_{30}^2. The structure of this ring can be easily derived from the structure of the graded ring $\mathcal{A}(\Gamma_{A_1^{(2)}})$.

Theorem 5.15 *Let $S = A_1^{(3)}$. The graded ring $\mathcal{A}(\Gamma_S)$ is a polynomial ring in*

$$h_4,\ h_6,\ \psi_3^2,\ h_8,\ h_{10} \text{ and } h_{12}.$$

PROOF Let $k \in \mathbb{Z}$ be even, and let $f \in [\Gamma_S, k, 1]$. By virtue of Theorem 2.40, the restrictions of the h_j generate the graded ring $\mathcal{A}(\Gamma_{A_1^{(2)}})$. Thus there exists a polynomial p such that

$$\widetilde{f} := f - p(h_4, h_6, h_8, h_{10}, h_{12})$$

vanishes on $\mathcal{H}_{A_1^{(2)}}$. Since the Borcherds product ψ_3 vanishes on $\mathcal{H}_{A_1^{(2)}}$ of first order we can divide $\widetilde{f}$ by ψ_3 and get

$$\widetilde{f}/\psi_3 \in [\Gamma_S, k-3, \nu_2\nu_\pi \det].$$

Due to Lemma 5.14 the quotient $\widetilde{f}/\psi_3$ also vanishes on $\mathcal{H}_{A_1^{(2)}}$. Hence we can divide a second time by ψ_3 and get

$$\widetilde{f}/\psi_3^2 \in [\Gamma_S, k-6, 1].$$

By induction we conclude that the graded ring is generated by the given functions. The algebraic independence of the generators follows from the algebraic independence of the restrictions of the h_j to $\mathcal{H}_{A_1^{(2)}}$ and the fact that ψ_3^2 vanishes on $\mathcal{H}_{A_1^{(2)}}$. ∎

Remark 5.16 *In a forthcoming paper (cf. [FSM]) Freitag and Salvati Manni determine the structure of this ring using completely different methods.*

Of course, it is possible to express the Eisenstein series E_4, E_6, E_{10} and E_{12} as polynomials in the generators. The result is

$$\begin{aligned}
E_4 &= h_4,\\
E_6 &= h_6 - 3456\psi_3^2,\\
17E_{10} &= 15h_{10} + 2h_4h_6 - 18432h_4\psi_3^2,\\
21421E_{12} &= 22050h_{12} + 400h_6^2 - 2764800h_6\psi_3^2 - 1029h_4^3 + 4777574400\psi_3^4.
\end{aligned}$$

Corollary 5.17 *The graded ring $\mathcal{A}(\Gamma_{A_1^{(3)}})$ is a polynomial ring in*

$$E_4,\ E_6,\ \psi_3^2,\ h_8,\ E_{10} \textit{ and } E_{12}.$$

Now we can determine the structure of the full ring $\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_S, k, 1]$ of modular forms with respect to Γ_S for $S = A_1^{(3)}$.

Theorem 5.18 *Let $S = A_1^{(3)}$.*

a) *The graded ring $\mathcal{A}(\Gamma'_S) = \bigoplus_{k\in\mathbb{Z}}[\Gamma'_S, k, 1]$ is generated by the modular forms*

$$\psi_3,\ E_4,\ E_6,\ h_8,\ E_{10},\ E_{12},\ \psi_{18} \textit{ and } \psi_{30}$$

of which ψ_3, E_4, E_6, h_8, E_{10} and E_{12} are algebraically independent.

b) *There are uniquely determined polynomials $p, q \in \mathbb{C}[X_1, \dots, X_6]$ such that*

$$\begin{aligned}
\psi_{18}^2 &= p(\psi_3, E_4, E_6, h_8, E_{10}, E_{12}),\\
\psi_{30}^2 &= q(\psi_3, E_4, E_6, h_8, E_{10}, E_{12}).
\end{aligned}$$

c) *We have*

$$\mathcal{A}(\Gamma'_S) \cong \mathbb{C}[X_1, \dots, X_8]/\left(X_7^2 - p(X_1, \dots, X_6),\ X_8^2 - q(X_1, \dots, X_6)\right)$$

and

$$\sum_{k=0}^{\infty} \dim[\Gamma'_S, k]t^k = \frac{(1+t^{18})(t+t^{30})}{(1-t^3)(1-t^4)(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})}.$$

PROOF a) This follows analogously to the corresponding result for $S = A_3$ from Theorem 5.15 and Lemma 5.14.

b) Theorem 5.15 yields existence and uniqueness of the polynomials.

c) Let $Q \in \mathbb{C}[X_1, \dots, X_8]$ such that $Q(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}, \psi_{18}, \psi_{30}) = 0$. There exist polynomials $Q_0, Q_1, Q_2, Q_3 \in \mathbb{C}[X_1, \dots, X_6]$ such that $Q - Q_0 - X_7Q_1 - X_8Q_2 - X_7X_8Q_3 \in (X_7^2 - p(X_1, \dots, X_6),\ X_8^2 - q(X_1, \dots, X_6))$, hence

$$\begin{aligned}
&Q_0(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) + \psi_{18}\cdot Q_1(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})\\
&\quad + \psi_{30}\cdot Q_2(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) + \psi_{18}\psi_{30}\cdot Q_3(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) = 0.
\end{aligned}$$

Applying the modular substitution $w \mapsto M\langle w\rangle$, $M = R_{\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)}$, to this equation we get

$$\begin{aligned}&Q_0(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) - \psi_{18}\cdot Q_1(E_4, E_6, \psi_8, \psi_9, E_{10}, E_{12})\\ &\quad + \psi_{30}\cdot Q_2(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) - \psi_{18}\psi_{30}\cdot Q_3(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) = 0,\end{aligned}$$

and thus

$$\begin{aligned}Q_0(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) + \psi_{30}\cdot Q_2(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) = 0,\\ Q_1(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) + \psi_{30}\cdot Q_3(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) = 0.\end{aligned}$$

Applying the modular substitution $w \mapsto M\langle w\rangle$, $M = T_{e_4}R_{\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right)}$, to those equations yields

$$\begin{aligned}Q_0(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) - \psi_{30}\cdot Q_2(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) = 0,\\ Q_1(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) - \psi_{30}\cdot Q_3(\psi_3, E_4, E_6, F_8, E_{10}, E_{12}) = 0.\end{aligned}$$

Now the algebraic independence of ψ_3, E_4, E_6, F_8, E_{10} and E_{12} implies that Q_0, Q_1, Q_2 and Q_3 vanish identically. Thus $Q \in (X_7^2 - p(X_1,\dots,X_6),\ X_8^2 - q(X_1,\dots,X_6))$. The dimension formula is a direct consequence of the algebraic structure of $\mathcal{A}(\Gamma'_S)$. ∎

As in the case $S = A_3$ we can apply the Rankin-Cohen type differential operator we introduced in Section 2.2 to the algebraically independent primary generators of $\mathcal{A}(\Gamma'_{A_1^{(3)}})$. The result is

Corollary 5.19 *There exists a constant $c \in \mathbb{C}$, $c \neq 0$, such that*

$$\{\psi_3, E_4, E_6, h_8, E_{10}, E_{12}\} = c\psi_{18}\psi_{30}.$$

PROOF Since ψ_3, E_4, E_6, h_8, E_{10} and E_{12} are algebraically independent, we have $0 \neq g := \{\psi_3, E_4, E_6, h_8, E_{10}, E_{12}\} \in [\Gamma_{A_1^{(3)}}, 48, \nu_2\nu_\pi]$ according to Proposition 2.14. Due to the character Lemma 5.14 yields $g/(\psi_{18}\psi_{30}) \in [\Gamma_{A_1^{(3)}}, 0, 1] = \mathbb{C}$. ∎

Just as in the case $S = A_3$ we can replace some of the generators by cusp forms. We replace the Eisenstein series E_{10} and E_{12} by the cusp forms

$$f_{10} := E_{10} - E_4\cdot E_6 \quad \text{and} \quad f_{12} := E_{12} - \frac{441}{691}E_4^3 - \frac{250}{691}E_6^2,$$

and we replace h_8 by the cusp form

$$f_8 := h_8 - E_4^2.$$

Since the constant term of f_8 vanishes and due to

$$f_8|\Phi \in [\mathrm{SL}(2;\mathbb{Z}), 8] = \mathbb{C}\cdot G_8,$$

where G_8 denotes the normalized elliptic Eisenstein series of weight 8, we conclude

$$f_8|\Phi = 0,$$

and thus f_8 is indeed a cusp form.

Analogously to the corresponding result for $S = A_3$ we can now determine the generators of the ideal of cusp forms in $\mathcal{A}(\Gamma'_{A_1^{(3)}})$.

Corollary 5.20 *The ideal of cusp forms in $\mathcal{A}(\Gamma'_S)$ is generated by*

$$\psi_3,\ f_8,\ f_{10},\ f_{12},\ \psi_{18}\ \textit{and}\ \psi_{30}.$$

Finally, we can again determine the algebraic structure of the field of orthogonal modular functions.

Theorem 5.21 *Let $S = A_1^{(3)}$.*

a) The field $\mathcal{K}(\Gamma_S)$ of orthogonal modular functions with respect to Γ_S and the trivial character is a rational function field in the generators

$$\frac{\psi_3^2}{E_6},\quad \frac{h_8}{E_4^2},\quad \frac{E_{10}}{E_4E_6},\quad \frac{E_{12}}{E_4^3}\quad \textit{and}\quad \frac{E_6^2}{E_4^3}.$$

b) The field $\mathcal{K}(\Gamma'_S)$ of all orthogonal modular functions with respect to Γ'_S is an extension of degree 4 over $\mathcal{K}(\Gamma_S)$ generated by ψ_{18}/E_6^3 and ψ_{30}/E_6^5.

PROOF a) Let $f \in \mathcal{K}(\Gamma_S)$. Due to Baily-Borel ([BB66, Cor. 10.12]) there exist $g, h \in [\Gamma'_S, k]$ such that $f = g/h$. Just as in the case $S = A_3$ we can assume that g and h are modular forms of even weight with respect to the trivial character. Thus f is a quotient of polynomials in E_4, E_6, ψ_3^2, h_8, E_{10} and E_{12}. Again it remains to be shown that all monomials $E_4^{k_4}E_6^{k_6}\psi_3^{2k_3}h_8^{k_8}E_{10}^{k_{10}}E_{12}^{k_{12}}$ with $k_j \in \mathbb{Z}$ and $3k_3 + \sum_j j\cdot k_j = 0$ can be written in the above generators. This follows from

$$E_4^{k_4}E_6^{k_6}\psi_3^{2k_3}h_8^{k_8}E_{10}^{k_{10}}E_{12}^{k_{12}} = \left(\frac{E_6^2}{E_4^3}\right)^{-k_4-k_6-k_3-2k_8-2k_{10}-3k_{12}} \left(\frac{\psi_3^2}{E_6}\right)^{k_3} \left(\frac{h_8}{E_4^2}\right)^{k_8} \left(\frac{E_{10}}{E_4E_6}\right)^{k_{10}} \left(\frac{E_{12}}{E_4^3}\right)^{k_{12}}.$$

Hence $\mathcal{K}(\Gamma_S)$ is a function field in the above generators which are algebraically independent according to Theorem 5.15.

b) We have $g := \psi_{18}/E_6^3 \in [\Gamma_S, 0, \nu_\pi]_{\mathrm{mer}}$ and $h := \psi_{30}/E_6^5 \in [\Gamma_S, 0, \nu_2]_{\mathrm{mer}}$. Just as in the case of holomorphic modular forms the vector space of modular functions splits into the eigenspaces of the characters of Γ_S. Since some eigenspaces vanish we have

$$\begin{aligned}\mathcal{K}(\Gamma'_S) &= [\Gamma_S, 0, 1]_{\mathrm{mer}} \oplus [\Gamma_S, 0, \nu_\pi]_{\mathrm{mer}} \oplus [\Gamma_S, 0, \nu_2]_{\mathrm{mer}} \oplus [\Gamma_S, 0, \nu_2\nu_\pi]_{\mathrm{mer}}\\ &= \mathcal{K}(\Gamma_S) \oplus g\cdot\mathcal{K}(\Gamma_S) \oplus h\cdot\mathcal{K}(\Gamma_S) \oplus gh\cdot\mathcal{K}(\Gamma_S) = \mathcal{K}(\Gamma_S)[g,h].\end{aligned}$$

Due to Theorem 5.18 we have $g^2, h^2 \in \mathcal{K}(\Gamma_S)$, and thus $\mathcal{K}(\Gamma'_S)$ is an extension of degree 4 over $\mathcal{K}(\Gamma_S)$. ■

A. Orthogonal and Symplectic Transformations

We use the notation introduced in Section 2.7. Moreover, we denote the most common elements of the symplectic group $\mathrm{Sp}(2;\mathbb{H})$ by

$$\begin{aligned}\mathrm{Trans}(H) &= \begin{pmatrix} I_2 & H \\ 0 & I_2 \end{pmatrix} \quad \text{for } H \in \mathrm{Her}(2;\mathbb{H}), \\ \mathrm{Rot}(U) &= \begin{pmatrix} {}^t\overline{U} & 0 \\ 0 & U^{-1} \end{pmatrix} \quad \text{for } U \in \mathrm{GL}(2;\mathbb{H}).\end{aligned}$$

According to [Kr85], $\mathrm{Sp}(2;\mathcal{O})$ is generated by $J_{\mathbb{H}}$, $\mathrm{Trans}(H)$, $H \in \mathrm{Her}(2;\mathcal{O})$, and $\mathrm{Rot}(U)$, $U \in \mathrm{GL}(2;\mathcal{O})$ where $U = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\varepsilon \in \mathcal{O}^\times = \langle \omega \mathrm{i}_2, \omega \mathrm{i}_3 \rangle$. Thus the extended modular group

$$\Gamma_{\mathbb{H}} = \langle \{ Z \mapsto M\langle Z \rangle;\ M \in \mathrm{Sp}(2;\mathcal{O}) \text{ or } M = \rho I \},\ I_{\mathrm{tr}} \rangle,\ \rho = \frac{1+\mathrm{i}_1}{\sqrt{2}},$$

is generated by the following biholomorphic transformations of $H(2;\mathbb{H})$:

$$\begin{aligned} Z &\mapsto J_{\mathbb{H}}\langle Z \rangle = -Z^{-1}, \\ Z &\mapsto \mathrm{Trans}(H)\langle Z \rangle = Z + H,\ H \in \mathrm{Her}(2;\mathcal{O}), \\ Z &\mapsto \mathrm{Rot}\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)\langle Z \rangle = \begin{pmatrix} \tau_1 & \overline{\varepsilon}(x+iy) \\ (\overline{x}+i\overline{y})\varepsilon & \tau_2 \end{pmatrix},\ \varepsilon \in \{\omega \mathrm{i}_2, \omega \mathrm{i}_3\}, \\ Z &\mapsto (\rho I)\langle Z \rangle = \mathrm{Rot}\left(\begin{smallmatrix} \overline{\rho} & 0 \\ 0 & \rho \end{smallmatrix}\right)\langle Z \rangle = \begin{pmatrix} \tau_1 & \rho(x+iy)\overline{\rho} \\ \rho(\overline{x}+i\overline{y})\overline{\rho} & \tau_2 \end{pmatrix}, \\ Z &\mapsto I_{\mathrm{tr}}(Z) = {}^tZ, \end{aligned}$$

where $Z = \begin{pmatrix} \tau_1 & x+iy \\ \overline{x}+i\overline{y} & \tau_2 \end{pmatrix}$, $\tau_1, \tau_2 \in \mathcal{H}$, $x, y \in \mathbb{H}$.

A.1. The case $S = D_4$

The orthogonal half-space $\mathcal{H}_{D_4}$ is biholomorphically mapped to $H(2;\mathbb{H})$ by

$$\varphi_{\mathbb{H}} : \mathcal{H}_{D_4} \to H(2;\mathbb{H}),\ (x_1,u,x_2)+i(y_1,v,y_2) \mapsto \begin{pmatrix} x_1+iy_1 & \iota_{D_4}(u)+i\ \iota_{D_4}(v) \\ \overline{\iota_{D_4}(u)+i\ \iota_{D_4}(v)} & x_2+iy_2 \end{pmatrix}$$

where $\iota_{D_4} : \mathbb{R}^4 \to \mathbb{H}$ is given by

$$(x_1,x_2,x_3,x_4) \mapsto x_1 + x_2\mathrm{i}_1 + x_3\mathrm{i}_2 + x_4\omega.$$

This map allows us to identify the corresponding elements of $\Gamma_{\mathbb{H}}$ and Γ_{D_4} (or more precisely of $\Gamma_{D_4}/\{\pm I_8\}$) considered as subgroup of $\mathrm{Bihol}(\mathcal{H}_{D_4})$. The following table lists the generators of $\Gamma_{D_4}/\{\pm I\}$ and the elements of $\Gamma_{\mathbb{H}}$ those generators correspond to, and vice versa.

$M \in \Gamma_{D_4}$	$\gamma \in \Gamma_{\mathbb{H}}$
J	$J_{\mathbb{H}}$
$T_g, g = (g_1,\widetilde{g},g_2) \in \Lambda_0$	$\mathrm{Trans}(H), H = \begin{pmatrix} g_1 & \iota_{D_4}(\widetilde{g}) \\ * & g_2 \end{pmatrix} \in \mathrm{Her}(2;\mathcal{O})$
$M_{\mathrm{tr}} = R_{\left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right)}$	I_{tr}
$R_{\left(\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \overline{\rho} & 0 \\ 0 & \overline{\rho} \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \overline{\omega} & 0 \\ 0 & \overline{\omega} \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \overline{\omega - \mathrm{i}_1} & 0 \\ 0 & \omega - \mathrm{i}_3 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ -1 & 1 & 1 & 0 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \omega\mathrm{i}_2 & 0 \\ 0 & 1 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -2 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \omega\mathrm{i}_3 & 0 \\ 0 & 1 \end{pmatrix}$

A.2. The case $S = A_1^{(3)}$

The orthogonal half-space $\mathcal{H}_{A_1^{(3)}}$ is biholomorphically mapped to the submanifold

$$H(2;\mathbb{H}_{A_1^{(3)}}) = \left\{ \begin{pmatrix} \tau_1 & z \\ * & \tau_2 \end{pmatrix} \in H(2;\mathbb{H});\ z = z_1 + z_2\mathrm{i}_1 + z_3\mathrm{i}_2 + z_4\mathrm{i}_3,\ z_4 = 0 \right\}$$

of $H(2;\mathbb{H})$ by

$$\varphi_{A_1^{(3)}} : \mathcal{H}_{A_1^{(3)}} \to H(2;\mathbb{H}_{A_1^{(3)}}),\ (x_1, u, x_2) + i(y_1, v, y_2) \mapsto \begin{pmatrix} x_1 + iy_1 & \iota_{A_1^{(3)}}(u) + i\ \iota_{A_1^{(3)}}(v) \\ \overline{\iota_{A_1^{(3)}}(u) + i\ \iota_{A_1^{(3)}}(v)} & x_2 + iy_2 \end{pmatrix}$$

where $\iota_{A_1^{(3)}} : \mathbb{R}^3 \to \mathbb{H}_{A_1^{(3)}}$ is given by

$$(x_1, x_2, x_3) \mapsto x_1 + x_2\mathrm{i}_1 + x_3\mathrm{i}_2.$$

The following table lists elements of the orthogonal modular group $\Gamma_{A_1^{(3)}}$ and corresponding elements of $\Gamma_{\mathbb{H}} \cap \mathrm{Bihol}(H(2;\mathbb{H}_{A_1^{(3)}}))$, i.e., if $M \in \Gamma_{A_1^{(3)}}$ then the corresponding element $\gamma \in \Gamma_{\mathbb{H}}$ satisfies

$$\gamma(\varphi_{A_1^{(3)}}(w)) = M\langle w\rangle \qquad \text{for all } w \in \mathcal{H}_{A_1^{(3)}}.$$

$M \in \Gamma_{A_1^{(3)}}$	$\gamma \in \Gamma_{\mathbb{H}} \cap \mathrm{Bihol}(H(2;\mathbb{H}_{A_1^{(3)}}))$
J	$J_{\mathbb{H}}$
$T_g, g = (g_1, \widetilde{g}, g_2) \in \Lambda_0$	$\mathrm{Trans}(H), H = \begin{pmatrix} g_1 & \iota_{A_1^{(3)}}(\widetilde{g}) \\ * & g_2 \end{pmatrix} \in \mathrm{Her}(2;\mathcal{O}_{A_1^{(3)}})$
$M_{\mathrm{tr}} = R_{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)}$	$I_{\mathrm{tr}} = \mathrm{Rot}\begin{pmatrix} \mathrm{i}_3 & 0 \\ 0 & \mathrm{i}_3 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} (\mathrm{i}_2 - \mathrm{i}_1)/\sqrt{2} & 0 \\ 0 & (\mathrm{i}_1 - \mathrm{i}_2)/\sqrt{2} \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \overline{\omega} + \mathrm{i}_2 & 0 \\ 0 & \overline{\omega} + \mathrm{i}_1 \end{pmatrix}$

A.3. The case $S = A_3$

The orthogonal half-space $\mathcal{H}_{A_3}$ is biholomorphically mapped to the submanifold

$$H(2;\mathbb{H}_{A_3}) = \left\{ \begin{pmatrix} \tau_1 & z \\ * & \tau_2 \end{pmatrix} \in H(2;\mathbb{H});\ z = z_1 + z_2 \mathrm{i}_1 + z_3 \mathrm{i}_2 + z_4 \mathrm{i}_3,\ z_3 = z_4 \right\}$$

of $H(2;\mathbb{H})$ by

$$\varphi_{A_3} : \mathcal{H}_{A_3} \to H(2;\mathbb{H}_{A_3}),\ (x_1, u, x_2) + i(y_1, v, y_2) \mapsto \begin{pmatrix} x_1 + iy_1 & \iota_{A_3}(u) + i\ \iota_{A_3}(v) \\ \overline{\iota_{A_3}(u) + i\ \iota_{A_3}(v)} & x_2 + iy_2 \end{pmatrix}$$

where $\iota_{A_3} : \mathbb{R}^3 \to \mathbb{H}_{A_3}$ is given by

$$(x_1, x_2, x_3) \mapsto x_1 + x_2\omega + x_3\mathrm{i}_1.$$

The following table lists elements of the orthogonal modular group Γ_{A_3} and corresponding elements of $\Gamma_{\mathbb{H}} \cap \mathrm{Bihol}(H(2;\mathbb{H}_{A_3}))$, i.e., if $M \in \Gamma_{A_3}$ then the corresponding element $\gamma \in \Gamma_{\mathbb{H}}$ satisfies

$$\gamma(\varphi_{A_3}(w)) = M\langle w\rangle \qquad \text{for all } w \in \mathcal{H}_{A_3}.$$

$M \in \Gamma_{A_3}$	$\gamma \in \Gamma_{\mathbb{H}} \cap \mathrm{Bihol}(H(2;\mathbb{H}_{A_3}))$
J	$J_{\mathbb{H}}$
$T_g, g = (g_1, \widetilde{g}, g_2) \in \Lambda_0$	$\mathrm{Trans}(H), H = \begin{pmatrix} g_1 & \iota_{A_3}(\widetilde{g}) \\ * & g_2 \end{pmatrix} \in \mathrm{Her}(2;\mathcal{O}_{A_3})$
$M_{\mathrm{tr}} := R_{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)}$	$I_{\mathrm{tr}} = \mathrm{Rot}\begin{pmatrix} (\mathrm{i}_2 - \mathrm{i}_3)/\sqrt{2} & 0 \\ 0 & (\mathrm{i}_2 - \mathrm{i}_3)/\sqrt{2} \end{pmatrix}$
$R_{\left(\begin{smallmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} -\mathrm{i}_1 & 0 \\ 0 & \mathrm{i}_1 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)}$	$\mathrm{Rot}\begin{pmatrix} \omega - \mathrm{i}_1 & 0 \\ 0 & \overline{\omega} + \mathrm{i}_1 \end{pmatrix}$

B. Orthogonal and Unitary Transformations

Let $\mathbb{K}$ be an imaginary quadratic number field. We use the following abbreviations for the most common elements of the unitary group $\mathrm{U}(2;\mathbb{K})$

$$\mathrm{Trans}(H) = \begin{pmatrix} I_2 & H \\ 0 & I_2 \end{pmatrix} \quad \text{for } H \in \mathrm{Her}(2;\mathbb{K}),$$
$$\mathrm{Rot}(U) = \begin{pmatrix} {}^t\overline{U} & 0 \\ 0 & U^{-1} \end{pmatrix} \quad \text{for } U \in \mathrm{GL}(2;\mathbb{K}).$$

According to [De01, Lem. 1.4] $\mathrm{SU}(2;\mathfrak{o}_{\mathbb{K}})$ is generated by J_{Her} and $\mathrm{Trans}(H)$, $H \in \mathrm{Her}(2;\mathfrak{o}_{\mathbb{K}})$, and $\Gamma(2;\mathbb{K}) = \mathrm{U}(2;\mathfrak{o}_{\mathbb{K}})$ is generated by J_{Her}, $\mathrm{Trans}(H)$, $H \in \mathrm{Her}(2;\mathfrak{o}_{\mathbb{K}})$ and $\mathrm{Rot}(U)$, $U = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\varepsilon \in \mathfrak{o}_{\mathbb{K}}^{\times}$. Thus

$$\Gamma_{\mathbb{K}} = \left\langle \{Z \mapsto M\langle Z\rangle;\ M \in \widetilde{\Gamma(2;\mathbb{K})}\},\ I_{\mathrm{tr}} \right\rangle$$

is generated by the following biholomorphic transformations of $H(2;\mathbb{C})$:

$$\begin{gathered} Z \mapsto J_{\mathrm{Her}}\langle Z\rangle = -Z^{-1}, \\ Z \mapsto \mathrm{Trans}(H)\langle Z\rangle = Z + H,\ H \in \mathrm{Her}(2;\mathfrak{o}_{\mathbb{K}}), \\ Z \mapsto \mathrm{Rot}\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)\langle Z\rangle = \left(\begin{smallmatrix} \tau_1 & \overline{\varepsilon} z_1 \\ \varepsilon z_2 & \tau_2 \end{smallmatrix}\right),\ \varepsilon \in \mathfrak{o}_{\mathbb{K}}^{\times} \cap \{\pm 1, \pm i\}, \\ Z \mapsto I_{\mathrm{tr}}(Z) = {}^tZ, \end{gathered}$$

where $Z = \begin{pmatrix} \tau_1 & z_1 \\ z_2 & \tau_2 \end{pmatrix} \in H(2;\mathbb{C})$.

B.1. The case $S = A_1^{(2)}$

Let $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. Then $\mathfrak{o}_\mathbb{K} = \mathbb{Z} + \mathbb{Z}i$, $S = S^\mathbb{K} = A_1^{(2)}$ and

$$w = (\tau_1, w_1, w_2, \tau_2) = (x_1, u_1, u_2, x_2) + i(y_1, v_1, v_2, y_2) \in \mathcal{H}_{A_1^{(2)}}$$

corresponds to

$$\begin{aligned} Z &= \begin{pmatrix} \tau_1 & (u_1 - v_2) + i(u_2 + v_1) \\ (u_1 + v_2) - i(u_2 - v_1) & \tau_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & u_1 + iu_2 \\ u_1 - iu_2 & x_2 \end{pmatrix} + i \begin{pmatrix} y_1 & v_1 + iv_2 \\ v_1 - iv_2 & y_2 \end{pmatrix} \in H(2; \mathbb{C}). \end{aligned}$$

The generators of $\Gamma_{A_1^{(2)}}$ and $\Gamma_\mathbb{K}$

$M \in \Gamma_{A_1^{(2)}}$	$M\langle w\rangle$	$\gamma(Z)$	$\gamma \in \Gamma_{\mathbb{Q}(\sqrt{-1})}$
J	$J\langle w\rangle$	$-Z^{-1}$	J_{Her}
$T_{(g_0,\dots,g_3)}$	$w + (g_0, \dots, g_3)$	$Z + H$	$\text{Trans}(H), H = \begin{pmatrix} g_0 & g_1 + ig_2 \\ * & g_3 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$	$(\tau_1, -w_2, w_1, \tau_2)$	$\begin{pmatrix} \tau_1 & iz_1 \\ -iz_2 & \tau_2 \end{pmatrix}$	$\text{Rot}\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$
$M_{\text{tr}} = R_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$	$(\tau_1, w_1, -w_2, \tau_2)$	${}^tZ$	I_{tr}

The Abelian characters of $\Gamma_\mathbb{K}$

We have $\mathrm{U}(2; \mathfrak{o}_\mathbb{K})^{\text{ab}} = \langle \det, \nu_\wp \rangle$. We can extend $\det$ and $\nu_\wp$ to $\Gamma_\mathbb{K}$ by defining $\det(I_{\text{tr}}) := \nu_\wp(I_{\text{tr}}) := 1$. Moreover, we define $\nu_{\text{skew}} : \Gamma_\mathbb{K} \to \mathbb{C}$ by $\nu_{\text{skew}}(I_{\text{tr}}) := -1$ and $\nu_{\text{skew}}(M) := 1$ for $M \in \mathrm{U}(2; \mathfrak{o}_\mathbb{K})$. Considering that $M\langle I_{\text{tr}}(Z)\rangle = I_{\text{tr}}(\overline{M}\langle Z\rangle)$ for all $M \in \mathrm{U}(2; \mathbb{C})$ we can easily verify that $\Gamma_\mathbb{K}^{\text{ab}} = \langle \det, \nu_\wp, \nu_{\text{skew}} \rangle$.

$\gamma \in \Gamma_{\mathbb{Q}(\sqrt{-1})}$	$\det(\gamma)$	$\nu_\wp(\gamma)$	$\nu_{\text{skew}}(\gamma)$
J_{Her}	1	1	1
$\text{Trans}\begin{pmatrix} g_0 & g_1 + ig_2 \\ * & g_3 \end{pmatrix}$	1	$(-1)^{\sum g_j}$	1
$\text{Rot}\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$	-1	1	1
I_{tr}	1	1	-1

B.2. The case $S = A_2$

Let $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. Then $\mathfrak{o}_\mathbb{K} = \mathbb{Z} + \mathbb{Z}\omega$, $\omega = \frac{1}{2}(1 + i\sqrt{3})$, $S = S^\mathbb{K} = A_2$ and

$$w = (\tau_1, w_1, w_2, \tau_2) = (x_1, u_1, u_2, x_2) + i(y_1, v_1, v_2, y_2) \in \mathcal{H}_{A_2}$$

corresponds to

$$Z = \begin{pmatrix} x_1 & u_1 + \omega u_2 \\ u_1 + \overline{\omega} u_2 & x_2 \end{pmatrix} + i \begin{pmatrix} y_1 & v_1 + \omega v_2 \\ v_1 + \overline{\omega} v_2 & y_2 \end{pmatrix} \in H(2;\mathbb{C}).$$

The generators of Γ_{A_2} and $\Gamma_\mathbb{K}$

$M \in \Gamma_{A_2}$	$M\langle w\rangle$	$\gamma(Z)$	$\gamma \in \Gamma_{\mathbb{Q}(\sqrt{-3})}$
J	$J\langle w\rangle$	$-Z^{-1}$	J_{Her}
$T_{(g_0,\ldots,g_3)}$	$w + (g_0, \ldots, g_3)$	$Z + H$	$\mathrm{Trans}(H), H = \begin{pmatrix} g_0 & g_1 + \omega g_2 \\ * & g_3 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)}$	$(\tau_1, w_1 + w_2, -w_1, \tau_2)$	$\begin{pmatrix} \tau_1 & \overline{\omega} z_1 \\ \omega z_2 & \tau_2 \end{pmatrix}$	$\mathrm{Rot}\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$
$M_{\mathrm{tr}} = R_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)}$	$(\tau_1, w_1 + w_2, -w_2, \tau_2)$	${}^tZ$	I_{tr}

The Abelian characters of $\Gamma_\mathbb{K}$

We have $\mathrm{U}(2;\mathfrak{o}_\mathbb{K})^{\mathrm{ab}} = \langle\det\rangle \cong C_3$. Because of $M\langle I_{\mathrm{tr}}(Z)\rangle = I_{\mathrm{tr}}(\overline{M}\langle Z\rangle)$ for all $M \in \mathrm{U}(2;\mathbb{C})$ we have

$$[I_{\mathrm{tr}}, \mathrm{Rot}\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}] = I_{\mathrm{tr}} \circ \mathrm{Rot}\begin{pmatrix} \overline{\omega} & 0 \\ 0 & 1 \end{pmatrix} \circ I_{\mathrm{tr}} \circ \mathrm{Rot}\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Rot}\begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma'_\mathbb{K}.$$

Since we also have $\mathrm{Rot}\begin{pmatrix} \omega^3 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Rot}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma'_\mathbb{K}$ we get $(Z \mapsto M\langle Z\rangle) \in \Gamma'_\mathbb{K}$ for all $M \in \mathrm{U}(2;\mathfrak{o}_\mathbb{K})$, and thus $[\Gamma_\mathbb{K} : \Gamma'_\mathbb{K}] \leq 2$. We define $\nu_{\mathrm{skew}} : \Gamma_\mathbb{K} \to \mathbb{C}$ by $\nu_{\mathrm{skew}}(I_{\mathrm{tr}}) := -1$ and $\nu_{\mathrm{skew}}(M) := 1$ for $M \in \mathrm{U}(2;\mathfrak{o}_\mathbb{K})$. Due to $M\langle I_{\mathrm{tr}}(Z)\rangle = I_{\mathrm{tr}}(\overline{M}\langle Z\rangle)$ for all $M \in \mathrm{U}(2;\mathbb{C})$ we get $\Gamma_\mathbb{K}^{\mathrm{ab}} = \langle\nu_{\mathrm{skew}}\rangle$.

$\gamma \in \Gamma_{\mathbb{Q}(\sqrt{-3})}$	$\nu_{\mathrm{skew}}(\gamma)$
J_{Her}	1
$\mathrm{Trans}\begin{pmatrix} g_0 & g_1 + \omega g_2 \\ * & g_3 \end{pmatrix}$	1
$\mathrm{Rot}\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	1
I_{tr}	-1

B.3. The case $S = S_2$

Let $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. Then $\mathfrak{o}_\mathbb{K} = \mathbb{Z} + \mathbb{Z}i\sqrt{2}$, $S = S^\mathbb{K} = S_2$ and

$$w = (\tau_1, w_1, w_2, \tau_2) = (x_1, u_1, u_2, x_2) + i(y_1, v_1, v_2, y_2) \in \mathcal{H}_{S_2}$$

corresponds to

$$\begin{aligned} Z &= \begin{pmatrix} \tau_1 & (u_1 - \sqrt{2}v_2) + i(v_1 + \sqrt{2}u_2) \\ (u_1 + \sqrt{2}v_2) + i(v_1 - \sqrt{2}u_2) & \tau_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & u_1 + i\sqrt{2}u_2 \\ u_1 - i\sqrt{2}u_2 & x_2 \end{pmatrix} + i \begin{pmatrix} y_1 & v_1 + i\sqrt{2}v_2 \\ v_1 - i\sqrt{2}v_2 & y_2 \end{pmatrix} \in H(2;\mathbb{C}). \end{aligned}$$

The generators of Γ_{S_2} and $\Gamma_\mathbb{K}$

$M \in \Gamma_{S_2}$	$M\langle w\rangle$	$\gamma(Z)$	$\gamma \in \Gamma_{\mathbb{Q}(\sqrt{-2})}$
J	$J\langle w\rangle$	$-Z^{-1}$	J_{Her}
$T_{(g_0,\ldots,g_3)}$	$w + (g_0, \ldots, g_3)$	$Z + H$	$\mathrm{Trans}(H), H = \begin{pmatrix} g_0 & g_1 + ig_2 \\ * & g_3 \end{pmatrix}$
$R_{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$	$(\tau_1, -w_1, -w_2, \tau_2)$	$\begin{pmatrix} \tau_1 & -z_1 \\ -z_2 & \tau_2 \end{pmatrix}$	$\mathrm{Rot}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$M_{\mathrm{tr}} = R_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$	$(\tau_1, w_1, -w_2, \tau_2)$	${}^tZ$	I_{tr}

The characters of Γ_S and $\Gamma_\mathbb{K}$

We have $\mathrm{U}(2;\mathfrak{o}_\mathbb{K})^{\mathrm{ab}} = \langle \nu_\wp \rangle$. We can extend $\nu_\wp$ to $\Gamma_\mathbb{K}$ by defining $\nu_\wp(I_{\mathrm{tr}}) := 1$. Moreover, we define $\nu_{\mathrm{skew}} : \Gamma_\mathbb{K} \to \mathbb{C}$ by $\nu_{\mathrm{skew}}(I_{\mathrm{tr}}) := -1$ and $\nu_{\mathrm{skew}}(M) := 1$ for $M \in \mathrm{U}(2;\mathfrak{o}_\mathbb{K})$. Considering that $M\langle I_{\mathrm{tr}}(Z)\rangle = I_{\mathrm{tr}}(\overline{M}\langle Z\rangle)$ for all $M \in \mathrm{U}(2;\mathbb{C})$ we can easily verify that $\Gamma_\mathbb{K}^{\mathrm{ab}} = \langle \nu_\wp, \nu_{\mathrm{skew}} \rangle$.

$\gamma \in \Gamma_{\mathbb{Q}(\sqrt{-2})}$	$\nu_\wp(\gamma)$	$\nu_{\mathrm{skew}}(\gamma)$
J_{Her}	1	1
$\mathrm{Trans}\begin{pmatrix} g_0 & g_1 + i\sqrt{2}g_2 \\ * & g_3 \end{pmatrix}$	$(-1)^{g_0+g_1+g_3}$	1
$\mathrm{Rot}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1
I_{tr}	1	-1

C. Eichler Transformations

We want to show that a group which is nicely generated in the sense of Definition 1.18 is also nicely generated in the sense of Freitag/Hermann [FH00, Def. 4.7]. We use the notation we introduced in Chapter 1. Freitag/Hermann call a subgroup Γ of $\mathrm{O}(S_1;\mathbb{R})$ nicely generated if it is generated by the group $\mathrm{EO}(\Lambda)$ of Eichler transformations and by the group $\mathrm{O}(\Lambda)$ considered as subgroup of $\mathrm{O}(S_1;\mathbb{R})$ via the embedding $A \mapsto R_A$. The group $\mathrm{EO}(\Lambda)$ is generated by all Eichler transformations of the form $E(f_j, v)$, $1 \le j \le 4$, where the pairs (f_1, f_2) and (f_3, f_4) span the two hyperbolic planes which are contained in Λ_1 and where $v \in \Lambda_1$ is orthogonal to f_j. In our terminology we have

$$f_1 = e_2,\ f_2 = e_{l+3},\ f_3 = e_1,\ f_4 = e_{l+4},$$

where $(e_j)_{1\le j\le l+4}$ is the standard basis of $V_1 = \mathbb{R}^{l+4}$. With this choice we obviously have

$$\Lambda_1 = H_1 \oplus H_2 \oplus \Lambda,$$

where $H_1 = \mathbb{Z}f_1 + \mathbb{Z}f_2$ and $H_2 = \mathbb{Z}f_3 + \mathbb{Z}f_4$ are two integral hyperbolic planes, that is

$$q_1(x_1f_1 + x_2f_2) = x_1x_2 \quad \text{and} \quad q_1(x_3f_3 + x_4f_4).$$

The Eichler transformations $E(f_j, v)$ are then defined for all $v \in \Lambda_1$ which are orthogonal to f_j by

$$E(f_j, v)(a) = a - (a, f_j)_1 v + (a, v)_1 f_j - q_1(v)(a, f_j)_1 f_j \quad \text{for all } a \in V_1.$$

In order to see how they act on $\mathcal{H}_S$ we have to apply them to $a = [(-q_0(w), w, 1)] \in \mathcal{K}^+$ (cf. Section 4.2). Then for $w = (\tau_1, z, \tau_2)$ and $h = (0, 0, \lambda, 0, 0)$, $\lambda \in \Lambda$, we get

$$\begin{aligned}
E(f_1, f_3)(w) &= (\tau_1 + 1, z, \tau_2) && = T_{e_1}\langle w\rangle,\\
E(f_1, f_4)(w) &= (-q_0(w) + \tau_1, z, \tau_2)\,(-\tau_2 + 1)^{-1} && = (JT_{e_{l+2}}J)\langle w\rangle,\\
E(f_2, f_3)(w) &= (\tau_1, z, \tau_2 + 1) && = T_{e_{l+2}}\langle w\rangle,\\
E(f_2, f_4)(w) &= (\tau_1, z, -q_0(w) + \tau_2)\,(-\tau_1 + 1)^{-1} && = (JT_{e_1}J)\langle w\rangle,\\
E(f_1, h)(w) &= \big(\tau_1 - {}^t\lambda S z + q(\lambda)\tau_2, z - \lambda\tau_2, \tau_2\big) && = U_{-\lambda}\langle w\rangle,\\
E(f_2, h)(w) &= \big(\tau_1, z - \lambda\tau_1, \tau_2 - {}^t\lambda S z + q(\lambda)\tau_1\big) && = (JU_\lambda J)\langle w\rangle,\\
E(f_3, h)(w) &= (\tau_1, z - \lambda, \tau_2) && = T_{(0,-\lambda,0)}\langle w\rangle,\\
E(f_4, h)(w) &= (\tau_1, z + q_0(w)\lambda, \tau_2)\,\big(-q(\lambda)q_0(w) - {}^t\lambda S z + 1\big)^{-1} && = (JT_{(0,\lambda,0)}J)\langle w\rangle.
\end{aligned}$$

Since

$$E(f_j, f_i) = E(f_i, f_j)^{-1}$$

for $i = 1, 2$ and $j = 3, 4$, and

$$E(f_j, v_1 + v_2) = E(f_j, v_1) \circ E(f_j, v_2)$$

for $1 \leq j \leq 4$ and all $v_1, v_2 \in \Lambda_1 \cap f_j^{\perp}$ we see that the above eight Eichler transformations generate the group $\mathrm{EO}(\Lambda)$. We conclude that a subgroup Γ of $\mathrm{O}(S_1; \mathbb{R})$ which is nicely generated in the sense of Freitag/Hermann is also nicely generated in the sense of Definition 1.18. On the other hand, we have

$$J\langle w\rangle = (E(f_1, f_3) \circ E(f_2, f_4) \circ E(f_2, f_3) \circ E(f_1, f_4) \circ E(f_2, f_3) \circ E(f_1, f_3))\,(w).$$

Thus the converse is also true. In fact we have shown even more, namely that the subgroup $\langle J,\ T_g;\ g \in \Lambda_0\rangle$ of Γ_S considered as subgroup of $\mathrm{Bihol}(\mathcal{H}_S)$ is isomorphic to the group $\mathrm{EO}(\Lambda)$.

D. Discriminant Groups

$S = D_4$

$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,0,0,0)]$	$[(\frac{1}{2},\frac{1}{2},0,0)]$	$[(\frac{1}{2},0,\frac{1}{2},0)]$	$[(0,\frac{1}{2},\frac{1}{2},0)]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$S = A_1^{(3)}$

$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,0,0)]$	$[(\frac{1}{2},0,0)]$	$[(0,\frac{1}{2},0)]$	$[(0,0,\frac{1}{2})]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,\frac{1}{2},\frac{1}{2})]$	$[(\frac{1}{2},0,\frac{1}{2})]$	$[(\frac{1}{2},\frac{1}{2},0)]$	$[(\frac{1}{2},\frac{1}{2},\frac{1}{2})]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$

$S = A_3$

$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,0,0)]$	$[(\frac{1}{4},\frac{1}{2},-\frac{1}{4})]$	$[(\frac{1}{2},0,\frac{1}{2})]$	$[(-\frac{1}{4},\frac{1}{2},\frac{1}{4})]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	0	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{8}$

$S = A_1^{(2)}$

$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,0)]$	$[(\frac{1}{2},0)]$	$[(0,\frac{1}{2})]$	$[(\frac{1}{2},\frac{1}{2})]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

$S = A_2$

$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,0)]$	$[(\frac{1}{3},\frac{1}{3})]$	$[(-\frac{1}{3},-\frac{1}{3})]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	0	$\frac{1}{3}$	$\frac{1}{3}$

$S = S_2$

$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(0,0)]$	$[(0,\frac{1}{4})]$	$[(0,-\frac{1}{4})]$	$[(\frac{1}{2},0)]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
$[\lambda] \in \mathrm{Dis}(\Lambda)$	$[(\frac{1}{2},\frac{1}{4})]$	$[(\frac{1}{2},-\frac{1}{4})]$	$[(0,\frac{1}{2})]$	$[(\frac{1}{2},\frac{1}{2})]$
$\overline{q}_S([\lambda]) \bmod \mathbb{Z}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$

E. Dimensions of Spaces of Vector-valued Modular Forms

In the following tables we list the dimensions of the spaces $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$ of vector-valued modular forms for some positive definite matrices S. For weights $k \in \frac{1}{2}\mathbb{Z}$ which do not occur in the tables the dimension is 0. We write $d(k) := \dim[\mathrm{Mp}(2;\mathbb{Z}), k, \rho_S]$.

$S = A_3$

k	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{13}{2}$	$\frac{17}{2}$	$\frac{21}{2}$	$\frac{25}{2}$	$2n+\frac{1}{2},\ n \geq 7$
$d(k)$	0	0	1	0	1	1	1	$d(k-12)+1$

k	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$	$\frac{15}{2}$	$\frac{19}{2}$	$\frac{23}{2}$	$\frac{27}{2}$	$2n+\frac{3}{2},\ n \geq 7$
$d(k)$	1	1	2	2	3	3	4	$d(k-12)+3$

$S = A_1^{(3)}$

k	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$	$\frac{15}{2}$	$2n+\frac{3}{2},\ n \geq 4$
$d(k)$	1	3	4	5	$d(k-6)+4$

$S = D_4$

k	0	2	4	6	8	10	12	$2n,\ n \geq 7$
$d(k)$	0	1	1	3	2	4	4	$d(k-12)+4$

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Notation

$M\langle Z\rangle$	$= (AZ+B)(CZ+D)^{-1}$ (pp. 53, 57)
$M\langle \tau\rangle$	$= (a\tau+b)/(c\tau+d)$
$M\langle w\rangle$	$= (-q_0(w)b + Aw + c)(M\{w\})^{-1}$ (p. 10)
$M\{w\}$	$= -\gamma q_0(w) + {}^t dw + \delta$ (p. 10)
$M \geq 0$	M is positive semi-definite
$M > 0$	M is positive definite
$a \geq 0$	$a \in \overline{\mathcal{P}_S}$ (p. 31)
$a > 0$	$a \in \mathcal{P}_S$ (p. 31)
$\lambda_0 > 0$	See p. 83
$(\cdot,\cdot)$	A bilinear form, usually $(\cdot,\cdot)_S$ (p. 9)
$(\cdot,\cdot)_0$	$= (\cdot,\cdot)_{S_0}$, the bilinear form associated to S_0 (p. 9)
$(\cdot,\cdot)_1$	$= (\cdot,\cdot)_{S_1}$, the bilinear form associated to S_1 (p. 9)
$(x,y)_S$	$= {}^t xSy$, the bilinear form associated to S (p. 8)
$\lfloor\cdot\rfloor$	The greatest integer function
$\sqrt{\cdot}$	The principal branch of the square root
$[g,h]$	The commutator $ghg^{-1}h^{-1}$ of g and h
G^{ab}	The commutator factor group G/G' of G
G'	The commutator subgroup of G
$H \leq G$	H is a subgroup of G
$A_1 \times \ldots \times A_n$	The block diagonal matrix with diagonal elements $A_1, \ldots, A_n$
$[a_1, \ldots, a_n]$	The diagonal matrix with diagonal elements $a_1, \ldots, a_n$
$A[B]$	$= {}^t\overline{B}AB$
${}^tM$	The transpose of M
${}^t\overline{M}$	The conjugate transpose of M
$\{f_1, \ldots, f_n\}$	A certain Rankin-Cohen type differential operator (p. 36)
$\vert\Phi$	Siegel's Φ-operator (p. 33)
$\vert X$	Restriction of a function to a subspace or subgroup X
$\vert_k$	The Petersson slash operator of weight k (pp. 29, 54, 57, 66)
$\vert_{k,m,S}$	The slash operator of weight k and index (m,S) (p. 40)

$\left(\frac{\partial F}{\partial z}\right)$	$= \left(\frac{\partial(F_1,\ldots,F_n)}{\partial(z_1,\ldots,z_n)}\right)$, the Jacobian matrix of $F : \mathbb{C}^n \to \mathbb{C}^n$ (p. 35)
$\det\left(\frac{\partial F}{\partial z}\right)$	$= \det\left(\frac{\partial(F_1,\ldots,F_n)}{\partial(z_1,\ldots,z_n)}\right)$, the Jacobian (determinant) of $F : \mathbb{C}^n \to \mathbb{C}^n$ (p. 35)
$[\Gamma, k, \nu]$	The space of modular forms of weight k with respect to Γ and ν (p. 29)
$[\Gamma_S, k, \nu]_0$	The subspace of cusp forms in $[\Gamma_S, k, \nu]$ (p. 33)
$[\Gamma, k]$	$= [\Gamma, k, 1]$
$[\Gamma_{\mathbb{H}}, k, \chi]$	The space of quaternionic modular forms of weight k with respect to χ (p. 58)
$[\Gamma_{\mathbb{K}}, k, \chi]$	The space of Hermitian modular forms of weight k with respect to χ (p. 54)
$[\Gamma_{\mathbb{K}}, k, \chi]_0$	The subspace of cusp forms in $[\Gamma_{\mathbb{K}}, k, \chi]$ (p. 54)
$[\Gamma_S, k, \chi]_{\mathrm{mer}}$	The space of meromorphic modular forms of weight k with respect to χ (p. 97)
$[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$	The space of modular forms of weight k with respect to ρ (p. 66)
$[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]_0$	The subspace of cusp forms of $[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]$
$[\mathrm{Mp}(2;\mathbb{Z}), k, \rho]_\infty$	The space of nearly holomorphic modular forms of weight k (p. 68)
$[\mathrm{SL}(2;\mathbb{Z}), k]$	The space of elliptic modular forms of weight k
$A(n)$	The alternating group of degree n
$\mathcal{A}(\Gamma)$	$= \bigoplus_{k\in\mathbb{Z}}[\Gamma, k, 1]$, the graded ring of modular forms with respect to Γ (p. 30)
$\alpha(x)$	$= (1, \ldots, x^{l-1})$ (p. 81)
$\alpha_f(\mu)$	The Fourier coefficients of the orthogonal modular form f
B_n	A Bernoulli number
$\mathrm{Bihol}(X)$	The group of biholomorphic automorphisms on the space X
C	$= \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$, the generator of the center of $\mathrm{Mp}(2;\mathbb{Z})$
$\mathbb{C}$	The complex numbers
$\mathbb{C}[\Lambda^\sharp/\Lambda]$	The group ring of the discriminant group $\Lambda^\sharp/\Lambda$
$\mathbb{C}[X_1, \ldots, X_n]$	The polynomial ring in n variables
$c_\mu(n)$	The Fourier coefficients of a vector-valued modular form
C_n	The cyclic group of order n
χ	An Abelian character of Γ_S
D_n	The dihedral group of order n
$\delta(\lambda^\perp)$	The discriminant of $\lambda^\perp$ (p. 84)
$\det$	The determinant map or the determinant character of a modular group
$\mathrm{diag}(M)$	The column vector consisting of the diagonal elements of the matrix M
$\mathrm{Dis}(\Lambda)$	The discriminant group $\Lambda^\sharp/\Lambda$ of Λ (p. 7)
e	$= {}^t(1, 0, \ldots, 0, 1) \in \mathbb{R}^{l+2}$ (p. 9)
e_j	An element of the standard basis $(e_1, \ldots, e_l)$ of $\mathbb{R}^l$
$E_k(\tau;\ v, S)$	A vector-valued Eisenstein series of weight k (p. 73)
$E_k^{A_1^{(3)}}$	The normalized orthogonal Eisenstein series of weight k for $\Gamma_{A_1^{(3)}}$ (p. 60)

$E_k^{A_3}$	The normalized orthogonal Eisenstein series of weight k for Γ_{A_3} (p. 60)
$E_k^{D_4}$	The normalized orthogonal Eisenstein series of weight k for Γ_{D_4} (p. 60)
$E_k^{\mathbb{H}}$	The normalized quaternionic Eisenstein series of weight k for $\Gamma_{\mathbb{H}}$ (p. 58)
$E_k^{\mathbb{K}}$	The normalized Hermitian Eisenstein series of weight k for $\Gamma_{\mathbb{K}}$ (p. 54)
$E_k^{S_{\mathbb{K}}}$	The normalized orthogonal Eisenstein series of weight k for $\Gamma_{S^{\mathbb{K}}}$ (p. 56)
e_μ	An element of the standard basis $(e_\mu)_{\mu\in\Lambda^\sharp/\Lambda}$ of $\mathbb{C}[\Lambda^\sharp/\Lambda]$
η	The Dedekind eta function (p. 67)
$\mathbb{F}_q$	The field of two elements
f_8, f_{10}, f_{12}	Certain cusp forms for Γ_{A_3} and/or $\Gamma_{A_1^{(3)}}$ (pp. 94, 104)
f_μ	A component of a vector-valued modular form f
G_k	The normalized elliptic Eisenstein series of weight k (p. 45)
Γ	A subgroup of finite index of Γ_S
$\Gamma(2;\mathbb{K})$	$= \mathrm{U}(2;\mathfrak{o}_{\mathbb{K}})$, the Hermitian modular group
$\widetilde{\Gamma(2;\mathbb{K})}$	A certain subgroup of $\Gamma(2;\mathbb{K})$ (p. 54)
$\Gamma_{\mathbb{H}}$	The extended quaternionic modular group (p. 57)
Γ_∞	$= \{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right);\ n\in\mathbb{Z}\} \le \mathrm{SL}(2;\mathbb{Z})$
$\widetilde{\Gamma}_\infty$	$= \{(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right), 1);\ n\in\mathbb{Z}\} \le \mathrm{Mp}(2;\mathbb{Z})$
$\Gamma_{\mathbb{K}}$	The extended Hermitian modular group (p. 54)
Γ_S	$= \mathrm{O}(\Lambda_1)\cap\mathrm{O}^+(S_1;\mathbb{R})$, the orthogonal modular group with respect to S (p. 11)
Γ_S^{ab}	The group of Abelian characters of Γ_S (p. 22)
$\mathrm{GL}(n;R)$	The group of invertible $n\times n$ matrices with elements in R
$h(\Delta_{\mathbb{K}})$	Class number of an imaginary quadratic number field with discriminant $\Delta_{\mathbb{K}}$
$H(2;\mathbb{C})$	The Hermitian half-space of degree 2 (p. 53)
$H(2;\mathbb{H})$	The half-space of quaternions of degree 2 (p. 57)
$\mathbb{H}$	The Hamilton quaternions
$\mathbb{H}_S$	A subspace of $\mathbb{H}$ (p. 13)
$\mathcal{H}$	The complex upper half plane $\{\tau\in\mathbb{C};\ \mathrm{Im}(\tau)>0\}$
$\mathcal{H}_S$	The (orthogonal) half-space associated to S (p. 9)
$H_0(\mu,n)$	A certain subset of $\mathcal{P}_S^1$ (p. 78)
h_k	Certain modular forms of weight $k\in\{4,6,8,10,12\}$ for $\Gamma_{A_1^{(3)}}$ (p. 62)
$H_S(\mathbb{R})$	The Heisenberg group (p. 27)
$\mathrm{Her}(n;R)$	The set of Hermitian $n\times n$ matrices with elements in R
I	An identity matrix
$\mathrm{i}_1,\mathrm{i}_2,\mathrm{i}_3$	The canonical non-real basis elements of $\mathbb{H}$
I_n	The identity matrix in $\mathrm{Mat}(n;R)$
I_{tr}	The involution on $H(2;\mathbb{C})$ or $H(2;\mathbb{H})$ mapping Z to ${}^tZ$

ι_S	The isomorphism $\mathbb{R}^l \to \mathbb{H}_S$ (p. 13)
ι_T^S	An isometric embedding of Λ_T in Λ_S (p. 45)
$\mathrm{Im}(z)$	The imaginary part of $z \in \mathbb{C}$
J	A certain element of Γ_S, or the element $\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right) \in \mathrm{Mp}(2;\mathbb{Z})$ (p. 11)
$j_{k,m,S}(g,(\tau,z))$	A factor of automorphy on $J_S(\mathbb{R}) \times (\mathcal{H} \times \mathbb{C}^l)$ (p. 40)
$j(M,w)$	$= M\{w\}$, the factor of automorphy on $\mathrm{O}^+(S_1;\mathbb{R}) \times \mathcal{H}_S$ (p. 29)
$\tilde{J}$	A certain element of $\mathrm{O}(\Lambda_0)$ (p. 11)
$J_{\mathbb{H}}$	$= \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right)$
$j_{\mathbb{H}}(M,Z)$	A factor of automorphy on $\langle \mathrm{Sp}(2;\mathcal{O}), \rho I_4 \rangle \times H(2;\mathbb{H})$ (p. 59)
J_{Her}	$= \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right)$
$j_{\mathrm{Her}}(M,Z)$	A factor of automorphy on $\widetilde{\Gamma(2,\mathbb{K})} \times H(2;\mathbb{K})$ (p. 55)
$J_k(m,S)$	$= J_k(m,S,1)$
$J_k(m,S,\nu)$	The space of Jacobi forms of index (m,S) and weight k with respect to ν (p. 41)
$J_k^0(m,S)$	$= J_k^0(m,S,1)$
$J_k^0(m,S,\nu)$	The subspace of Jacobi cusp forms in $J_k(m,S,\nu)$ (p. 41)
$J_S(\mathbb{R})$	The Jacobi group (p. 27)
$J_S(\mathbb{Z})$	The integral Jacobi group (p. 28)
$\mathbb{K}$	An imaginary quadratic number field (p. 53)
$\mathcal{K}$	$= \{[Z] \in \mathcal{N};\ (Z,\overline{Z})_1 > 0\}$ (p. 83)
$\mathcal{K}(\Gamma_S)$	The field of orthogonal modular functions for Γ_S (p. 97)
$\mathcal{K}^+$	A component of $\mathcal{K}$ (p. 84)
l	A positive integer, usually the rank of S
Λ	A lattice, usually $\mathbb{Z}^l$ and even of signature $(0,l)$
λ	Usually an element of Λ or $\Lambda^\sharp$
Λ_0	$= \mathbb{Z} \times \Lambda \times \mathbb{Z}$
Λ_1	$= \mathbb{Z} \times \Lambda_0 \times \mathbb{Z}$
$\Lambda_{\mathbb{Q}}$	$= \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$
Λ_T	The lattice associated to T (p. 45)
$\lambda^\perp$	A rational quadratic divisor (p. 84)
$\Lambda^\sharp$	The dual lattice of Λ (p. 7)
$\mathcal{M}(\Gamma_{\mathbb{H}},k)$	The Maaß space in $[\Gamma_{\mathbb{H}},k,1]$ (p. 58)
$\mathcal{M}(\Gamma_S,k)$	$= \mathcal{M}(\Gamma_S,k,1)$
$\mathcal{M}(\Gamma_S,k,\nu)$	The Maaß space in $[\Gamma_S,k,\nu]$
M_D	A certain element of Γ_S (p. 12)
M_D^*	A certain element of Γ_S (p. 12)
M_{tr}	A certain element of Γ_S (a rotation) (p. 20)
μ	Usually an element of $\Lambda^\sharp$ or $\Lambda^\sharp/\Lambda$

μ_0	An element of $\Lambda_0^\sharp$ or $\Lambda_0^\sharp/\Lambda_0$
$\mathrm{Mat}(n, m; R)$	The group of $n \times m$ matrices with elements in R
$\mathrm{Mat}(n; R)$	The ring of $n \times n$ matrices with elements in R
$\mathrm{Mp}(2; \mathbb{R})$	The metaplectic cover of $\mathrm{SL}(2; \mathbb{R})$ (p. 65)
$\mathrm{Mp}(2; \mathbb{Z})$	The integral metaplectic group (p. 65)
$\mathrm{Mp}(2; \mathbb{Z})[N]$	The principal congruence subgroup of $\mathrm{Mp}(2; \mathbb{Z})$ of level N
$N(z)$	$= z\overline{z}$, the norm on $\mathbb{H}$ (p. 13)
$\mathbb{N}$	The natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$= \mathbb{N} \cup \{0\}$
$\mathcal{N}$	The zero-quadric in $P(V_1(\mathbb{C}))$ (p. 83)
ν	An Abelian character of Γ_S
ν_2	The Siegel character of Γ_S (p. 24)
ν_η	The character of the Dedekind eta function (p. 67)
$\nu_\wp$	The Siegel character of $\Gamma_\mathbb{K}$ (p. 55)
ν_π	The orthogonal character of Γ_S (p. 22)
ν_ρ	A certain Abelian character of $\Gamma_\mathbb{H}$ (p. 59)
ν_{skew}	The symmetry character of $\Gamma_\mathbb{K}$ (p. 55)
ν_{tr}	A certain Abelian character of $\Gamma_\mathbb{H}$ (p. 59)
$\mathrm{O}(b^+, b^-)$	The real orthogonal group of signature (b^+, b^-) (p. 8)
$\mathrm{O}(\Lambda)$	The orthogonal group of Λ (p. 8)
$\mathrm{O}_{\mathrm{d}}(\Lambda)$	The discriminant kernel of $\mathrm{O}(\Lambda)$ (p. 8)
$\mathrm{O}^+(\Lambda_0)$	$= \{A \in \mathrm{O}(\Lambda_0);\ A \cdot \mathcal{H}_S = \mathcal{H}_S\}$ (p. 15)
$\mathrm{O}(S; \mathbb{R})$	The real orthogonal group with respect to S (p. 8)
$\mathrm{O}^+(S_1; \mathbb{R})$	The connected component of the identity of $\mathrm{O}(S_1; \mathbb{R})$ (p. 10)
$\mathcal{O}$	The Hurwitz order (p. 13)
$\mathcal{O}_S$	$= \mathcal{O} \cap \mathbb{H}_S$ (p. 13)
$\mathfrak{o}_\mathbb{K}$	The ring of integers of the imaginary quadratic number field $\mathbb{K}$ (p. 53)
ω	$= \frac{1}{2}(1 + \mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_3)$
P	A certain element of Γ_S (a rotation) (p. 12)
$P(V_1(\mathbb{C}))$	The projective space of $V_1(\mathbb{C})$
$\widetilde{P}$	A certain element of $\mathrm{O}^+(\Lambda_0)$ (p. 12)
$\mathcal{P}_S$	The positive cone associated to S (p. 9)
$\mathcal{P}_S^1$	$= \{v \in \mathcal{P}_S;\ q_0(v) = 1\}$ (p. 77)
$\overline{\mathcal{P}_S}$	The closure of $\mathcal{P}_S$ (p. 31)
$P_S(\mathbb{R})$	The parabolic subgroup of $\mathrm{O}^+(S_1; \mathbb{R})$ (p. 26)
$P_S(\mathbb{Z})$	$= P_S(\mathbb{R}) \cap \Gamma_S$ (p. 28)
Φ	Siegel's Φ-operator (p. 33)

ϕ_4	A Borcherds product of weight 4 for $\Gamma_{A_1^{(2)}}$ (p. 56)
ϕ_9	A Borcherds product of weight 9 for Γ_{A_2} (p. 56)
ϕ_{10}	A Borcherds product of weight 10 for $\Gamma_{A_1^{(2)}}$ (p. 56)
ϕ_{30}	A Borcherds product of weight 30 for $\Gamma_{A_1^{(2)}}$ (p. 56)
ϕ_{45}	A Borcherds product of weight 45 for Γ_{A_2} (p. 56)
$\varphi_{\mathbb{H}}$	A certain biholomorphic isomorphism from $\mathcal{H}_{D_4}$ to $H(2;\mathbb{H})$ (p. 58)
$\varphi_{\mathbb{K}}$	A certain biholomorphic isomorphism from $\mathcal{H}_{S^{\mathbb{K}}}$ to $H(2;\mathbb{C})$ (p. 54)
φ_m	A Fourier-Jacobi coefficient of index m (p. 38)
ψ_3	A Borcherds product of weight 3 for $\Gamma_{A_1^{(3)}}$ (p. 100)
ψ_8	A Borcherds product of weight 8 for Γ_{A_3} (p. 92)
ψ_9	A Borcherds product of weight 9 for Γ_{A_3} (p. 92)
ψ_{18}	A Borcherds product of weight 18 for $\Gamma_{A_1^{(3)}}$ (p. 100)
ψ_{20}	A Borcherds product of weight 20 for $\Gamma_{A_1^{(3)}}$ (p. 100)
ψ_{30}	A Borcherds product of weight 30 for $\Gamma_{A_1^{(3)}}$ (p. 100)
ψ_{54}	A Borcherds product of weight 54 for Γ_{A_3} (p. 92)
ψ_k	A Borcherds product of weight k (p. 85)
$\mathrm{PO}(S_1;\mathbb{R})$	$= \mathrm{O}(S_1;\mathbb{R})/\{\pm I\}$ (p. 10)
$\mathrm{PO}^+(S_1;\mathbb{R})$	$= \mathrm{O}^+(S_1;\mathbb{R})/\{\pm I\}$ (p. 10)
$\mathrm{Pos}(n;R)$	The ring of positive definite Hermitian $n\times n$ matrices with elements in R
q	$e^{2\pi i\tau}$ for $\tau\in\mathcal{H}$, or a quadratic form and then usually q_S (p. 9)
$\overline{q}(\mu+\Lambda)$	$= q(\mu)+\mathbb{Z}$ (p. 7)
$\mathbb{Q}$	The rational numbers
q_0	$= q_{S_0}$, the quadratic form associated to S_0 (p. 9)
q_1	$= q_{S_1}$, the quadratic form associated to S_1 (p. 9)
$q_S(x)$	$= \frac{1}{2}(x,x)_S$, the quadratic form associated to S (p. 8)
$\mathbb{R}$	The real numbers
R_A	A certain element of Γ_S (a rotation) (p. 12)
R_g	A certain element of Γ_S (a rotation) (p. 12)
ρ	$= (1+\mathrm{i}_1)/\sqrt{2}$, or a finite representation of $\mathrm{Mp}(2;\mathbb{Z})$
ρ_S	The Weil representation attached to $(\Lambda^\sharp/\Lambda, q_S)$ (p. 68)
$\rho_S^\sharp$	The dual representation of ρ_S
ρ_S^-	The induced Weil representation on $\{e_\mu - e_{-\mu};\ \mu\in\Lambda^\sharp/\Lambda\}$ (p. 71)
ρ_S^+	The induced Weil representation on $\{e_\mu + e_{-\mu};\ \mu\in\Lambda^\sharp/\Lambda\}$ (p. 71)
ϱ	A component of $\varrho_f(W)$
ϱ_f	The Weyl vector of f
$\varrho_f(W)$	The Weyl vector associated to W and f (p. 79)

ϱ_{z_0}	A component of $\varrho_f(W)$
$\varrho_{z_0^\sharp}$	A component of $\varrho_f(W)$
$\mathrm{Re}(z)$	The real part of $z \in \mathbb{C}$
$\mathrm{Rot}(U)$	$= \left(\begin{smallmatrix} {}^t\overline{U} & 0 \\ 0 & U^{-1} \end{smallmatrix}\right)$ for $U \in \mathrm{GL}(2;\mathbb{H})$ or $U \in \mathrm{GL}(2;\mathbb{K})$
S	A nonsingular real symmetric matrix, usually even
$S(n)$	The symmetric group of degree n
S_0	An extension of $-S$ of signature $(1, l+1)$
S_1	An extension of S_0 of signature $(2, l+2)$
$S^{\mathbb{K}}$	The even matrix associated to the imaginary quadratic field $\mathbb{K}$ (p. 54)
sign	The sign function
$\mathrm{SL}(n; R)$	The group of $n \times n$ matrices with elements in R and determinant 1
$\mathrm{SO}(\Lambda)$	The special orthogonal group of Λ (p. 19)
$\mathrm{Sp}(2;\mathbb{H})$	The symplectic group of degree 2 over $\mathbb{H}$ (p. 57)
$\mathrm{Sp}(n; R)$	The symplectic group of degree n over R
$\mathrm{Stab}_G(X)$	$= \{g \in G;\ gx \in X \text{ for all } x \in X\}$, the stabilizer of X in G
$\mathrm{SU}(2;\mathbb{K})$	$= \mathrm{U}(2;\mathbb{K}) \cap \mathrm{SL}(4;\mathbb{K})$, the special unitary group of degree 2 over $\mathbb{K}$ (p. 53)
$\mathrm{Sym}(n; R)$	The set of symmetric $n \times n$ matrices with elements in R
T	$= \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \in \mathrm{Mp}(2;\mathbb{Z})$
T_g	A certain element of Γ_S (a translation) (p. 11)
$\Theta(\tau; S, p_r)$	A vector-valued theta series (p. 75)
Θ_a	A quaternionic theta series (p. 60)
$\theta_\mu(\tau; S, p_r)$	A component of $\Theta(\tau; S, p_r)$
τ, τ_1, τ_2	Usually elements of $\mathcal{H}$
$\mathrm{trace}(M)$	The trace of the matrix M
$\mathrm{Trans}(H)$	$= \left(\begin{smallmatrix} I_2 & H \\ 0 & I_2 \end{smallmatrix}\right)$ for $H \in \mathrm{Her}(2;\mathbb{H})$ or $H \in \mathrm{Her}(2;\mathbb{K})$
$\mathrm{U}(2;\mathbb{K})$	The unitary group of degree 2 over $\mathbb{K}$ (p. 53)
U_λ	A certain element of Γ_S (a rotation) (p. 12)
${}_\lambda U$	A certain element of Γ_S (a rotation) (p. 12)
$\widetilde{U}_\lambda$	A certain element of $\mathrm{O}^+(\Lambda_0)$ (p. 12)
V	$= \Lambda \otimes \mathbb{R}$, usually $\mathbb{R}^l$
$V(\mathbb{C})$	$= V \otimes \mathbb{C}$
$v(x)$	$= (1, -x^2\alpha(x), x)$ (p. 81)
V_0	$\Lambda_0 \otimes \mathbb{R}$
V_1	$\Lambda_1 \otimes \mathbb{R}$
$v_1(x)$	$= v(x)/\sqrt{q_0(v(x))}$
w	Usually an element of $\mathcal{H}_S$ of the form (τ_1, z, τ_2)
W_f	The Weyl chamber of f (p. 82)

$Y_1, \ldots, Y_6$	Certain quaternionic theta series (p. 60)
Z	An element of $H(2;\mathbb{C})$ or $H(2;\mathbb{H})$
z	Usually an element of $\mathbb{C}^l$
$\mathbb{Z}$	The integers

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