Productivity numbers in topological and topological linear spaces

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Dedicated to Nico Pumplün, on the occasion of his unbelievable birthday.

Abstract: It is shown that productivity numbers of coreflective subcategories of topological linear spaces are precisely submeasurable cardinals (unlike locally convex spaces, where such numbers are measurable). A similar result is expected in topological spaces (only partial results are given here).
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1 Introduction

In my contribution to a similar (but 5 years younger) Festschrift ([7]), a survey of known results about productivity numbers of coreflective subcategories of various categories was given. One of the results asserts that those numbers in the category of topological linear spaces (shortly `TLS`) are submeasurable cardinals. Although that result is said to have been published in [6], only a special case was proved there, namely the case for the first sequential cardinal. The original idea of the author was that the published special proof can be almost automatically transferred to higher cardinals. But it is not the case and we shall provide a full general proof here.

A similar situation in the category of topological spaces (shortly `Top`) has not yet been completely solved. Nevertheless, a progress was done and we shall describe the nowadays situation.

We shall now briefly recall some concepts and terminology.

Every subcategory will be full and so it suffices to speak about subclasses of objects instead of subcategories. In the category `TLS` of topological linear spaces over $\mathbb{R}$ and continuous linear maps. we shall always assume that our coreflective classes $\mathcal{C}$ contain $\mathbb{R}$ or, equivalently, that $\mathcal{C}$ are bicoreflective. Bicoreflectivity means that the coreflective maps are linear isomorphisms, i.e., that for every space $X \in$ `TLS` there exists a finer space cX belonging to $\mathcal{C}$ such that every continuous linear mapping from a space in $\mathcal{C}$ to X is continuous already into the finer space cX. Equivalently, $\mathcal{C}$ are closed under inductive generation, i.e., under quotients, direct sums (all in `TLS`) and contain the finest topological linear spaces. Every class of spaces from `TLS` has a coreflective hull in `TLS`.

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Since finite products coincide with finite direct sums, every coreflective class is finitely productive (in TLS). We say that a subclass $\mathcal{C}$ of a category $\mathcal{K}$ is *κ-productive* if every product (in $\mathcal{K}$) of less than κ members of $\mathcal{C}$ belongs to $\mathcal{C}$; finite (or countable) productivity is another expression for ω- (or ω_1- , resp.) productivity. *Productivity number* of a subclass $\mathcal{C}$ of $\mathcal{K}$ is the smallest cardinal κ (if it exists) such that a product in $\mathcal{K}$ of κ-many objects from $\mathcal{C}$ does not belong to $\mathcal{C}$, otherwise it is a symbol ∞ that we consider to be bigger than any cardinal in this case. Productivity number of $\mathcal{C}$ will be denoted by $p_{\mathcal{C}}$ or just p (thus, $\mathcal{C}$ is $p_{\mathcal{C}}$-productive). Very often (and it is our case of coreflective classes in TLS or Top) one may take powers of a single space instead of products of spaces in the definition of productivity numbers (take the sum of the coordinate spaces and realize that the original product is a retract of the power of the sum). In the case of TLS, productivity numbers p have one more property, namely no product of at least p many spaces from $\mathcal{C}$ having nontrivial separated modifications belongs to $\mathcal{C}$; that follows from an important result of P. and S.Dierolf [2] that a coreflective class $\mathcal{C}$ is κ-productive in TLS iff $\mathbb{R}^\lambda \in \mathcal{C}$ for all $\lambda < \kappa$.

In [6] we have proved that productivity numbers of coreflective subclasses of LCS are measurable cardinals and that the productivity number of a countably productive coreflective class in TLS is sequential or ∞ if sequential cardinals do not exist.

Let us recall that sequential cardinal is a cardinal κ such that there exists a sequentially continuous noncontinuous real-valued map on the Cantor space 2^κ. Those cardinals were dealt with in the classical Mazur's paper [8] and in [10]. Mazur showed that the first sequential cardinal is weakly inaccessible and that every sequentially continuous map on a product of less than κ-many metrizable separable spaces into a metrizable space is continuous (even a little more general spaces can be used). Noble [10] generalized the class of metrizable separable spaces used in the last mentioned result to a bigger class including first countable spaces.

The first sequential cardinal has its continuation in a hierarchy of similar cardinals. To define it, we must start with some basic concepts.

A *submeasure* μ on an algebra $\mathcal{B}$ of sets is a real-valued mapping defined on $\mathcal{B}$ and having the following properties:

$\mu(\emptyset) = 0$,
$\mu(A) \leq \mu(B)$ for $A \subset B, A, B \in \mathcal{B}$,
$\mu(A \cup B) \leq \mu(A) + \mu(B)$ for $A, B \in \mathcal{B}$.

In the last property we may assume that A, B are disjoint.

A submeasure μ on κ is said to be *λ-subadditive* if $\mu(\bigcup_\tau A_\alpha) \leq \sum_\tau \mu(A_\alpha)$, whenever $\tau < \lambda$ and $\{A_\alpha\}$ is a (disjoint) family in $\mathcal{B}$.

A mapping f between topological spaces is said to be *τ-continuous*, for a cardinal $\tau > \omega$, if it preserves limits of nets of lengths less than τ; f is *monotonically τ-continuous* if it preserves limits of well-ordered nets of lengths less than τ (by length of a net we mean cardinality of the index set of the net, i.e., of the domain of the net). In our case we shall use the above kind of continuity in algebras of subsets of a set. If not said otherwise, we shall always have in mind the order–convergence, i.e., a net $\{A_i\}$ converges to A if $\limsup\{A_i\} = \liminf\{A_i\} = A$. Using this convergence, it is not difficult to show that a submeasure is (monotonically) τ-continuous on a τ-complete algebra if it is (monotonically) τ-continuous at 0. For instance, it is monotonically τ-continuous iff $\mu(A_\alpha) \to 0$ for $\{A_\alpha\}_{\alpha<\lambda} \searrow 0$, where $\lambda < \tau$.

A submeasure μ on $\mathcal{B}$ is said to be *τ-additive on null sets* if $\mu(\bigcup_\lambda A_\alpha) = 0$ provided $\lambda < \tau$ and $\mu(A_\alpha) = 0$ for every $\alpha < \lambda$ (one may assume that $\{A_\alpha\}$ is a disjoint system). We should realize that every monotonically τ-continuous submeasure is τ-subadditive and thus is τ-additive on null sets. But τ-subadditivity does not imply monotonical τ-continuity. If we add sequential continuity to τ-subadditivity, we already get the τ-continuity: *For $\tau > \omega$, a submeasure on a τ-complete B is monotonically τ-continuous iff it is sequentially continuous and τ-additive on null sets.*

Definition 1 An infinite cardinal κ is said to be *submeasurable* if there exists a nonzero κ-continuous submeasure on the algebra of all subsets of κ vanishing at singletons.

The first submeasurable cardinal equals to ω and the second one coincides with the first sequential cardinal (if it exists – see [6]).

In case of $\tau = \omega_1$, the τ-continuity and monotonical τ-continuity coincide (with sequential continuity). It is not the case for higher cardinals τ in general topological spaces. But for submeasures both types of continuity coincide, so we could use *monotone κ-continuity* in the previous definition. Other characterizations of submeasurable cardinals from [6], [4] and [1]: *a cardinal κ is submeasurable iff there is a noncontinuous map $g : 2^\kappa \to \mathbb{R}$ (or $g : \mathbb{N}^\kappa \to \mathbb{R}$) that is κ-continuous, or iff there is a noncontinuous homomorphism $g : \mathbb{Z}_2^\kappa \to G$ (or $g : \mathbb{Z}^\kappa \to G$) into a topological group G that is κ-continuous* (instead of homomorphisms into G we can use group–pseudonorms into $\mathbb{R}$, and again monotone κ-continuity instead of κ-continuity). Later, we shall show modified characterizations in `TLS`.

2 Topological linear spaces

We shall now prove that submeasurable cardinals coincide with productivity numbers of coreflective non-productive classes in `TLS`.

Theorem 2 *Productivity numbers of coreflective subcategories in* `TLS` *are submeasurable cardinals or ∞.*

Proof: Let $\mathcal{C}$ be a coreflective class in `TLS` and κ be its productivity number. We may suppose that $\omega < \kappa < \infty$. We know that there is a non-continuous linear map $f : \mathbb{R}^\kappa \to E$, where E is a Fréchet space, such that f is continuous on the coreflection $c\mathbb{R}^\kappa$ of $\mathbb{R}^\kappa$ in $\mathcal{C}$. It was proved by Noble in [10] that a map f on a product of spaces into a regular space is continuous iff its restrictions to any canonical image of 2^κ and to a Σ-product (in fact, σ-product suffices) are continuous. Thus, in our case, either a restriction to a canonical image of 2^κ is not continuous, or the restriction to the σ-product Y of κ-copies of $\mathbb{R}$ at the point 0 is not continuous. We shall show that the latter case is not possible. Otherwise there is an $\varepsilon > 0$ such that every neighborhood of 0 in Y contains a point x with $|f(x)| \geq \varepsilon$ (by $|.|$ we denote the distance to 0 in E). Start with a finite set C_0 and find a point $x_0 \in \mathbb{R}^\kappa$ having a finite support C_1 such that $|f(x_0)| \geq \varepsilon$ and $|\mathrm{pr}_i(x_0)| < 1$ for $i \in C_0$. Next, we can find a point

x_1 having a finite support C_2 such that $|f(x_1)| \geq \varepsilon$ and $|\mathrm{pr}_i(x_1)| < 1/2$ for $i \in C_1$. Continuing in the same way, we get a sequence $\{x_n\}$ of points of Y having supports C_n and such that $|f(x_n)| \geq \varepsilon$ and $|\mathrm{pr}_i(x_n)| < 1/(n+1)$ for each $i \in C_{n-1}$ and each n. Denote by S the union of all C_n's. Thus, S is countable and $\mathbb{R}^S$ canonically embeds into $\mathbb{R}^\kappa$. The image contains all points x_n. Since the restriction of f to that canonical image must be continuous (because $\mathbb{R}^\omega$ belongs to $\mathcal{C}$) and $x_n \to 0$ we have $f(x_n) \to 0$, which contradicts our assumption.

It follows that f is not continuous on some canonical image of 2^κ into $\mathbb{R}^\kappa$. We remind that the canonical image means in our case that there is a point $\{r_\alpha\} \in \mathbb{R}^\kappa$ and a map $\psi = \Pi_\kappa \psi_\alpha : 2^\kappa \to \mathbb{R}^\kappa$ with $\psi_\alpha(0) = 0, \psi_\alpha(1) = r_\alpha$. It remains to show that the composition $f\psi$ is κ-continuous. To simplify notation we shall forget about ψ and shall assume that 2^κ is a part of $\mathbb{R}^\kappa$ – equivalently, that all $r_\alpha \neq 0$. Take some $\lambda < \kappa$ and a net $\{x_\alpha\}_{\alpha\in\lambda}$ in 2^κ converging to 0. Suppose that $|f(x_\alpha)| \geq \varepsilon$ for all $\alpha \in \lambda$ and some $\varepsilon > 0$. For every coordinate $\beta \in \kappa$, the net $\{\mathrm{pr}_\beta(x_\alpha)\}_\alpha$ is eventually 0 starting with an index α_β. Therefore, the set κ decomposes into λ many sets $B_\alpha = \{\beta \in \kappa : \alpha_\beta = \alpha\}$ (some of the sets B_α may be empty).

For $y \in 2^\kappa$ we may take a special net $\{y^\alpha\}_{\alpha\in\lambda}$ in 2^κ converging to y, namely $\mathrm{pr}_{B_\gamma}(y^\alpha) = \mathrm{pr}_{B_\gamma}(y)$ if $\gamma \leq \alpha$ and $\mathrm{pr}_{B_\gamma}(y^\alpha) = 0$ otherwise. If $\{f(y^\alpha)\}$ does not converge to $f(y)$ for some y, we may assume in our case that $x_\alpha = y - y^\alpha$. In this case we may define a linear continuous map $h = \Pi h_\alpha : \mathbb{R}^\lambda \to \Pi_{\alpha\in\lambda}\mathbb{R}^{B_\alpha}$ with $h_\alpha(1) = \mathrm{pr}_{B_\alpha}(y)$. Then the net $\{\chi_{\lambda\setminus\alpha}\}_{\alpha\in\lambda}$ converges to 0 in $\mathbb{R}^\lambda$ and its h-image is the net $\{x_\alpha\}$, which contradicts our assumption that $f(x_\alpha)$ does not converge to 0.

So, we may assume that $f(y^\alpha)$ converges to $f(y)$ for every y. It follows that we may substitute the original points x_α by x_α^γ for some γ (depending on α) and take $\varepsilon/2$ instead of ε. It follows we may assume that x_α is 0 on $\bigcup\{B_\delta : \delta > \gamma\}$ for some γ. On the other hand, for every $\gamma \in \lambda$ the restriction of the net $\{x_\alpha\}$ to $\bigcup\{B_\delta : \delta \leq \gamma\}$ is eventually 0. Consequently, we may choose a cofinal set S in λ such that the supports of x_α's are disjoint for different $\alpha \in S$. Now it is easy to follow the construction of h from the previous paragraph and to get a contradiction.

Hence, the mapping f on our 2^κ is κ-continuous and not continuous, which means that κ is a submeasurable cardinal. $\diamond$

The next result shows that every submeasurable cardinal is attained as a productivity number of a coreflective class in `TLS`. Our procedure is a modification of the proof of Theorem 3 from [6].

Theorem 3 *For every submesurable cardinal κ there is a coreflective class $\mathcal{C}$ in* `TLS` *having κ for its productivity number.*

Proof: Take a κ-continuous submeasure μ on κ vanishing at singletons and having $\mu(\kappa) = 1$. Denote by E the linear space $\mathbb{R}^\kappa$ endowed with the metric

$$d(x, y) = \int \frac{|x - y|}{1 + |x - y|}\, d\mu\,.$$

Then (E, d) is a topological linear space and the identity mapping $f : \mathbb{R}^\kappa \to E$ is not continuous since the net $\{\chi_K\}_{K\in[\kappa]^{<\omega}}$ of characteristic functions of finite sets converges to χ_κ in $\mathbb{R}^\kappa$ but $d(\chi_K, \chi_\kappa) = 1/2$ for every finite set K. We shall now prove that fg is continuous for every continuous linear map $g : \mathbb{R}^\lambda \to \mathbb{R}^\kappa, \lambda < \kappa$.

Take a net $\{x_i\}_I$ converging to 0 in $\mathbb{R}^\lambda$. Because of weight of $\mathbb{R}^\lambda$ we may assume that $|I| < \kappa$. Choose $\varepsilon > 0$ and define

$$A_i = \{\beta \in \kappa : |pr_\beta(g(x_j))| < \varepsilon/2 \text{ for } j > i\}.$$

Then $\{A_i\}_I$ is a cover of κ and $A_i \subset A_j$ for $i < j$. Since μ is κ-continuous, there is some $i_0 \in I$ such that $\mu(\kappa \setminus A_{i_0}) < \varepsilon/2$. Then

$$d(g(x_j), 0) \leq \int_{A_{i_0}} \frac{|g(x_j)|}{1 + |g(x_j)|}\, d\mu + \int_{\kappa \setminus A_{i_0}} \frac{|g(x_j)|}{1 + |g(x_j)|}\, d\mu \leq \varepsilon/2 + \varepsilon/2$$

for $j > i_0$. Consequently, $g(x_i)$ converges to 0 in (E, d).

We have proved that the coreflective hull $\mathcal{C}$ of $\{\mathbb{R}^\lambda\}_{\lambda<\kappa}$ does not contain $\mathbb{R}^\kappa$ and, thus, its productivity number equals to κ. $\diamond$

Since the coreflective hull (in TLS) of all powers of reals is productive, we have the following result.

Corollary 4 *The class of productivity numbers of coreflective classes in* TLS *coincides with the class consisting of all of submeasurable cardinals and of* ∞.

As an interesting result we shall prove now the following modification of characterizations of submeasurable cardinals by group homomorphisms on $\mathbb{Z}^\kappa$. By a pseudonorm p on a topological linear space E we mean the distance to 0 of points of E for a translation invariant metric on E. Clearly, compositions of pseudonorms with linear maps are pseudonorms and, conversely, every pseudonorm is a composition of a linear map into a metrizable topological linear space F (or a Fréchet space) and the canonical pseudonorm of F.

Proposition 5 *The following conditions are equivalent for an infinite cardinal* κ*:* 1. κ *is submeasurable;*

2. *there is a noncontinuous linear map on* $\mathbb{R}^\kappa$ *into a topological linear space that is* κ*-continuous;*

3. *there is a noncontinuous pseudonorm on* $\mathbb{R}^\kappa$ *that is* κ*-continuous;*

4. *there is a noncontinuous real-valued function on* $\mathbb{R}^\kappa$ *that is* κ*-continuous.*

Proof: Clearly, $2 \Rightarrow 3 \Rightarrow 4$. The implication $1 \Rightarrow 2$ is shown in the proof of Theorem 3. It remains to prove $4 \Rightarrow 1$. Assuming 4, we shall show that there is a noncontinuous, κ-continuous real-valued function on $\mathbb{Z}^\kappa$.

Let $f : \mathbb{R}^\kappa \to \mathbb{R}$ be a κ-continuous non-continuous function. We want to find a continuous map $h : \mathbb{Z}^\kappa \to \mathbb{R}^\kappa$ such that the composition $fh : \mathbb{Z}^\kappa \to \mathbb{R}$ is κ-continuous and non-continuous. By a classical result $\mathbb{R}$ is a continuous image of $\mathbb{Z}^\omega$ (see, e.g., [9]) by a mapping g, and the composition fg^κ is thus κ-continuous. It remains to show that we can find g in such a way that fg^κ is not continuous. Since f is not continuous, there is a net $\{x_i\}_I$ in $\mathbb{R}^\kappa$ converging to some point x such that $f(x_i)$ does not converge to $f(x)$. Clearly, we may assume that $x = 0$ So, it suffices to find g such that for every net $\{a_i\}_I$ in $\mathbb{R}$ converging to 0 there is a net $\{b_i\}_I$ in $\mathbb{Z}^\omega$ converging to 0 and $g(b_i) = a_i$ for every $i \in I$.

Take the usual continuous map m_1 of $\{-1, 0, 1\}^\omega$ onto $[-1, 1]$ assigning to $\{c_n\}$ the point $\sum c_n/2_n\}$. The subspace P of $\mathbb{Z}^\omega$ of all sequences having 2 for their first coordinate is homeomorphic to the space of irrationals and, thus, there is a continuous map m_2 of P onto $\mathbb{R}$. The space $\{-1,0,1\}^\omega \cup P$ is a closed subspace of $\mathbb{Z}^\omega$ and the mapping $m_1 \cup m_2$ can be extended continuously to a map $g : \mathbb{Z}^\omega \to \mathbb{R}$. Now, every net $\{a_i\}_I$ converging to 0 in $\mathbb{R}$ must be eventually in $[-1,1]$. For those points a_i belonging to $[-1,1]$ we find b_i with $m_1(b_i) = a_i$ and for the remaining points a_i we take arbitrary points b_i with $g(b_i) = a_i$. Then b_i converges to 0 and we are done. $\diamond$

3 Topological spaces

First we repeat some basic facts. The problem whether there is a productive coreflective nontrivial subcategory of `Top` was posed in 1978 in [5]. At a conference in Oxford in 1989, Dow and Watson gave some partial solutions of that problem and they published them in [3]. We shall mention here some of their results:

1. If GCH holds and there are no inaccessible cardinals, there is no productive coreflective nontrivial subcategory of `Top`.

2. If $\square(\lambda)$ holds for all regular uncountable cardinals λ then there is no productive coreflective nontrivial subcategory of `Top`.

3. The existence of a productive coreflective nontrivial subcategory of `Top` implies consistency of a weakly compact cardinal.

4. If real-measurable cardinals exist then there is a finitely productive coreflective nontrivial subcategory of `Top` containing the topological ordered space $\omega_0 + 1$.

As far as I know, the original problem is still open in ZFC. To get closer to a solution, it seemed convenient to consider how much productive are coreflective classes. This was a start for the results about productivity numbers in topological categories. Although in the cases of topological linear spaces, topological groups, uniform spaces, and in their convenient subcategories, productivity numers are fully described now, the original case of topological spaces has not yet been solved. The last mentioned result of Dow and Watson suggests that there may be connections to measurability in `Top`, too. We shall modify the example from [3] to get that every submeasurable cardinal is a productivity number of some coreflective class in `Top`. It is not clear at all that the converse holds, too. Maybe, if one restricts to coreflective classes of completely regular spaces, there is a chance to show that their productivity numbers are submeasurable cardinals (or 2, or ∞).

Theorem 6 *Every submeasurable cardinal is a productivity number of some coreflective class in* `Top`.

Proof: Let κ be a submeasurable cardinal and μ be a corresponding submeasure on κ with $\mu(\kappa) = 1$. For a family $\mathcal{G} = \{G_\alpha : \alpha \in \kappa\}$ of open sets in a topological space X and for $r > 0$ define

$$\mathcal{G}_r = \{x \in X : \mu\{\alpha : x \notin G_\alpha\} < r\}.$$

Of course, $\mathcal{G}_r = X$ for $r > 1$ and so it suffices to take $r \in]0,1]$. Now we can define our main class

$$\mathcal{C} = \{X : \mathcal{G}_r \text{ is open for every } \mathcal{G} \text{ and every } r > 0\}.$$

We shall prove that $\mathcal{C}$ is reflective in $\mathtt{Top}$ that is κ-productive and not κ^+-productive. It is clear that $\mathcal{C}$ contains all discrete spaces and is closed under disjoint sums.

1. *$\mathcal{C}$ is closed under quotients.* Take a quotient $f : X \to Y$, where $X \in \mathcal{C}$, and a family $\mathcal{G} = \{G_\alpha : \alpha \in \kappa\}$ of open sets in Y. The family $\mathcal{H} = \{f^{-1}(G_\alpha) : \alpha \in \kappa\}$ of open sets in X has all the required $\mathcal{H}_r$ open. It suffices to show that $\mathcal{H}_r = f^{-1}(\mathcal{G}_r)$, which is easy.

2. *$\mathcal{C}$ is finitely productive.* Take $X, Y \in \mathcal{C}$, an open family $\mathcal{G} = \{G_\alpha\}_\kappa$ in $X \times Y$ and $r \in]0,1]$. Choose $(x_0, y_0) \in \mathcal{G}_r$ and denote $A = \{\alpha : (x_0, y_0) \in G_\alpha\}$ (thus $\mu(\kappa \setminus A) < r$). If $\alpha \in A$, then $(x_0, y_0) \in G_\alpha$ and, hence, there are open sets U_α, V_α in X, Y resp., such that $(x_0, y_0) \in U_\alpha \times V_\alpha \subset G_\alpha$. If $\alpha \notin A$ we put $U_\alpha = X, V_\alpha = Y$. We shall show that for some $\varepsilon > 0$, the open set $\mathcal{U}_\varepsilon \times \mathcal{V}_\varepsilon$ contains (x_0, y_0) and is contained in $\mathcal{G}_r$. It will suffice to take $\varepsilon > 0$ such that $\mu(\kappa \setminus A) + 2\varepsilon < r$.

Clearly, $(x_0, y_0) \in \mathcal{U}_t \times \mathcal{V}_t$ for every $t > 0$. Take now $(x, y) \in \mathcal{U}_\varepsilon \times \mathcal{V}_\varepsilon$; we want to show that $\mu\{\alpha : (x, y) \notin G_\alpha\} < r$. Our assumption implies that $\mu\{\alpha : x \notin U_\alpha\} < \varepsilon, \mu\{\alpha : y \notin V_\alpha\} < \varepsilon$. Since

$$\{\alpha : (x,y) \notin G_\alpha\} \subset \kappa \setminus A \cup \{\alpha \in A : (x,y) \notin U_\alpha \times V_\alpha\} \subset$$

$$\kappa \setminus A \cup \{\alpha \in A : x \notin U_\alpha\} \cup \{\alpha \in A : x \notin V_\alpha\},$$

we have $\mu\{\alpha : (x, y) \notin G_\alpha\} \le \mu(\kappa \setminus A) + 2\varepsilon < r$, which was to prove.

3. *$\mathcal{C}$ is κ-productive.* Take $X \in \mathcal{C}$, an open family $\mathcal{G} = \{G_\alpha\}_\kappa$ in X^λ for a $\lambda < \kappa$, and $r \in]0,1]$. Choose $x \in \mathcal{G}_r$ (thus $\mu(\kappa \setminus A) < r$, where $A = \{\alpha : x \in G_\alpha\}$). For a finite subset F of λ denote $A_F = \{\alpha : x \in G_\alpha$ and there exists a canonical open neighborhood $U_\alpha \subset G_\alpha$ of x in X^λ with $\mathrm{pr}_{\lambda \setminus F}(U_\alpha) = X^{\lambda \setminus F}$. Clearly, the net $\{A \setminus A_F : F \in [\lambda]^{<\omega}\}$ converges to $\emptyset$ and since μ is κ-continuous, there is a finite F such that $\mu(A \setminus A_F) < \varepsilon$, where we choose ε in such a way that $\mu(\kappa \setminus A) + 2\varepsilon < r$. For $\alpha \in A_F$ take the corresponding neighborhood U_α from the definition of A_F and define $V_\alpha = \mathrm{pr}_F(U_\alpha)$; for $\alpha \notin A_F$ define $V_\alpha = X^F$. We know from the previous part 2 that $X^F \in \mathcal{C}$ and so, the sets $\{V_\alpha\}_s$ are open for every $s > 0$. Moreover, $\mathrm{pr}_F(x) \in \{V_\alpha\}_s$ for every $s > 0$. Take now $y \in X^\lambda$ with $\mathrm{pr}_F(y) \in \{V_\alpha\}_\varepsilon$. Since

$$\{\alpha : y \notin G_\alpha\} \subset \{\alpha : x \notin G_\alpha\} \cup \{\alpha : x \in G_\alpha, \mathrm{pr}_F(y) \notin V_\alpha\} \subset$$

$$\{\kappa \setminus A\} \cup (A \setminus A_F) \cup \{\alpha \in A_F : \mathrm{pr}_F(y) \notin V_\alpha\}$$

and, thus, $\mu\{\alpha : y \notin G_\alpha\} \le \mu(\kappa \setminus A) + \varepsilon + \varepsilon < r$. Consequently, $\mathrm{pr}_F^{-1}(\{V_\alpha\}_\varepsilon)$ is a neighborhood of x contained in $\mathcal{G}_r$.

4. *$\mathcal{C}$ is not κ^+-productive.* We shall show that $2^\kappa \notin \mathcal{C}$. For $\alpha \in \kappa$ take $G_\alpha = \mathrm{pr}_\alpha^{-1}(0)$. Clearly, $0 \in \mathcal{G}_r$ for every $r > 0$. It suffices to prove that no neighborhood of 0 is contained in $\mathcal{G}_{1/2}$. Take a finite set $F \subset \kappa$ and the basic neighborhood $U_F = \{x \in 2^\kappa : \mathrm{pr}_F(x) = 0\}$ of 0. If $y \in 2^\kappa$ then $\{\alpha : y \notin G_\alpha\} = \{\alpha : \mathrm{pr}_\alpha(y) = 1\}$. So, if we take for $y \in U_F$ the characteristic function of $\kappa \setminus F$, we have $\mu\{\alpha : y \notin G_\alpha\} = 1$ and, hence $y \notin \mathcal{G}_{1/2}$. $\diamond$

When we try to prove that productivity number of a coreflective class $\mathcal{C}$ in $\mathtt{Top}$ is submeasurable, we meet several difficulties. If κ is the productivity number of $\mathcal{C}$, there is a noncontinuous mapping f of a power X^κ (for some

$X \in \mathcal{C}$) into a topological space Y that is continuous on the coreflection of X^κ in $\mathcal{C}$. To prove that κ is submeasurable, we need Y to be $\mathbb{R}$ or its subspace and, also, we need a nice space X. If we work in completely regular spaces, then we can choose $\mathbb{R}$ as the space Y. Noble's result from [10] asserts that in that case the mapping f is not continuous either on 2^κ (and then we are done) or on a Σ-product of X's. One can prove that the latter case cannot occur (as we did in Theorem 2) for spaces X with characters less than κ only. How to proceed for X having large characters is not clear now.

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