Reduced product objects in model categories

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0. Introduction

In his influential 1967 monograph *Homotopical Algebra*, D. Quillen [8] described an abstract approach to homotopy theory enabling analogous theories to be defined in categories other than the category of spaces and continuous maps. Although the starting point in the classical homotopy theory of spaces and maps is the equivalence relation (between maps) of *homotopy*, in Quillen's approach it is that of a *model category*, that is to say, a category **C**, together with three distinguished classes of morphisms, *we*, *cof*, *fib*, called weak equivalences, cofibrations, and fibrations, respectively. These are required to satisfy certain axioms which reflect typical properties of the classes of such maps in topology and they enable the construction of much of the basic machinery of homotopy theory in the category **C**. However, it is not possible in a model category to introduce all possible concepts and prove analogs of all possible theorems that hold in the homotopy theory of spaces: for the simple reason that the axioms of a model category are self-dual whereas the Eckmann-Hilton duality in spaces is known not to be perfect.

Nevertheless additional axioms, if enjoyed by a particular model category, sometimes enable further classical concepts and results to be introduced in **C**. A relatively recent instance of this has been the successful definition by Doerane [2] of a notion of *Lusternik-Schnirelmann category* in a type of model category satisfying the so-called *cube axiom*. In such cases there is a price to be paid: the richer theory is only available in categories **C** for which the additional axioms can be verified.

The purpose of this paper is to show that the cube axiom permits the development of another powerful feature of the homotopy theory of spaces: the existence of James spaces [6]. The classical James space construction associates with a locally countable pointed CW-complex X a space X_∞ and a homotopy equivalence $X_\infty \to \Omega\Sigma X$, where Ω and Σ refer to the loop and (reduced) suspension endofuctors of the category of pointed spaces. It opens the door to the study of the suspension operation via the inclusion $X \to X_\infty$ and to the detection of elements in the cokernel of suspension via the James map $X_\infty \to (X \wedge X)_\infty$. Such considerations have hitherto been out of the reach of abstract homotopy theory, although generalizations, fibrewise and equivariant, of the equivalence $X_\infty \to \Omega\Sigma X$ have been obtained, [3].

In this paper we define the reduced powers X_n of an object X in a suitable model category, or more generally, the objects $(X, A)_n$ and $(X, A)_\infty$ as in the work [4] of Gray, associated with a cofibration $A \to X$. If, in particular, a certain cube axiom is satisfied, we prove the weak equivalence of the object X_∞ to $\Omega\Sigma X$ generalizing work of I. M. James and others.

1. Model categories

Many authors have found it convenient to modify Quillen's axioms as presented in [8]. We use the version given by Hovey [5]:

1.1 Definition A *model category* is a category $\mathbf{C}$ with all small limits and colimits together with a model structure on $\mathbf{C}$.
A *model structure* on a category $\mathbf{C}$ consists of three classes of morphisms of $\mathbf{C}$ called *weak equivalences, cofibrations* and *fibrations*, and two functorial factorisations $(\alpha,\ \beta)$ and $(\gamma,\ \delta)$ satisfying the following properties:

1. (2-out-of-3) If f and g are morphisms of $\mathbf{C}$ such that gf is defined and two of f, g and gf are weak equivalences then so is the third.

2. (Retract) If f and g are morphisms of $\mathbf{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

3. (Lifting) Define a map (i.e. morphism of $\mathbf{C}$) to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

4. (Factorisation) For any map $f : A \to B$,

$$f \;=\; A \xrightarrow{\alpha(f)} B' \xrightarrow[\sim]{\beta(f)} B \qquad \text{and} \qquad f \;=\; A \xrightarrow[\sim]{\gamma(f)} A' \xrightarrow{\delta(f)} B \;,$$

indicating that $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

The retract property enables the statement of the lifting property to be strengthened:

1.2 Lemma ([5, Lemma 1.1.10]). *A map in a model category is a cofibration (trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a map is a fibration (trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations).*

In particular, every isomorphism in a model category is a trivial cofibration and a trivial fibration.

1.3 Corollary ([5, 1.1.11]). *The cofibrations (trivial cofibrations) in a model category are closed under pushout. That is, if*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout square, where f is a cofibration (trivial cofibration), then g is a cofibration (trivial cofibration). Dually, fibrations (trivial fibrations) are closed under pullback.

Every model category $\mathbf{C}$ has an initial object 0 (the colimit of the empty diagram) and a terminal object $*$. If 0 and $*$ are isomorphic then $\mathbf{C}$ is *pointed.* Since our goal is a James construction in $\mathbf{C}$ we assume henceforth that $\mathbf{C}$ is pointed. We assume throughout that every object is cofibrant.

A commutative diagram in $\mathbf{C}$

(1.4)
$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a *homotopy pullback* if the induced map (shown dotted) in the following diagram is a weak equivalence.

(1.5)
$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dd, "k"'] \arrow[dr, dotted] & & C \arrow[dd] \arrow[dr, tail, "\sim"] & \\ & A \times_B C' \arrow[rr] \arrow[dl] & & C' \arrow[dl, two heads, "\delta(g)"] \\ A \arrow[rr, "f"'] & & B & \end{tikzcd}$$

Here it is to be understood that the square with source $A \times_B C'$ is a pullback. The special case $C = *$ of 1.4 is of some significance, for then we call D the *homotopy fibre* of f and denote it by F_f. If both $C = A = *$ then we say that D is a loop object of B and denote it by ΩB.

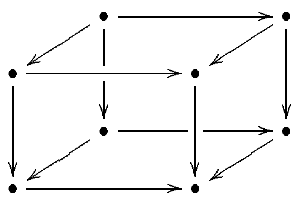
Dually, we define the notions of *homotopy pushout* square and *homotopy cofibre* (i.e. *mapping cone*): specifically the square 1.4 is a homotopy pushout if the induced dotted arrow in the following diagram is a weak equivalence.

$$
\begin{tikzcd}
D \arrow[rrrr, "h"] \arrow[rr, "\alpha(h)"] \arrow[d, "k"'] & & C' \arrow[rr, "\sim"] \arrow[dd] & & C \arrow[d, "g"] \\
A \arrow[rrrr] \arrow[drr] & & & & B \\
& & A \vee_D C' \arrow[urr, dotted] & &
\end{tikzcd}
\tag{1.6}
$$

In the case $C = *$ of 1.6, we call C' a *cone* on D and $A \vee_D C'$ a *mapping cone* of k. If there is a weak equivalence $X \to *$ then we say that X is *weakly contractible*. In particular each mapping cone of $1 : X \to X$ is a cone on X and is weakly contractible.

In order to prove the weak equivalence of infinite reduced power with loops-suspension, we require the following cube axiom and the conditions in 1.11 below.

1.7 Cube Axiom. Suppose that we have a commutative diagram as follows.



(1.8)

If the top and bottom faces are homotopy push-outs and the left and rear faces are homotopy pull-backs, then the remaining two faces are homotopy pull-backs.

This axiom is exactly the same as the axiom [2, Cube axiom on p220] in the paper of Doeraene. In the topological case it is very similar to the first cube theorem [7, Theorem 18] of Mather, except that we assume strict commutativity of the diagram. Given this axiom, one can easily prove the following proposition, and we omit the routine proof.

1.9 Proposition. *Suppose that* $\mathbf{C}$ *is a model category satisfying the Cube Axiom 1.7. Then in* $\mathbf{C}$ *the following condition holds:*
Given a commutative diagram as in diagram 1.8 in which the bottom face is a homotopy push-out and the vertical faces are homotopy pull-backs, then the top face is a homotopy push-out square.

The following Lemma is an indication of the power of the cube axiom in our context.

1.10 Lemma. *Taking the product of a rectangular diagram with a fixed fibrant object F of $\mathbf{C}$ preserves the property of being a homotopy pushout.*

Proof. There is a cubical diagram whose base is the original rectangle and whose upper face is the desired homotopy pushout. The vertical arrows are product projections which, under our assumptions, are fibrations. Hence the vertical faces are homotopy pullback rectangles and we may apply Proposition 1.9. $\square$

Conditions 1.11:

(we) Given any object X and weak equivalence $f : A \to B$ in $\mathbf{C}$, then the morphism $X \times f : X \times A \to X \times B$ is a weak equivalence.

(cof) Given any object X and cofibration $f : A \to B$ in $\mathbf{C}$, then the morphism $X \times f : X \times A \to X \times B$ is a cofibration.

(cotriad) Given any commutative diagram of solid arrows as below, in which every morphism is a cofibration and the upper quadrilateral is a pushout, then the induced map $B +_A C \to D$ is a cofibration.

$$\begin{tikzcd}
B \arrow[dr] \arrow[ddr] & A \arrow[l] \arrow[r] & C \arrow[dl] \arrow[ddl] \\
& B+_A C \arrow[d, dashed] & \\
& D &
\end{tikzcd}$$

1.12 Remark. (a) If Condition 1.11*(we)* is satisfied, then Lemma 1.10 will hold more generally, without F having to be fibrant.

(b) The cube axiom and Conditions 1.11 can be seen to hold, for instance, in the category of topological spaces $\mathbf{Top}_*$ with model structure as in [5, Cor 2.4.20] and in the category of pointed simplicial sets $\mathbf{SSet}_*$ with the model structure as in [5, Cor 3.6.6].

(c) The definition of model category of [5] adopted here is perhaps more restrictive than what we actually need for our main results. However, since this is an initial study of reduced product objects in an axiomatic setting, we considered it convenient to work with a definiton which is familiar to the readership.

2. The Gray construction

For a cofibration $i : A \to X$ in $\mathbf{C}$, we construct objects $(X,A)_n$ in $\mathbf{C}$, for positive integers n, which are analogues of the relevant subspaces of the spaces $(X,A)_\infty$ defined by Gray [4]. Fat wedge maps $w_n : W_n(X,A) \to X \times A^n$ are defined as follows (consistent with the construction of Doeraene [2, Definition 3.1]). We take w_0 to be the map $* \to X$ and, for $n > 0$, we define w_n inductively as follows. Let

$$e = 1 \times (* \to A) : V \to V \times A$$

be the natural map and note that the outside of the following diagram is commutative.

$$\text{(2.1)} \qquad \begin{tikzcd} W_{n-1}(X,A) \arrow[r, "e"] \arrow[d, "w_{n-1}"'] & W_{n-1}(X,A) \times A \arrow[d, dotted, "\bar{w}"'] \arrow[ddr, "w_{n-1} \times A"] & \\ X \times A^{n-1} \arrow[r, dotted, "\bar{e}"] \arrow[drr, "e"'] & W_n(X,A) \arrow[dr, dotted] & \\ & & X \times A^n \end{tikzcd}$$

The desired map w_n is then the induced map from the pushout (recall that $\mathbf{C}$ has small colimits) of the top left corner. In the case $X = A$ we denote $(X,A)_n$ simply by A_n and likewise $W_n(X,A)$ by W_nA.

We now define the objects $(X,A)_n$ $(n \geq 0)$, *folding maps* ϕ_n $(n \geq 0)$ and *identification maps* μ_n $(n \geq 1)$

$$\phi_n : W_n(X,A) \to (X,A)_n \quad ; \quad \mu_n : X \times A^{n-1} \to (X,A)_n$$

inductively as follows. We start off by setting $(X,A)_0 = *$, $\phi_0 = * \to *$ and $\mu_1 = 1$. Given ϕ_{n-1}, then μ_n is defined by forming a pushout square

$$\text{(2.2)} \qquad \begin{tikzcd} W_{n-1}(X,A) \arrow[r, "w_{n-1}"] \arrow[d, "\phi_{n-1}"'] & X \times A^{n-1} \arrow[d, "\mu_n"] \\ (X,A)_{n-1} \arrow[r, "j_n"'] & (X,A)_n \end{tikzcd}$$

Given μ_n, we now define ϕ_n. To do so we require an alternative way of constructing the object $W_n(X, A)$, this time as the colimit of a larger diagram. The construction, although somewhat technical, is conceptually simple.

Fix any $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, let S_k be the collection of all proper nonempty subsets of $\{1, 2, ..., k\}$. Given any $\sigma \in S_{n+1}$, let $Z_\sigma = Y \times A^{|\sigma|-1}$, where $Y = X$ if $1 \in \sigma$ and otherwise $Y = A$. Let $\sigma_1, \sigma_2, ..., \sigma_k$ be the distinct elements of σ in increasing order. For any $\sigma, \tau \in S_{n+1}$ with $\sigma \subset \tau$, let $f_{\sigma,\tau} : Z_\sigma \to Z_\tau$ be a section to the relevant projection map so that $f_{\sigma,\tau}$ 'inserts the base point $*$ in the i'th place' for every i which is such that $\tau_i \in \tau \backslash \sigma$. Note that if $\sigma, \tau, \rho \in S_{n+1}$ with $\sigma \subset \tau \subset \rho$, then $f_{\tau,\rho} \circ f_{\sigma,\tau} = f_{\sigma,\rho}$. Now let D be the diagram formed by all the maps $f_{\sigma,\tau}$ for $\sigma, \tau \in S_{n+1}$, together with the initial maps into the objects Z_σ. Let D_0 be the "sub-diagram" of D obtained by removing the object $U = Z_{\{1,2,...,n\}}$ and every arrow having U as its target. Then the colimit of diagram D is (isomorphic to) $W_n(X, A)$. To this end, note that the colimit of the diagram D_0 exists and coincides with $W_{n-1}(X, A) \times A$. Augment the diagram D with maps $g_\sigma : Z_\sigma \to (X, A)_n$ which coincide with μ_n (in the obvious sense) for all maximal members σ of S_{n+1}. By induction on n one can prove that the augmented diagram is commutative. Then since $W_n(X, A)$ is a colimit of D there exists a unique map ϕ_n such that $\phi_n \circ g_\sigma$ agrees with f_σ for all maximal σ, where $f_\sigma : Z_\sigma \to W_n(X, A)$ refers to the maps into the colimiting object.

We shall need to recognise $W_n(X, A)$ also as the pushout

(2.3)
$$\begin{tikzcd}
W_{n-1}A \arrow[r, "f"] \arrow[d, "w_{n-1}"'] & X \times W_{n-1}A \arrow[d, dotted, "w'"] \arrow[ddr, "X \times w_{n-1}"] & \\
A^n \arrow[r, dotted, "\bar{f}"] \arrow[drr, "f \times A^n"'] & W_n(X, A) \arrow[dr, dotted, "w_n"] & \\
 & & X \times A^n
\end{tikzcd} ,$$

where $f = (* \to X) \times 1 : V \to X \times V$.

We also define a *multiplication* map

$$\nu_n : X \times A_{n-1} \to (X, A)_n \ ,$$

as follows. Let $\nu_2 = \mu_2$ and then define ν_{n+1} inductively to be the unique map (dotted arrow) determined by pushout in the following diagram which may be checked to be commutative by considering the appropriate 'diagram D'.

$$\begin{tikzcd}
X \times W_{n-1}(A) \arrow[r] \arrow[d, "X\times\phi_{n-1}"'] & X \times A^n \arrow[d, "X\times\mu_n"'] \arrow[ddr, "\mu_{n+1}"] & \\
X \times A_{n-1} \arrow[r] \arrow[drr, "j_{n-1}\circ\nu_n"'] & X \times A_n \arrow[dr, dotted, "\nu_{n+1}"] & \\
 & & (X,A)_{n+1}
\end{tikzcd} \tag{2.4}$$

2.5 Proposition. *If Conditions 1.11(cof* and *cotriad) hold, then for any cofibration $A \to X$, the map $w_n : W_n(X,A) \to X \times A^n$ is a cofibration.*

Proof. The proof is by induction on n. The map $w_0 : * \to CA$ is a cofibration.
Now assume that for some $n \in \mathbb{N}$ we know that w_{n-1} is a cofibration. Note that in the following commutative diagram, the rows present cotriads of which the pushouts are objects $W_n(X,A)$ and $X \times A^n$, and the induced map is w_n.

$$\begin{tikzcd}
X \times A^{n-1} \arrow[d, "1"'] & W_{n-1}(X,A) \arrow[l, "w_{n-1}"'] \arrow[r] \arrow[d, "w_{n-1}"'] & W_{n-1}(X,A) \times A \arrow[d, "w_{n-1}\times A"] \\
X \times A^{n-1} & X \times A^{n-1} \arrow[l, "1"] \arrow[r] & X \times A^n
\end{tikzcd} \tag{2.6}$$

Due to Condition 1.11*(cof)*, all of the maps in diagram 2.6 are cofibrations. Condition 1.11*(cotriad)* ensures that w_n is a cofibration. □

2.7 Proposition. *If Conditions 1.11 hold, then for every object A and for every $n \in \mathbb{N}$, the object $(CA,A)_n$ is weakly contractible.*

For the proof of this proposition, we require the following.

2.8 Proposition. *If Conditions 1.11 hold, then for each $n \in \mathbb{N}$, the map $w_n : W_n(CA,A) \to CA \times A^n$ is a weak equivalence.*

Proof. The proof is by induction on n. We first note that by Proposition 2.5 the maps w_n are cofibrations. The map $w_0 : * \to CA$ and $w_1 : CA \vee A \to CA \times A$ is a weak equivalence. Given $n > 0$, suppose that w_{n-1} is a weak equivalence. Then in diagram 2.6, with $X = CA$, the vertical arrows are weak equivalences (Condition 1.11*(we)* applies to the arrow $w_{n-1} \times A$) and the pushouts of the cotriads in the top and bottom rows are $W_n(CA, A)$ and $CA \times A^n$ respectively. The horizontal arrows pointing to the right hand side are cofibrations. The unique map $W_n(CA, A) \to CA \times A^n$ determined by push-out coincides with w_n. Hence, by the cube lemma [5, Lemma 5.2.6], w_n is a weak equivalence. $\square$

Proof of Proposition 2.7. For each $n \in \mathbb{N} \cup \{0\}$ there is a cofibration $j_n : (CA, A)_n \to (CA, A)_{n+1}$. The map $j_0 : (CA, A)_0 \to (CA, A)_1$ coincides with $* \to CA$, and is therefore a weak equivalence. Since $(CA, A)_0$ is weakly contractible, it suffices to prove that each j_n is a weak equivalence. By Proposition 2.8, the map w_{n-1} in diagram 2.2 is a weak equivalence if $X = CA$. Since the square is a push-out, it follows that j_n is a weak equivalence, completing the proof of Proposition 2.7. $\square$

3. The Main Theorem

In order to prove for an object A that A_∞ is weakly equivalent to $\Omega\Sigma A$, we need to impose certain conditions on A. Thus we shall assume throughout this section that we work with a fixed cofibration $A \to X$, and that our model category satisfies Condition 1.11. Under these conditions, by Proposition 2.5 the maps w_n are cofibrations, and we can now identify many other cofibrations, such as the maps w' of diagram 2.3 for instance.

The object obtained as the push-out of the cotriad $* \leftarrow A \to X$ is denoted by X/A and is weakly equivalent to a mapping cone of i.

Definition 3.1. For a cofibration $A \to X$, we define $V_1(X, A) = W_1(X, A)$ and, for $n \geq 2$, the object $V_n(X, A)$ so that the following square is a push-out.

(3.2)
$$\begin{tikzcd} X \times W_{n-1}(A) \arrow[r, "X \times \phi_{n-1}"] \arrow[d, "w'"'] & X \times A_{n-1} \arrow[d] \\ W_n(X, A) \arrow[r] & V_n(X, A) \end{tikzcd}$$

For the trivial cofibration $A = A$ we write $V_n(A)$ instead of $V_n(A, A)$.

Proposition 3.3. *The object P obtained in the following push-out square, is (isomorphic to) $V_n(X, A)$.*

$$\begin{array}{ccc} * \times A_{n-1} & \longrightarrow & * \times A_n \\ \downarrow & & \downarrow \\ X \times A_{n-1} & \longrightarrow & P \end{array}$$

Proof. Consider the commutative diagram below.

$$\begin{array}{ccc} * \times W_{n-1}(A) & \longrightarrow & * \times A^n \\ \downarrow & & \downarrow \\ X \times W_{n-1}(A) & \longrightarrow & W_n(X, A) \\ \downarrow & & \downarrow \\ X \times A_{n-1} & \longrightarrow & V_n(X, A) \end{array}$$

We note (comparing with 2.3) that the upper square is a push-out. The lower square is a push-out, by definition of $V_n(X, A)$. Thus the outer square is a push-out. Now we turn to the following commutative diagram.

$$\begin{array}{ccc} * \times W_{n-1}(A) & \longrightarrow & * \times A^n \\ \downarrow & & \downarrow \\ * \times A_{n-1} & \longrightarrow & * \times A_n \\ \downarrow & & \downarrow \\ X \times A_{n-1} & \longrightarrow & V_n(X, A) \end{array}$$

We have shown above that the outer square is a push-out. By the definition of the objects A_n it follows that the upper square is a push-out and hence the lower square is also. This completes the proof of Proposition 3.3. □

Theorem 3.4. *If X is an object satisfying Condition 1.11, then for any cofibration $A \to X$, the following square in which the vertical arrows are versions of ν and the horizontal arrows are the obvious maps, is a homotopy push-out square.*

$$\begin{array}{ccc} A \times A_n & \longrightarrow & X \times A_n \\ \downarrow & & \downarrow \\ A_{n+1} & \longrightarrow & (X, A)_{n+1} \end{array}$$

Proof. The morphism $X \times w_{n-1} : X \times W_{n-1}(A) \to X \times A^n$ can be factorized as follows (see 2.2):

$$X \times W_{n-1}(A) \xrightarrow{w'} W_n(X, A) \xrightarrow{w_n} X \times A^n \cdot$$

We have two commutative diagrams inducing arrows via 3.2 :

A
$$\begin{tikzcd}
X\times W_{n-1}A \arrow[r, "X\times\phi_{n-1}"] \arrow[d, "w'"'] & X\times A_{n-1} \arrow[d] \arrow[dd, bend left=60, "X\times j_n"] \\
W_n(X,A) \arrow[r] \arrow[d, "w_n"'] & V_n(X,A) \arrow[d, dotted, "v_n"'] \\
X\times A^n \arrow[r, "X\times\mu_n"'] & X\times A_n
\end{tikzcd}$$

B
$$\begin{tikzcd}
X\times W_{n-1}A \arrow[r, "X\times\phi_{n-1}"] \arrow[d, "w'"'] & X\times A_{n-1} \arrow[d] \arrow[dr, "\nu_n"] & \\
W_n(X,A) \arrow[r] \arrow[rr, bend right=15, "\phi_n"'] & V_n(X,A) \arrow[r, dotted, "\alpha_n"] & (X,A)_n
\end{tikzcd}$$

By Lemma 1.10 (see also Remark 1.12(a)) the vertical composite of the two squares in **A** is a homotopy pushout. Since the upper square is a homotopy pushout, the lower square is also a homotopy pushout. Next, considering the diagram

(3.5)
$$\begin{tikzcd}
W_n(X,A) \arrow[rr] \arrow[dr, "\phi_n"] \arrow[dd, "w_n"'] & & V_n(X,A) \arrow[dl, "\alpha_n"'] \arrow[dd, "v_n"] \\
& (X,A)_n \arrow[dd, "j_{n+1}"' near start] & \\
X\times A^n \arrow[rr, "X\times\mu_n" near end] \arrow[dr, "\mu_{n+1}"'] & & X\times A_n \arrow[dl, "\nu_{n+1}"] \\
& (X,A)_{n+1} &
\end{tikzcd}$$

we recognize the push-out square defining the object $(X,A)_{n+1}$. In view of **B** the top triangle is commutative and we may check that the remainder of the diagram is commutative. We have shown that the square at the back is a homotopy pushout. It now follows that the right front square is a homotopy pushout, since the back and left front squares are. Note that the right front square in diagram 3.5 is also the right hand face of the following commutative cube.

(3.6)
$$\begin{tikzcd}
& V_nA \arrow[rr] \arrow[dl] \arrow[dd] & & V_n(X,A) \arrow[dl, "\alpha_n"'] \arrow[dd, "v_n"] \\
A_n \arrow[rr] \arrow[dd] & & (X,A)_n \arrow[dd] & \\
& A\times A_n \arrow[rr] \arrow[dl] & & X\times A_n \arrow[dl, "\nu_{n+1}"] \\
A_{n+1} \arrow[rr] & & (X,A)_{n+1} &
\end{tikzcd}$$

In the left face of diagram 3.6 we have a similar square (for the special case $X = A$). Thus the proof will be complete if we can show that the upper face of 3.6 is a homotopy pushout (since the left and right faces are homotopy

pushout squares). This we now prove by induction.
In the case $n = 1$, the relevant square is as follows and is obviously a pushout.

$$\begin{tikzcd} W_1(A) \arrow[r] \arrow[d] & W_1(X, A) \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

So the case $n = 1$ of the theorem follows. Now assume that $n \geq 2$ and consider the following commutative diagram. By Proposition 3.3, the top left is a pushout, so that the composed square on the top row is a homotopy pushout.

$$\begin{tikzcd} * \times A_{n-1} \arrow[r] \arrow[d] & A \times A_{n-1} \arrow[r] \arrow[d] & X \times A_{n-1} \arrow[d] \\ * \times A_n \arrow[r] & V_n(A) \arrow[r] \arrow[d] & V_n(X, A) \arrow[d] \\ & A_n \arrow[r] & (X, A)_n \end{tikzcd}$$

Thus the top right square (2) is a homotopy pushout. By the inductive hypothesis the vertical composite of the right hand squares is a homotopy pushout. Thus the lower square is a homotopy pushout and the induction is complete. □

In any category, the direct limit of any sequence of maps,

$$A_1 \to A_2 \to A_3 \to \ldots,$$

is defined and it may or may not exist for a given sequence. For convenience (and a little abusively) we suppress the role of the morphisms in the sequence above, and denote the direct limit by $\lim(A_n)$ if it exists. In a model category the direct limits exist for all sequences of maps, and direct limit is functorial in the obvious way. Nevertheless, the following proposition is much more generally applicable. Its proof is simple and we omit it.

3.7 Proposition. *Suppose that (in any category) we have a sequence of pushout squares*

$$\begin{tikzcd} A_n^{(0)} \arrow[r] \arrow[d] & A_n^{(1)} \arrow[d] \\ A_n^{(2)} \arrow[r] & A_n^{(3)} \end{tikzcd},$$

and for each $i \in \{0, 1, 2, 3\}$ and $n \in \mathbb{N}$, a given map

$$f_n^{(i)} : A_n^{(i)} \to A_{n+1}^{(i)},$$

which is such that we actually have a sequence of maps of squares. If each of the four direct limits lim$(A_n^{(i)})$ *exist, then they form a pushout square.*

For our final result we need the relevant model category to satisfy the following condition on direct limits.

Condition 3.8: *Given any sequence of cofibrations as below, in which* A_n *is weakly contractible for each* $n \in \mathbb{N}$,

$$\begin{tikzcd} A_1 \arrow[r, tail] & A_2 \arrow[r, tail] & A_3 \arrow[r, tail] & \cdots \end{tikzcd}$$

then $\lim(A_n)$ *is weakly contractible.*

The limit of the sequence, $(X,A)_1 \to (X,A)_2 \to (X,A)_3 \to \ldots$, we denote by $(X,A)_\infty$, and the object $(X,X)_\infty$ is denoted by X_∞.

3.9 Theorem. *Suppose that* $\mathbf{C}$ *satisfies the cube axiom (and Conditions 1.11). Then for any cofibration* $A \to X$, *there is a map* $f_\infty : (X,A)_\infty \to X/A$ *having* A_∞ *as its homotopy fibre.*

Proof. For the following commutative diagram we form the pushouts of the cotriads in the top row and in the bottom row. Condition 1.11*(cof)* ensures that $A \times A_n \to X \times A_n$ is a cofibration.

$$\begin{tikzcd}
A_{n+1} \arrow[d] & A \times A_n \arrow[l, "\mu_{n+1}"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\mathrm{proj}"'] & X \times A_n \arrow[d, "\mathrm{proj}"] \\
* & A \arrow[l] \arrow[r] & X
\end{tikzcd}$$

By Theorem 3.4, the object obtained as the pushout of the upper cotriad is $(X,A)_{n+1}$. The object obtained as the pushout of the cotriad in the bottom cotriad is X/A. Then there exists a unique map $f_{n+1} : (X,A)_{n+1} \to X/A$, completing a commutative cube diagram. We have a sequence (indexed by n) of such cubes, and the limit of this sequence is a cube as below:

$$\begin{tikzcd}
& A \times A_\infty \arrow[rr, "\mathrm{incl}"] \arrow[dl, "\mu_\infty"'] \arrow[dd] & & X \times A_\infty \arrow[dl] \arrow[dd] \\
A_\infty \arrow[rr] \arrow[dd] & & (X,A)_\infty \arrow[dd, "f_\infty"] & \\
& A \arrow[rr] \arrow[dl] & & X \arrow[dl] \\
* \arrow[rr] & & X/A &
\end{tikzcd}$$

The proof is completed through application of Proposition 3.7, Condition 3.8, and the Cube Axiom 1.7. □

3.10 Corollary. *Suppose that* **C** *satisfies the cube axiom and Condition 3.8 (together with Condition 1.11).*
Then for any object A, the homotopy fibre of the initial morphism $* \to \Sigma A$ is A_∞, i.e., A_∞ is weakly equivalent to $\Omega\Sigma A$.

Proof. This is deduced from Theorem 3.9, using Proposition 2.7 and Condition 3.8. □

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