VECTOR BUNDLES OVER CURVES OF GENUS ONE AND ARBITRARY INDEX

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Abstract Vector bundles on curves of genus one and arbitrary index are investigated and classified, extending Atiayh's results on curves of genus one over algebraically closed fields. Absolutely indecomposable bundles have certain admissible slopes which depend on the index of the curve.

Introduction

There exist only few examples of algebraic varieties over fields where vector bundles are classified explicitly. The (absolutely) indecomposable vector bundles on elliptic curves over algebraically closed fields are determined by Atiyah [At] (cf. [At], Theorem 7 for the main result) in 1957. Using methods of descent, his results are extended by A. Tillman [T] in 1983, who classifies the indecomposable vector bundles on elliptic curves with rational points over perfect base fields. The (unpublished) results of her thesis on indecomposable vector bundles on curves, are generalized and extended to arbitrary proper algebraic schemes by Arason, Elman and Jacob [AEJ1] in 1994. One of the main theorems shows that indecomposable vector bundles on a proper scheme over a field k, can be realized as traces of absolutely indecomposable vector bundles on the scheme, obtained by extending the base field from k to a maximal subfield of the reduced endomorphism ring of the bundle ([AEJ1], 1.8).

An indecomposable vector bundle need not be indecomposable any more after an extension of the base field. This phenomenon is studied both in [T] and [AEJ1]. Their approach relies on the assumption that the vector bundles on the corresponding scheme $X:= X \times_k \overline{k}$ have been classified already, where $\overline{k}$ denotes the algebraic closure of k. Therefore, all of the above use methods of descent, i.e., change bases in their proofs, and rely on Atiyah's classification of the indecomposable vector bundles over the elliptic curve $\overline{X}$, to classify the indecomposable vector bundles in the special case where X is a curve of genus 1. In this context, it makes sense to only study curves of genus 1 over perfect fields which have rational points, that means are of index 1.

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Here, a different approach to study the indecomposable vector bundles on a curve of genus 1 is presented. It is possible to obtain classification results for absolutely indecomposable vector bundles on a curve of genus 1 by directly adapting Atiyah's original proofs to a more general setting. We are able to omit the assumption that the considered curve needs to have rational points, and often also that its base field needs to be perfect. This shows how canonical Atiyah's methods of proof are. Hence we obtain classification results for vector bundles on any curve of genus 1. These are the basis for a quick overview on results on symmetric bilinear forms [Pu1] generalized from [AEJ1, 2] and a comprehensive study of quaternion algebras over curves of genus 1 and arbitrary index [Pu2].

The paper is organized as follows. After some preliminaries, section 2 deals with splittings of vector bundles on a curve, introduced in [At] for curves over algebraically closed fields. We investigate *maximal* splittings for bundles over arbitrary base fields (2.2), adapting [At], Part I, Section 4. This yields a generalization of [At], Theorem 1, which is a refinement of Serre's Theorems A and B for indecomposable vector bundles (2.13). In section 3, maximal splittings are used to understand absolutely indecomposable vector bundles on curves of genus 1 and arbitrary index, over an arbitrary base field k. The evaluation map $\alpha\colon H^0(X,\mathcal{E})\otimes\mathcal{O}_X\to\mathcal{E}$ contains information about the behaviour of an indecomposable bundle $\mathcal{E}$ under base changes (cf. 3.2, 3.7). We prove a generalization of [At], Theorem 5 (i) by adapting Atiyah's original method of proof to our more general setting without using descent: For any curve of genus 1, there exists an absolutely indecomposable bundle $\mathcal{F}_r$ of rank r and degree 0 on X, with nontrivial global sections, which is unique up to isomorphism (3.12). Moreover, the absolutely indecomposable bundles of rank r and degree 0 are classified (3.13), in particular the selfdual ones among these (3.14). Atiyah's thoughts about the structure of the subring of the set of isomorphism classes of vector bundles on X which is generated by the classes of these $\mathcal{F}_r$, $r\geq 1$, carry over without change provided that $H^0(\mathrm{Pic}^0(X^{sep}))\cong\mathrm{Pic}^0X$ holds (see 3.17, 3.18). In section 4, an algorithm inspired by Atiyah's inductive definition of the bijection he constructs in the proof of [At], Theorem 6 is developped: (For the special case $i=1$ it is the classical euclidean algorithm with a "little twist".) A pair (r,d) with $r\geq 1$ is called *admissible* with respect to an integer $i\geq 1$ if the algorithm stops with a pair $(h,0)$. By construction, h is the greatest common divisor of r and d. This algorithm is applied to classify absolutely indecomposable vector bundles over curves of genus 1 and index i, which have rank r and degree d, where (r,d) is an admissible pair with respect to i (4.1, 4.2, 4.3, 4.4). As a consequence, the determinant is a bijective map from the set of isomorphism classes of absolutely indecomposable vector bundles on X of rank r and degree d to the set of line bundles of degree d, for any admissible pair (r,d), with r and d coprime (4.7). Furthermore, ([At], Corollary to Theorem 7) does not generalize to curves over fields that are not algebraically closed. A weaker result is obtained instead (4.8 and 4.9). For curves of genus 1 with index greater than 1, a G-invariant isomorphism class of an indecomposable vector bundle is not necessarily defined over X where $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ is the Galois group of the separable closure k^{sep} over k. (This holds, if X has index 1, see [T], 6.9). Such a G-invariant isomorphism class is defined over X in the general case as well, provided that the vector bundle has an admissible slope with respect to the index of X and provided that $H^0(\mathrm{Pic}^0(X^{sep})) \cong \mathrm{Pic}^0 X$ holds (4.11). For a perfect base field, $D(\mathcal{M}) = \mathrm{End}(\mathcal{M})/\mathrm{rad}(\mathrm{End}(\mathcal{M}))$ is a field, for an indecomposable vector bundle $\mathcal{M}$ on X, if one of the conditions in (4.12) hold. We show when there indeed are no absolutely indecomposable vector bundles on a curve of genus 1 with non-admissible rank and degree (r, d) with respect to the index (4.15). If X is a curve over a perfect field satisfying $H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$, for any admissible pair (r, d) with respect to the index of X, there is a bijection between the set of all isomorphism classes of indecomposable vector bundles of fixed rank r and degree d on X, and the set of (closed) points on E, whose degree divides r (4.16). In section 5, the structure of absolutely indecomposable vector bundles of rank r and degree d, with r and d coprime, is investigated. Generalizing lemmata in [At], Part III.2. we obtain a description of vector bundles on X when $k = \mathbb{R}$. Their rank and degree need not be coprime anymore, but have to be an admissible pair with respect to the index of X (5.8). This adapts [At], Theorem 10.

Throughout the paper, R is a commutative associative ring with a unit element, and k a field. We use the standard terminology of algebraic geometry from Hartshorne [H]. Some results and terminology from Arason, Elman and Jacob [AEJ1] are used. Most facts about elliptic curves used here can be found in Silverman's book [Si].

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1. Generalities

Let X be an integral noetherian regular scheme of dimension 1 with generic point ξ, and $\mathcal{E}$ a vector bundle on X of rank r. Write $\underset{\sim}{K}$ for the constant sheaf $K := k(X) = \mathcal{O}_{\xi,X}$ on X, and $\underset{\sim}{V} = \underset{\sim}{V}(\mathcal{E})$ for the constant sheaf $V := V(\mathcal{E}) := \mathcal{E}_\xi$ on X. We identify $\mathcal{O}_X \subset \underset{\sim}{K}$ and $\mathcal{E} \subset \underset{\sim}{V}$ via the canonical monomorphisms, and also $\mathcal{E}^\vee \subset \underset{\sim}{V}(\mathcal{E}^\vee) = ([V(\mathcal{E})]^\vee)_\sim$. Each nonzero global section $s \in H^0(X,\mathcal{E})$ corresponds to an $\mathcal{O}_X$-linear map
$0 \neq s\colon \mathcal{E}^\vee \to \mathcal{O}_X$. Its image $\mathcal{L} := \operatorname{im} s \subset \mathcal{O}_X$ is an $\mathcal{O}_X$-submodule, and indeed an invertible sheaf of ideals. The exact sequence

$$\mathcal{E}^\vee \xrightarrow[s]{} \mathcal{L} \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow \mathcal{L}^\vee \xrightarrow[s^\vee]{} \mathcal{E}.$$

Define

$$[s] := s^\vee(\mathcal{L}^\vee),$$

then $[s]$ is a subsheaf of $\mathcal{E}$ which is an invertible sheaf of ideals. The set $\{P \in X \mid s_P\colon \mathcal{E}_P^\vee \to \mathcal{O}_{P,X} \textit{ surjective}\} = \{P \in X \mid s(P) \neq 0\}$ is open and dense in X.

From now on let X be a curve over the field k, i.e., a geometrically integral, complete, smooth scheme of finite type over k of dimension 1. Let $\overline{X} := X \times_k \overline{k}$ be the base change of X from k to the algebraic closure $\overline{k}$ of k, and let $X' := X \times_k k'$ be the base change of X from k to a field extension k' of k. For a vector bundle $\mathcal{E}$ on X we usually write $\mathcal{E}' := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ and $\overline{\mathcal{E}} := \mathcal{E} \otimes \mathcal{O}_{\overline{X}}$ for the corresponding vector bundle on X', respectively on $\overline{X}$. Recall that $H^i(X,\mathcal{E}) \otimes_k k' \cong H^i(X',\mathcal{E}')$. For any finite separable field extension k'/k, nonisomorphic vector bundles on X extend to nonisomorphic vector bundles on X' (see for instance [AEJ 1], p.1325). For two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}(X)$ we write $\mathcal{L}_1 \geq \mathcal{L}_2$ if $H^0(X, \mathcal{H}om_X(\mathcal{L}_2,\mathcal{L}_1)) = \operatorname{Hom}(\mathcal{L}_2,\mathcal{L}_1) \neq 0$. Thus $\mathcal{L}_1 \geq \mathcal{L}_2$ if and only if $\mathcal{L}_2^\vee \otimes \mathcal{L}_1 \geq \mathcal{O}_X$. In particular, $\mathcal{L} \geq \mathcal{O}_X$ is equivalent to $H^0(X,\mathcal{L}) \neq 0$. That means, $\mathcal{L}_1 \geq \mathcal{O}_X$ if and only if there is an effective divisor D on X such that $\mathcal{L}_1 \cong \mathcal{L}(D)$. Moreover, if $\mathcal{L}_1 \geq \mathcal{L}_2$, then $\deg \mathcal{L}_1 \geq \deg \mathcal{L}_2$. Also, $\mathcal{L}_1 \geq \mathcal{L}_2$ if and only if $\mathcal{L}_1 \otimes \mathcal{O}_{X'} \geq \mathcal{L}_2 \otimes \mathcal{O}_{X'}$.

1.1 Remark (i) Let $\mathcal{N} \in \operatorname{Pic}(X)$ be a line bundle on X and $0 \neq s \in H^0(X,\mathcal{N})$ a nonzero global section of $\mathcal{N}$. Then $\mathcal{N} \cong [s]$.
(ii) Let $0 \neq s \in H^0(X,\mathcal{E})$ be a nonzero global section of a vector bundle $\mathcal{E}$ on X. Then $[s] \otimes \mathcal{O}_{X'} \cong [s \otimes \mathcal{O}_{X'}]$, where $X' = X \times_k k'$ is the base change of X from k to some base field extension k'.
(iii) For each $0 \neq s \in H^0(X,\mathcal{E})$ we know that $[s] \geq \mathcal{O}_X$, thus $\deg s := \deg[s] \geq 0$, and that $H^0(X,[s]) \neq 0$.

(iv) (see also [H], IV.1.2.) Let X be a complete nonsingular curve over k. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle on X with $\deg \mathcal{L} = 0$ and $H^0(X, \mathcal{L}) \neq 0$. Then $\mathcal{L} \cong \mathcal{O}_X$.
(v) Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle such that $\deg \mathcal{L} > 0$. Then $\deg \mathcal{L}^\vee < \deg \mathcal{O}_X = 0$ and thus $H^0(X, \mathcal{L}^\vee) = 0$. On the other hand, if $\deg \mathcal{L} < 0$, then $H^0(X, \mathcal{L}) = 0$.
(vi) For a nonzero global section $0 \neq s \in H^0(X, \mathcal{E})$ of a vector bundle $\mathcal{E}$ on X the morphism $0 \neq s\colon \mathcal{E}^\vee \to \mathcal{O}_X$ is surjective if and only if $\deg s = 0$.

2. Splittings of vector bundles on arbitrary curves

Let X be a nonsingular curve over an arbitrary field k. Following [At], p.419 a *splitting* of a vector bundle $\mathcal{E}$ on X of rank r is a sequence of vector bundles on X

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{L}_i := \mathcal{E}_i / \mathcal{E}_{i-1}$ is a line bundle on X for $1 \leq i \leq r$. We also write $\mathcal{E} = (\mathcal{L}_1, \ldots, \mathcal{L}_r)$. For $1 \leq i \leq r$,

$$\mathcal{E}^{(i)} := \mathcal{E}/\mathcal{E}_{i-1}$$

is a vector bundle of rank $r + 1 - i$, and by defining

$$\mathcal{E}_j^{(i)} := \mathcal{E}_{j+i-1}/\mathcal{E}_{i-1} \quad , \quad 0 \leq j \leq r + 1 - i$$

we obtain a splitting of $\mathcal{E}^{(i)}$ such that

$$\mathcal{L}_j^{(i)} := \mathcal{E}_j^{(i)}/\mathcal{E}_{j-1}^{(i)} \cong \mathcal{L}_{j+i-1}$$

for $1 \leq j \leq n + 1 - i$. In particular, given a splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ of $\mathcal{E}$, $\mathcal{L}_1$ is a line bundle which is a subbundle of $\mathcal{E}$, and $(\mathcal{L}_2, \ldots, \mathcal{L}_r)$ is up to isomorphism a splitting of the rank $r - 1$ vector bundle $\mathcal{E}/\mathcal{L}_1$. The short exact sequence

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{L}_1 \longrightarrow 0$$

yields $\det \mathcal{E} = \Lambda^r \mathcal{E} \cong \det \mathcal{L}_1 \otimes \det \mathcal{E}/\mathcal{L}_1 \cong \mathcal{L}_1 \otimes \det \mathcal{E}/\mathcal{L}_1$. Therefore $\det \mathcal{E} \cong \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r$ by induction on the rank r of $\mathcal{E}$, and $\deg \mathcal{E} = \sum_{i=1}^{n} \deg \mathcal{L}_i$. Every vector bundle on X has a splitting (the proof in [At] generalizes immediately).

2.1 Lemma *Let $\mathcal{E}$ be a vector bundle of rank r on a nonsingular curve X of genus g.*
(i) Let $H^0(X, \mathcal{E}) \neq 0$. Then

$$\deg s \leq \sup_{1 \leq i \leq r} \{\deg \mathcal{L}_i\}$$

for each $0 \neq s \in H^0(X, \mathcal{E})$ *and each splitting* $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *of* $\mathcal{E}$.
(ii) $h^0(X, \mathcal{E}) \leq \sum_{i=1}^{r} h^0(X, \mathcal{L}_i)$, *for each splitting* $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *of* $\mathcal{E}$.

The proof of (i) is analogous to the one given in [At], Lemma 2, (ii) is proved by induction on the rank of $\mathcal{E}$.

2.2 Definition [At] (i) A nonzero global section $s \in H^0(X, \mathcal{E})$ is called *maximal*, if $\deg s$ is maximal.
(ii) $[s]$ is a *maximal line bundle*, if $s \in H^0(X, \mathcal{E})$ is a maximal nonzero global section.
(iii) $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ is a *maximal splitting* of $\mathcal{E}$ if

(a) $\mathcal{L}_1$ is a maximal line bundle of $\mathcal{E}$,

(b) $(\mathcal{L}_2, \ldots, \mathcal{L}_r)$ is a maximal splitting of $\mathcal{E}/\mathcal{L}_1$.

The maximal degree of a section $0 \neq s \in H^0(X, \mathcal{E})$ is always finite by Lemma 2.1. A maximal splitting of $\mathcal{E}$ always exists if $\mathcal{E}$ has sufficient sections.

2.3 Example Let $\mathcal{E}$ be a vector bundle of rank 2 with $H^0(X, \mathcal{E}) \neq 0$. For each $0 \neq s \in H^0(X, \mathcal{E})$ the series $\{0\} = \mathcal{E}_0 \subset \mathcal{E}_1 = [s] \subset \mathcal{E}_2 = \mathcal{E}$ is a splitting of $\mathcal{E}$ with $\mathcal{L}_1 = [s]$, $\mathcal{L}_2 = \mathcal{E}/[s]$. In particular, we obtain the short exact sequence $0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$. If we assume that $s \in H^0(X, \mathcal{E})$ has maximal degree, then $\mathcal{L}_1 = [s]$ is a maximal line bundle. Since $\deg \mathcal{E} = \deg \mathcal{L}_1 + \deg \mathcal{L}_2$ the degree of $\mathcal{L}_2$ is uniquely determined by the degree of the maximal global section s. In case $H^0(X, \mathcal{L}_2) \neq 0$, for each section $0 \neq t \in H^0(X, \mathcal{L}_2)$ we obtain $\mathcal{L}_2 \cong [t]$ and $\deg t = \deg \mathcal{L}_2$ is necessarily maximal, so $(\mathcal{L}_1, \mathcal{L}_2)$ is a maximal splitting of $\mathcal{E}$. In case $H^0(X, \mathcal{L}_2) = H^0(X, \mathcal{E}/[s]) = 0$ for every $0 \neq s \in H^0(X, \mathcal{E})$ of maximal degree, $\mathcal{E}$ does not have a maximal splitting.

2.4 Remark For a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ of $\mathcal{E}$ we know that $\deg \mathcal{L}_i \geq 0$ for $i = 1, \ldots, r$ and thus $d = \deg \mathcal{E} = \sum_{i=1}^{r} \deg \mathcal{L}_i \geq 0$ is a necessary condition for its existence. In particular, if $d = \deg \mathcal{E} = 0$, then $H^0(X, \mathcal{L}_i) \neq 0$ and $\deg \mathcal{L}_i$ has to be 0 for $i = 1, \ldots, r$. This means that $(\mathcal{L}_1, \ldots, \mathcal{L}_r) = (\mathcal{O}_X, \ldots, \mathcal{O}_X)$.

As long as we assume that the curve still has rational points, the proof of [At], Lemma 3 can be adapted to the more general situation considered here, and we obtain

2.5 Lemma *Let X be a curve over k of genus $g = h^1(X, \mathcal{O}_X)$ with a k-rational point, and let $\mathcal{E}$ be a vector bundle on X of rank 2. A maximal splitting $(\mathcal{L}_1, \mathcal{L}_2)$ of $\mathcal{E}$ satisfies*

$$\deg \mathcal{L}_2 - \deg \mathcal{L}_1 \leq 2g.$$

The proof of 2.5 uses the following observation, which we will use repeatedly later on: Let X be a curve over k with a k-rational point. Let $\mathcal{E}$ be a vector bundle on X such that

$\deg s = 0$ for each nonzero global section $0 \neq s \in H^0(X, \mathcal{E})$. Then the canonical morphism

$$\alpha(P)\colon H^0(X,\mathcal{E}) \otimes_k k(P) \longrightarrow \mathcal{E}_P \otimes_{\mathcal{O}_{P,X}} k(P)$$
$$s \otimes 1 \longmapsto s_P \otimes 1$$

is injective for each k-rational $P \in X$. Hence, $\alpha_P\colon H^0(X,\mathcal{E}) \otimes_k \mathcal{O}_{P,X} \to \mathcal{E}_P$ is injective for each k-rational point $P \in X$. In particular, $m = h^0(X,\mathcal{E}) \leq \operatorname{rank} \mathcal{E} = r$. In other words: If there is a k-rational point $P \in X$ such that $H^0(X,\mathcal{E}) \otimes \mathcal{O}_{P,X} \to \mathcal{E}_P$ is not injective, there exists a nonzero global section $0 \neq s \in H^0(X,\mathcal{E})$ such that $\deg s \geq 1$.

[At], Lemma 4 still holds for curves containing rational points, since its proof mainly relies on 2.5.

2.6 Lemma *Let X be a curve over k of genus g with a k-rational point. Let $\mathcal{E}$ be a vector bundle on X with a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$. Then*

$$\deg \mathcal{L}_i - \deg \mathcal{L}_{i-1} \leq 2g \qquad (i = 2, \ldots, r).$$

2.7 Corollary *Let X be a curve of genus g over k with a k-rational point and let $\mathcal{E}$ be a vector bundle on X with a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$. Then*

$$\deg \mathcal{L}_1 \geq \frac{d}{r} - g(r-1),$$

where $d = \deg \mathcal{E}$ and $r = \operatorname{rank} \mathcal{E}$.

Proof It follows from 2.6 that $\deg \mathcal{L}_i - \deg \mathcal{L}_1 = (\deg \mathcal{L}_i - \deg \mathcal{L}_{i-1}) + \cdots + (\deg \mathcal{L}_2 - \deg \mathcal{L}_1) \leq (i-1)2g$ and hence $-2g(i-1) \leq \deg \mathcal{L}_1 - \deg \mathcal{L}_i$ for $i = 2, \ldots r$. Summing up these inequalities for $i = 2, \ldots r$ we obtain $-2g\frac{r(r-1)}{2} \leq r \deg \mathcal{L}_1 - d$ or $\deg \mathcal{L}_1 \geq \frac{d}{r} - g(r-1)$. $\square$

2.8 Remark (i) Let $\mathcal{E}$ be an indecomposable vector bundle on X of rank 2 and let $(\mathcal{L}_1, \mathcal{L}_2)$ be a splitting of $\mathcal{E}$. Then $\deg \mathcal{L}_2 \geq \deg \mathcal{L}_1 - (2g-2)$, where g denotes the genus of X, and $\mathcal{L}_2 \geq \mathcal{L}_1 \geq \mathcal{O}_X$ if X has genus 1. (Applying [At], Lemma 5 to the short exact sequence $0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$ we obtain an $\mathcal{O}_X$-linear map $0 \neq f\colon \mathcal{L}_1 \to \mathcal{L}_2 \otimes \omega_X$, so $\mathcal{L}_2 \otimes \omega_X \geq \mathcal{L}_1$ and $\deg \mathcal{L}_2 + \deg \omega_X = \deg \mathcal{L}_2 + 2g - 2 \geq \deg \mathcal{L}_1$.)

(ii) Let X be a curve of genus g over k with a k-rational point, and $\mathcal{E}$ an indecomposable vector bundle of rank 2 on X with a maximal splitting $(\mathcal{L}_1, \mathcal{L}_2)$. Then $\deg \mathcal{L}_2 \geq \frac{d}{2} - (3g-2)$, where $d = \deg \mathcal{E}$. (This follows from (i) and 2.5 as in the proof of [At], Lemma 7.)

(iii) Let $\mathcal{E}$ be an indecomposable vector bundle on X with a splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ such that $\mathcal{E}_2$ and $\mathcal{E}/\mathcal{L}_1$ are indecomposable. Then it can be proved by induction that $\deg \mathcal{L}_i - \deg \mathcal{L}_{i-1} \geq 2g - 2$ for $i = 2, \ldots, r$, that means $\deg \mathcal{L}_i \geq \deg \mathcal{L}_1 + (i-1)(2g-2)$.

The above observations can be generalized for indecomposable vector bundles $\mathcal{E}$ on X of arbitrary rank r.

2.9 Theorem *(i) Let $\mathcal{E}$ be an indecomposable vector bundle with sufficient sections on a curve X of genus $g \geq 1$. Then there exists a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ of $\mathcal{E}$ such that*

$$\deg \mathcal{L}_i \geq \deg \mathcal{L}_1 - (i-1)(2g-2)$$

for $i = 2, \ldots, r$.
(ii) Let $\mathcal{E}$ be an indecomposable vector bundle on a curve X of genus $g = 1$ with $H^0(X, \mathcal{E}) \neq 0$. Then $\mathcal{E}$ has a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ such that $\mathcal{L}_i \geq \mathcal{L}_1 \geq \mathcal{O}_X$ for $i = 2, \ldots, r$.

The proof is analogous to the one of [At], Lemma 6 and $6'$.

It has been observed by Atiyah already that (for algebraically closed k) the inequality in Theorem 2.9 (i) is of course valid for any maximal splitting of a vector bundle $\mathcal{E}$ which is indecomposable with sufficient sections. This is probably true also in our situation, however, we do not pursue this question here, since the version at hand suffices for our purposes. The proof of 2.12(i) given by Atiyah does not work for genus 0 (the inequality at line -4 of [At], p. 422 is only correct as long as $2g-2 \geq 0$).

2.10 Example Let X be the nonrational curve of genus 0 over k which is associated with the quaternion division algebra $(a,b)_k$. Let $P_0 \in X$ be a point of minimal degree, i.e., $\deg P_0 = 2$. Let k'/k be a separable quadratic splitting field of $(a,b)_k$ and $X' := X \times_k k'$. The vector bundles $\mathcal{G}_0 := \mathrm{tr}_{k'/k}(\mathcal{O}_{X'}(1))$ and $\mathcal{G}_m := \mathcal{G}_0 \otimes \mathcal{L}(mP_0) \cong \mathrm{tr}_{k'/k}(\mathcal{O}_{X'}(2m+1))$ for $m \in \mathbb{Z}$, $m \neq 0$, are indecomposable. It is known that $H^0(X, \mathcal{G}_m) \neq 0$ iff $m \geq 0, \det \mathcal{G}_m = \mathcal{L}((2m+1)P_0)$, and that $\deg \mathcal{G}_m = 2(2m+1)$. The generalized Euler-sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{L}(P_0) \longrightarrow 0$$

induces the *canonical sequence* of $\mathcal{G}_m$,

$$0 \longrightarrow \mathcal{L}(mP_0) \longrightarrow \mathcal{G}_m \longrightarrow \mathcal{L}((m+1)P_0) \longrightarrow 0.$$

Obviously, $(\mathcal{L}(mP_0), \mathcal{L}((m+1)P_0)$ is a splitting of $\mathcal{E}_m$. By Serre's Theorem A, $\mathcal{G}_m$ has a maximal splitting for $m \gg 0$. Let $(\mathcal{L}_1, \mathcal{L}_2)$ be such a maximal splitting of $\mathcal{G}_m$, i.e., $\mathcal{L}_1 = \mathcal{L}(n_1 P_0)$ and $\mathcal{L}_2 = \mathcal{L}(n_2 P_0)$ with $n_1, n_2 \geq 0$. Applying 2.8(i) for $r = 2$ yields (1): $n_2 \geq n_1 + 2$, and (2): $2m+1 = n_1 + n_2$, $m \geq 0$. Now

$$h^0(X, \mathcal{G}_m \otimes \mathcal{L}_1^\vee) \geq \chi(\mathcal{G}_m \otimes \mathcal{L}_1^\vee) = \chi(\mathcal{O}_X) + \chi(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee) = 2 + 2(n_2 - n_1) \geq 6,$$

and therefore

$$H^0(X, \mathcal{G}_m \otimes \mathcal{L}_1^\vee) = H^0(X, tr_{k'/k}(\mathcal{O}_{X'}(2(m-n_1)+1)) \neq 0 \Longleftrightarrow m \geq n_1.$$

For $m \geq 0$, $H^0(X, \mathcal{L}(mP_0)) \neq 0$ and thus there exists a nonzero section $t \in H^0(X, \mathcal{G}_m)$ such that $\mathcal{L}(mP_0) \cong [t]$. In particular, we have $n_1 \geq m$ this way and therefore $m = n_1$. Hence $(\mathcal{L}_1, \mathcal{L}_2) = (\mathcal{L}(mP_0), \mathcal{L}((m+1)P_0)$ is the canonical splitting from above. Indeed, one can show that each $\mathcal{G}_m$ with $m \geq 0$ has a maximal splitting $(\mathcal{L}(mP_0), \mathcal{L}((m+1)P_0)$: Let $0 \neq s \in H^0(X, \mathcal{G}_m)$ be a maximal section, then $[s] = \mathcal{L}(n_1P_0)$ and $\deg s = 2n_1 \geq 2m = \deg \mathcal{L}(mP_0)$. Now by 2.8 we have $\deg \mathcal{L}(n_2P_0) \geq \deg \mathcal{L}(n_1P_0) + 2 \geq 2$ implying $h^0(X, \mathcal{L}(n_2P_0)) \geq \chi(\mathcal{L}(n_2P_0)) = \deg \mathcal{L}(n_2P_0) + 2$ which means $H^0(X, \mathcal{L}(n_2P_0)) \neq 0$ and the existence of a maximal splitting is proved (2.3). We summarize: For $m \geq 0$, the maximal splitting $(\mathcal{L}_1, \mathcal{L}_2) = (\mathcal{L}(mP_0), \mathcal{L}((m+1)P_0)$ of $\mathcal{G}_m$ satisfies $\deg \mathcal{L}_2 - \deg \mathcal{L}_1 = 2 \geq g = 0$, so 2.5 (which generalizes [At], Lemma 3), indeed does not hold for a nonrational curve X of genus 0. Moreover, if $\overline{k}$ is the algebraic closure of k, and if $\overline{X} := X \times_k \overline{k}$ denotes the base change from k to $\overline{k}$, then over $\overline{X}$ the above maximal splitting of $\mathcal{G}_m$ becomes the exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{X}}(2m) \longrightarrow \mathcal{O}_{\overline{X}}(2m+1) \oplus \mathcal{O}_{\overline{X}}(2m+1) \longrightarrow \mathcal{O}_{\overline{X}}(2(m+1)) \longrightarrow 0$$

with $\deg \mathcal{O}_{\overline{X}}(2(m+1)) = 2m+2 \geq \deg \mathcal{O}_{\overline{X}}(2m) = 2m$, thus it cannot be a maximal splitting for the rank two vector bundle $\mathcal{G}_m \otimes \mathcal{O}_{\overline{X}} \cong \mathcal{O}_{\overline{X}}(2m+1) \oplus \mathcal{O}_{\overline{X}}(2m+1)$ on $\overline{X}$ by [At], Lemma 3. Thus a maximal splitting of a vector bundle does not necessarily stay maximal under base field extensions.

2.11 Lemma *Let X be a curve over k with a k-rational point and $\mathcal{E}$ an indecomposable vector bundle on X with $r =$ rank $\mathcal{E}$ and $d = \deg \mathcal{E}$. Then $\mathcal{E}$ has a splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ satisfying*

$$\deg \mathcal{L}_i \geq \frac{d}{r} - (r-1)(3g-2).$$

Both the statement and the proof are analogous to the proof of [At], Lemma 7 and uses 2.5, 2.6 and 2.9.

2.12 Lemma *Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle with $d = \deg \mathcal{L}$, and let g be the genus of the curve X. If $d \geq 2g$ then $\mathcal{L}$ is ample.*

A vector bundle $\mathcal{E}$ on X is called *ample* [At] if $\mathcal{E}$ has sufficient sections and $H^i(X, \mathcal{E}) = 0$ for each $i > 0$.

Proof. By [At], Lemma 8, $\overline{\mathcal{L}} := \mathcal{L} \otimes \mathcal{O}_{\overline{X}}$ is ample. Hence $h^1(X, \mathcal{L}) = 0$ and the evaluation map $\overline{\alpha}: H^0(\overline{X}, \overline{\mathcal{L}}) \otimes \mathcal{O}_{\overline{X}} \to \overline{\mathcal{L}}$ of $\overline{\mathcal{L}}$ is surjective. Let $\alpha: H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ be the evaluation map of $\mathcal{L}$. Identify $\overline{\alpha} = \alpha \otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{X}}$. Since $\mathcal{O}_{Q,\overline{X}}$ is faithfully flat over $\mathcal{O}_{P,X}$, for each $Q \in \overline{X}$ with $\tau(Q) = P$, $\tau\colon \overline{X} \to X$ the projection morphism, it follows that α is surjective as well. $\square$

Recall that for a morphism of schemes $\sigma\colon X \to Y$ with $Y = \mathrm{Spec} A$ affine, and an $\mathcal{O}_X$-module $\mathcal{E}$, there always exists a canonical morphism $\alpha\colon H^0(X,\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$, called the *evaluation map* of $\mathcal{E}$.

For every absolutely indecomposable vector bundle $\mathcal{E}$ on X of degree d and rank r there exists an integer $N(g,r,d)$ such that $h^1(X,\mathcal{E}(n)) = 0$ and such that for $\mathcal{O}_{\overline{X}}(n) := \mathcal{O}_X(n) \otimes \mathcal{O}_{\overline{X}}$ the bundle $\overline{\mathcal{E}} \otimes \mathcal{O}_{\overline{X}}(n)$ has sufficient sections for all $n \geq N(g,r,d)$, i.e., the evaluation map $\overline{\alpha}\colon H^0(\overline{X},\overline{\mathcal{E}}(n)) \otimes \mathcal{O}_{\overline{X}} \to \overline{\mathcal{E}}(n)$ is an epimorphism. This is an immediate consequence of [At], Theorem 1. For indecomposable vector bundles on X, this theorem can be generalized as follows.

2.13 Theorem *Let X be a curve over k of genus g with a k-rational point. There exists an integer $N(g,r,d)$ such that $\mathcal{E}(n)$ is ample for every indecomposable vector bundle $\mathcal{E}$ on X of rank r and degree d, and all $n \geq N(g,r,d)$.*

Proof Define $N(g,r,d)$ to be the first integer greater than or equal to $-\frac{d}{r} + (r-1)(3g-2) + 2g$. Let $\mathcal{E}$ be any indecomposable vector bundle on X with $d = \deg \mathcal{E}$ and $r = \mathrm{rank}\, \mathcal{E}$. By 2.11, there exists a splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ of $\mathcal{E}$ such that

$$\deg \mathcal{L}_i \geq \frac{d}{r} - (r-1)(3g-2)$$

for $i = 1, \ldots, r$. Therefore $(\mathcal{L}_1(n), \ldots, \mathcal{L}_r(n))$ is a splitting of $\mathcal{E}(n)$ satisfying

$$d_i := \deg \mathcal{L}_i(n) \geq \frac{d}{r} + nh - (r-1)(3g-2),$$

where $h = \deg \mathcal{O}(1)$ again. For each $n \geq N(g,r,d)$

$$\begin{aligned} \deg \mathcal{L}_i(n) &\geq \frac{d}{r} + h(-\frac{d}{r} + (r-1)(3g-2) + 2g) - (r-1)(3g-2), \\ &\geq 2g \end{aligned}.$$

Thus $\mathcal{L}_i(n)$ is ample for $i = 1, \ldots, r$ (2.12) and so is $\mathcal{E}(n)$. $\square$

3. Absolutely indecomposable vector bundles of degree 0

Let X be a nonsingular curve over a field k, k^{sep} the separable closure of k, $\overline{k}$ the algebraic closure of k, and let $X^{\mathrm{sep}} := X \times_k k^{\mathrm{sep}}$ and $\overline{X} := X \times_k \overline{k}$ denote the base changes for X from k to k^{sep} respectively $\overline{k}$. Recall that the *index* of X (denoted $\mathrm{ind}(X)$) is the greatest common divisor of the degrees of the field extensions k'/k such that $X(k') \neq \emptyset$. A vector bundle $\mathcal{E}$ on X is called *separably indecomposable*, if $\mathcal{E}^{\text{sep}} := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{\text{sep}}}$ is an indecomposable vector bundle on X^{sep}. We define the following sets of isomorphism classes of indecomposable vector bundles of rank r and degree d:

$\Omega_X(r,d) = \Omega(r,d) = \{[\mathcal{E}] \mid \mathcal{E}$ an indecomposable vector bundle on X of degree d and rank $r\}$

$\overline{\Omega}_X(r,d) = \overline{\Omega}(r,d) = \{[\mathcal{E}] \mid \mathcal{E}$ an absolutely indecomposable vector bundle on X of degree d and rank $r\}$

To avoid complicated terminology we refrain from writing $[\mathcal{E}]$ when we mean an element of $\overline{\Omega}(r,d)$, but simply write $\mathcal{E}$ instead. Obviously, $\overline{\Omega}(r,d) \subset \Omega(r,d)$. There exists a bijective map between $\Omega(r,d)$ and $\Omega(r,d+nir)$ for any integer n, given by $\mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{A}^n$, where $\mathcal{A}$ is a line bundle on X of degree i. (This also applies to the set $\overline{\Omega}(r,d)$.) Therefore it suffices to investigate these sets for $0 \leq d < ir$, for instance.

From now on, X is a curve of genus 1. In this case we do not need to distinguish between absolutely indecomposable and separably indecomposable vector bundles. The proof of the following proposition was communicated to us by J.K. Arason.

3.1 Proposition *Let X be a proper scheme over a field k. Then a vector bundle $\mathcal{E}$ on X is separably indecomposable if and only if it is absolutely indecomposable.*

Proof Obviously, any absolutely indecomposable vector bundle on X is separably indecomposable. Now assume that k is separably closed, and $\mathcal{E}$ is indecomposable over X. Then $D(\mathcal{E}) := \text{End}(\mathcal{E})/\text{rad}(\text{End}(\mathcal{E}))$ is a finite field extension k' of k. Let $\ell := k(x_i)$ with $x_i^p = a$ be an inseparable extension of k. By extending scalars, we get the vector bundle $\mathcal{N} := \mathcal{E} \otimes \mathcal{O}_Y$ on $Y := X \times_k \ell$, and $D(\mathcal{N}) = D(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Y) = D(\mathcal{E}) \otimes_k \ell/\text{rad}(D(\mathcal{E}) \otimes_k \ell) = k'(x_i)$. If the polynomial $x^p - a$ is irreducible over k', this is already a field. Otherwise there is an element $c \in k'$ such that $c^p = a$ and the radical of $k'(x_i)$ is generated by $x_i - c$. Hence $k'(x_i)$ modulo its radical is k'. In any case, $D(\mathcal{N})$ is a field, thus $\mathcal{N} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is indecomposable over Y. It follows that $\overline{\mathcal{E}} := \mathcal{E} \otimes \mathcal{O}_{\overline{X}}$ stays indecomposable. $\square$

In the more general setting of curves of genus 1 and arbitrary index, Atiyah's Lemma 10 becomes

3.2 Lemma *Let X be a curve of genus 1 and index i over k. Let $\mathcal{E}$ be an indecomposable vector bundle on X of rank r such that $m = h^0(X,\mathcal{E}) > 0$ and $0 \leq d = \deg \mathcal{E} < ir$.*

(i) $\mathcal{E}$ has a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ such that $\mathcal{L}_j \geq \mathcal{L}_1 = \mathcal{O}_X$ for $j = 2, \ldots, r$. In particular, $\deg s = 0$ for each nonzero global section $s \in H^0(X,\mathcal{E})$. If $i = 1$ then

$$\alpha(P) \colon H^0(X,\mathcal{E}) \otimes k(P) \longrightarrow \mathcal{E}_P \otimes k(P)$$

is injective for each k-rational point $P \in X$.

(ii) Let k'/k be a field extension such that $X(k') \neq \emptyset$. Define $X' := X \times_k k'$. If $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is indecomposable over X' and of degree $d < r$, then

$$\alpha_P \colon H^0(X, \mathcal{E}) \otimes \mathcal{O}_{P,X} \longrightarrow \mathcal{E}_P$$

is injective for each k'-rational point $P \in X$.

(iii) If $\mathcal{E}$ is absolutely indecomposable of degree $d < r$, the evaluation map

$$\alpha \colon H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \longrightarrow \mathcal{E}$$

is injective, and $\mathcal{O}_X^{(m)}$ is a subbundle of $\mathcal{E}$.

In the cases (i) (for $i = 1$), and (ii) we get $m \leq r$, in case (iii), even $m < r$.

Proof (i) By 2.9 (ii), the vector bundle $\mathcal{E}$ has a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ satisfying $\mathcal{L}_j \geq \mathcal{L}_1 \geq \mathcal{O}_X$ for $j = 2, \ldots, r$. Assume that $\deg \mathcal{L}_1 > 0$, then $d = \deg \mathcal{E} = \sum_{i=1}^{r} \deg \mathcal{L}_i \geq r \deg \mathcal{L}_1 \geq ir$, a contradiction to the hypothesis that $d < ir$. Thus it follows that $\deg \mathcal{L}_1 = 0$ and $\mathcal{L}_1 \cong \mathcal{O}_X$ by 1.15. Since $\mathcal{L}_1$ is a maximal line bundle of $\mathcal{E}$, this implies $\deg s = 0$ for each nonzero global section $0 \neq s \in H^0(X, \mathcal{E})$. Therefore for $i = 1$ the morphism $\alpha(P)$ is injective for each k-rational point $P \in X$ (see the remark after 2.5).

(ii) Let $\tau \colon X' \to X$ be the projection morphism. For $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{X'}$ consider the evaluation map

$$\alpha' \colon H^0(X', \mathcal{E}') \otimes_{k'} \mathcal{O}_{X'} \longrightarrow \mathcal{E}'.$$

By (i), the map $\alpha'(Q)$ is injective for each k'-rational point $Q \in X'$. Furthermore, we may identify $\alpha' = \alpha \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ where $\alpha \colon H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$ is the evaluation map for $\mathcal{E}$, and obtain that $\alpha'_Q = \alpha_P \otimes \mathrm{id}_{\mathcal{O}_{Q,X'}}$ is injective for each $P \in X$ with $P = \tau(Q)$. This implies the injectivity of α_P, since $\mathcal{O}_{Q,X'}$ is faithfully flat over $\mathcal{O}_{P,X}$.

(iii) The proof is obvious. $\square$

By the theorem of Riemann-Roch, $m > 0$ always holds for $d > 0$. For a vector bundle $\mathcal{E}$ on X with $d = \deg \mathcal{E} > r = \operatorname{rank} \mathcal{E}$, there cannot exist a point $P \in X$ such that the map $\alpha(P)$ is injective (otherwise we obtain the contradiction $r \geq d > r$ by Riemann-Roch). Moreover in case $d = r$, the injectivity of $\alpha(P)$ implies that $\alpha(P)$ and thus α_P is an isomorphism, and that $h^0(X, \mathcal{E}) = r$. Thus $\mathcal{E}$ is globally free and generated by global sections in case $\alpha(P)$ is injective for all $P \in X$.

There are several ways to generalize Atiyah's Lemma 11:

3.3 Lemma *Let X be a curve of genus 1 and index i over k. Let $\mathcal{E}$ be an absolutely indecomposable vector bundle on X of rank r and degree $d \geq r$. Then $\overline{\mathcal{E}} := \mathcal{E} \otimes \mathcal{O}_{\overline{X}}$ has* *a maximal splitting* $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *such that* $\mathcal{L}_j \geq \mathcal{L}_1 > \mathcal{O}_{\overline{X}}$ *for* $j = 2, \ldots, r$*. In particular,* $h^0(X, \mathcal{E}) = d$*.*

Proof By 2.9, $\overline{\mathcal{E}}$ possesses a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ satisfying $\mathcal{L}_j \geq \mathcal{L}_1 \geq \mathcal{O}_{\overline{X}}$ for $j = 2, \ldots, r$. If $\deg \mathcal{L}_1 = 0$ then $\mathcal{L}_1 \cong \mathcal{O}_{\overline{X}}$ (1.6) and $\alpha\colon H^0(\overline{X}, \overline{\mathcal{E}}) \otimes \mathcal{O}_{\overline{X}} \to \overline{\mathcal{E}}$ is injective. Hence $m = h^0(X, \mathcal{E}) \leq r$. However, by Riemann-Roch $r \geq m \geq \deg \mathcal{E}$ and we obtain the contradictions that $r > r$ if $\deg \mathcal{E} > r$, and $\overline{\mathcal{E}} \cong \mathcal{O}_{\overline{X}}^{(m)}$ if $\deg \mathcal{E} = r$. Thus always $\deg \mathcal{L}_j \geq \deg \mathcal{L}_1 > 0$ for $j = 2, \ldots, r$. Since $\mathcal{L}_j > \mathcal{O}_{\overline{X}}$ for $j = 1, \ldots, r$ it follows that $h^1(\overline{X}, \mathcal{L}_j) = 0$ for $j = 1, \ldots, r$ and so $h^1(\overline{X}, \overline{\mathcal{E}}) = 0$. □

3.4 Corollary *Let* X *be a curve of genus 1 and index* i *over* k*. Let* $\mathcal{E}$ *be an absolutely indecomposable vector bundle on* X *of rank* r *and degree* r*. Then* $\overline{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}_{\overline{X}}$ *has a maximal splitting* $(\mathcal{L}, \ldots, \mathcal{L})$ *such that* $\deg \mathcal{L} = 1$*.*

Proof $\overline{\mathcal{E}}$ has a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ such that $\mathcal{L}_j \geq \mathcal{L}_1 > \mathcal{O}_{\overline{X}}$ for $j = 2, \ldots, r$ (3.3). Since $r = \deg \mathcal{E} = \sum_{j=1}^{r} \deg \mathcal{L}_j \geq r \deg \mathcal{L}_1 \geq r$ we know that $\deg \mathcal{L}_j = 1$, for $j = 2, \ldots, n$. Furthermore, $\mathcal{L}_j \geq \mathcal{L}_1$ is equivalent to $H^0(X, \mathcal{L}_1^\vee \otimes \mathcal{L}_j) \neq 0$ and since $\deg(\mathcal{L}_1^\vee \otimes \mathcal{L}_j) = 0$, we get $\mathcal{L}_j \cong \mathcal{L}_1$ for $j = 2, \ldots, n$. □

3.5 Lemma *Let* X *be a curve of genus 1 and index* i *over* k*. Let* $\mathcal{E}$ *be an indecomposable vector bundle on* X *of rank* r *and degree* d*.*

(i) If $i = 1$ *and* $d > r$ *then* $\mathcal{E}$ *has a maximal splitting* $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *such that* $\mathcal{L}_j \geq \mathcal{L}_1 > \mathcal{O}_X$ *for* $j = 2, \ldots, r$*, and* $h^0(X, \mathcal{E}) = d$*.*

(ii) If $d = ni$ *with* $0 < n < r$ *then* $\mathcal{E}$ *has a maximal splitting* $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *such that* $\mathcal{L}_j \geq \mathcal{L}_1 \cong \mathcal{O}_X$*.*

Proof By 2.9, there is a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ of $\mathcal{E}$ such that $\mathcal{L}_j \geq \mathcal{L}_1 \geq \mathcal{O}_X$ for $j = 2, \ldots, r$.

(i) Assume $\deg \mathcal{L}_1 = 0$ then $\mathcal{L}_1 \cong \mathcal{O}_X$ and since $ind(X) = 1$, this implies that α_P is injective for all k-rational points $P \in X$. Hence $r \geq m$ and since $m \geq \deg \mathcal{E} = d$ by Riemann-Roch, this is a contradiction. Therefore $\deg \mathcal{L}_j \geq \deg \mathcal{L}_1 > 0$ for $j = 2, \ldots, r$ and thus $h^1(X, \mathcal{L}_j) = 0$, which implies $h^1(X, \mathcal{E}) = 0$.

(ii) Assume that $\deg \mathcal{L}_1 > 0$, then $\deg \mathcal{L}_1 \geq i$ and $ni = \deg \mathcal{E} = \sum_{j=1}^{r} \deg \mathcal{L}_j \geq r \deg \mathcal{L}_1 \geq ir$ implies $n \geq r$, a contradiction. □

3.6 Lemma *Let* X *be a curve of genus 1 and index* i *over* k*. Let* $\mathcal{E}$ *be an indecomposable vector bundle on* X *of rank* r *and degree* $d = ir$*. Then one of the following holds:*

(i) $\mathcal{E}$ *has a maximal splitting* $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *such that* $\mathcal{L}_j \geq \mathcal{L}_1 \cong \mathcal{O}_X$ *for* $j = 2, \ldots, r$*.*

(ii) $\mathcal{E}$ has a maximal splitting $(\mathcal{L},\ldots,\mathcal{L})$ with $\deg\mathcal{L}=i$. In particular, $\deg s\in\{0,i\}$ for all nonzero global sections $s\in H^0(X,\mathcal{E})$, and $h^0(X,\mathcal{E})=ir$. Furthermore, $\det\mathcal{E}\cong\mathcal{L}^r$.

Proof By 2.9 there is a maximal splitting $(\mathcal{L}_1,\ldots,\mathcal{L}_r)$ for $\mathcal{E}$ such that $\mathcal{L}_j\geq\mathcal{L}_1\geq\mathcal{O}_X$. Again, if $\deg\mathcal{L}_1=0$ then $\mathcal{L}_1\cong\mathcal{O}_X$. Otherwise, $\deg\mathcal{L}_1\geq i$ and $ir=\deg\mathcal{E}=\sum_{j=1}^{r}\deg\mathcal{L}_j\geq r\deg\mathcal{L}_1\geq ir$ yields $\deg\mathcal{L}_1=i$, and $\deg\mathcal{L}_j\geq i$ implies that $\deg\mathcal{L}_j=i$, $j=2,\ldots,r$. As in the proof of 3.4, we get $\mathcal{L}_j\cong\mathcal{L}_1$ for $j=2,\ldots,r$. Moreover, since $\deg\mathcal{L}_j>0$ we have $h^1(X,\mathcal{L}_j)=0$ for $j=1,\ldots,r$ and thus $h^1(X,\mathcal{E})=0$. □

3.7 Corollary *Let X be a curve of genus 1 and index 1 over k. Let $\mathcal{E}$ be a vector bundle on X of rank r and degree r.*

(a) If $\mathcal{E}$ is indecomposable then $h^1(X,\mathcal{E})=0$ and one of the following is true:

(i) $\mathcal{E}$ has a maximal splitting $(\mathcal{L}_1,\ldots,\mathcal{L}_r)$ with $\mathcal{L}_j\geq\mathcal{L}_1\cong\mathcal{O}_X$ for $j=2,\ldots,r$, and the map

$$\alpha(P)\colon H^0(X,\mathcal{E})\otimes k(P)\longrightarrow\mathcal{E}_P\otimes k(P)$$

is injective, for each k-rational point $P\in X$.

(ii) $\mathcal{E}$ has a maximal splitting $(\mathcal{L},\ldots,\mathcal{L})$ such that $\deg\mathcal{L}=1$. In particular, then $\deg s\leq 1$ for all nonzero global sections $s\in H^0(X,\mathcal{E})$, and $\det\mathcal{E}\cong\mathcal{L}^r$.

(b) Let k'/k be a field extension, $X':=X\times_k k'$, and let $\mathcal{E}':=\mathcal{E}\otimes\mathcal{O}_{X'}$ be indecomposable. Then one of the following is true:

(i) $\mathcal{E}'$ has a maximal splitting $(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ such that $\mathcal{S}_j\geq\mathcal{S}_1\cong\mathcal{O}_{X'}$ for $j=2,\ldots,r$, and the map

$$\alpha_P\colon H^0(X,\mathcal{E})\otimes\mathcal{O}_{P,X}\longrightarrow\mathcal{E}_P$$

is injective, for each k'-rational point $P\in X$.

(ii) $\mathcal{E}'$ has a maximal splitting $(\mathcal{S},\ldots,\mathcal{S})$ such that $\deg\mathcal{S}=1$. In particular, then $\deg s\leq 1$ for all nonzero global sections $s\in H^0(X',\mathcal{E}')$, and $\det\mathcal{E}\cong\mathcal{S}^r$. (This happens for instance, if $\mathcal{E}$ already has such a splitting.)

(c) Let $\mathcal{E}$ be absolutely indecomposable. Then $\overline{\mathcal{E}}$ has a maximal splitting $(\mathcal{S},\ldots,\mathcal{S})$ with $\deg\mathcal{S}=1$. In particular, $\deg s\leq 1$ for all nonzero global sections $s\in H^0(\overline{X},\overline{\mathcal{E}})$.

In all these cases, the evaluation map $\alpha\colon H^0(X,\mathcal{E})\otimes\mathcal{O}_X\to\mathcal{E}$ itself is not injective.

Proof (a) is 3.6 for $i=1$. In particular, $\mathcal{L}_1\cong\mathcal{O}_X$ and hence $\deg s=0$ for each nonzero global section $s\in H^0(X,\mathcal{E})$ implies that $\alpha(P)\colon H^0(X,\mathcal{E})\otimes k(P)\to\mathcal{E}_P\otimes k(P)$ is injective for each k-rational point $P\in X$, and thus $h^0(X,\mathcal{E})=r$. (However, α itself cannot be injective, otherwise $\mathcal{E}\cong\mathcal{O}_X^{(r)}$ is decomposable.)

(b) Apply 3.6 to the vector bundle $\mathcal{E}'$ on X'.

(c) If $\mathcal{E}$ is absolutely indecomposable, the assertion follows from 3.4, or 3.6. □

This implies: Let X be a curve of genus 1 with a rational point, and $\mathcal{E}$ an indecomposable vector bundle on X with $r = \operatorname{rank} \mathcal{E} = \deg \mathcal{E}$. Let $X' = X \times_k k'$ be a base change, such that $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{X'}$ is still indecomposable. Let $(\tilde{\mathcal{L}}_1, \ldots, \tilde{\mathcal{L}}_r)$ be a maximal splitting of $\mathcal{E}'$ of the type described in 3.2 (b) (i) or (b) (ii). Let $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ be a maximal splitting of $\mathcal{E}$ with $\mathcal{L}_i \geq \mathcal{L}_1 \geq \mathcal{O}_X$ for $i = 2, \ldots, r$, then $(\mathcal{L}'_1, \ldots, \mathcal{L}'_r)$ with $\mathcal{L}'_i := \mathcal{L}_i \otimes \mathcal{O}_{X'}$ is a splitting of $\mathcal{E}'$ and $\deg \tilde{\mathcal{L}}_1 \geq \deg \mathcal{L}'_1 = \deg \mathcal{L}_1$. If $\deg \tilde{\mathcal{L}}_1 = 0$, then $\deg \mathcal{L}_1 = 0$ and $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ has to be of the type described in 3.2 (a)(i).

Moreover: For an indecomposable vector bundle $\mathcal{E}$ on X of rank r and degree r, either $\deg s = 0$ for each nonzero global section $s \in H^0(X, \mathcal{E})$ or there exists a nonzero global section $0 \neq s \in H^0(X, \mathcal{E})$ of (maximal) degree 1. If $\mathcal{E}$ is even absolutely indecomposable, then there always exists a nonzero global section of maximal degree 1 in $H^0(\overline{X}, \overline{\mathcal{E}})$.

3.8 Lemma *Let $\mathcal{E} \in \overline{\Omega}(r, d)$ with $d \geq 0$. Then*

(i) $$m = h^0(X, \mathcal{E}) = \begin{cases} d & \text{if } \ d > 0 \\ 0 \text{ or } 1 & \text{if } \ d = 0 \end{cases}$$

(ii) For $d < r$, the evaluation map $\alpha: H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \longrightarrow \mathcal{E}$ is injective. Thus $\mathcal{O}_X^{(m)}$ is a subbundle of $\mathcal{E}$, and $\mathcal{G} := \mathcal{E}/\mathcal{O}_X^{(m)} \in \overline{\Omega}(r - m, d)$ is absolutely indecomposable with $h^0(X, \mathcal{G}) = m$.

This is an immediate consequence from [At], Lemma 15 (see also 3.2(iii)), since $\mathcal{E} \otimes \mathcal{O}_{\overline{X}}$ is indecomposable. For a vector bundle on X which is indecomposable, but not necessarily absolutely indecomposable, a weaker version of part (i) is obtained:

3.9 Lemma *Let X be a curve of genus 1 and index i over k. Let $\mathcal{E} \in \Omega(r, d)$ with $d \geq 0$, and $m := h^0(X, \mathcal{E})$*

(a) Let $i = 1$. Then $h^0(X, \mathcal{E}) = d$ if $d \geq r$ and $d \leq h^0(X, \mathcal{E}) \leq r$ if $0 \leq d < r$. In particular, $\mathcal{E}$ is not absolutely indecomposable if $d = 0$ and $m \neq 0, 1$, or if $0 < d < r$ and $m \neq d$.

(b) Let $i > 1$. Then $h^0(X, \mathcal{E}) = d$ if $d \geq ir$ and if there exists a nonzero global section $s \in H^0(X, \mathcal{E})$ of degree greater or equal to i. Let $0 < d < ir$, or let $d = 0$ and $m \neq 0$. Then $\deg s = 0$ *for all nonzero global sections $s \in H^0(X, \mathcal{E})$.*

(c) Let $i > 1$. Let k'/k be a field extension such that $\mathcal{E}' := \mathcal{E} \otimes \mathcal{O}_{X'}$ is indecomposable on $X' := X \times_k k'$, and such that $X(k') \neq \emptyset$. Then $h^0(X, \mathcal{E}) = d$ if $d \geq r$ and $d \leq h^0(X, \mathcal{E}) \leq r$ if $\quad 0 \leq d < r$.

Proof (a) Assume $i = 1$. For $d > r$, $h^0(X, \mathcal{E}) = d$ by 3.5 (i), for $d = r$, $h^0(X, \mathcal{E}) = d$ by Lemma 3.7 (a). Now let $0 \leq d < r$. If $d = 0$ and $m = h^0(X, \mathcal{E}) = 0$ there is nothing to prove. Let either be $d = 0$ and $m > 0$, or let $d > 0$ and thus also $m > 0$. Then there exists a maximal splitting $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ of $\mathcal{E}$ such that $\mathcal{L}_j \geq \mathcal{L}_1 \cong \mathcal{O}_X$ (3.5), and $\alpha(P)$ is injective for each k-rational point $P \in X$ implying $m \leq r$. The last part of (a) follows from [At], Lemma 15.

(b) Let $i > 1$ and $d \geq ir$. Again there is a maximal splitting $(\mathcal{L}_1, \dots, \mathcal{L}_r)$ of $\mathcal{E}$ such that $\mathcal{L}_j \geq \mathcal{L}_1 \geq \mathcal{O}_X$ for $j = 2, \dots, r$. If $\deg \mathcal{L}_1 = 0$ then $\deg s = 0$ for all nonzero global sections $s \in H^0(X, \mathcal{E})$. If $\deg \mathcal{L}_1 > 0$ then $\deg \mathcal{L}_j > 0$ implies $h^1(X, \mathcal{L}_j) = 0$ for all j, and hence also $h^1(X, \mathcal{E}) = 0$. Let $0 \leq d < ir$. For $d = 0$ and $m = 0$ there is nothing to prove, thus let either be $d = 0$ and $m \neq 0$, or $d > 0$ and hence $m = h^0(X, \mathcal{E}) > 0$. Again there exists a maximal splitting $(\mathcal{L}_1, \dots, \mathcal{L}_r)$ such that $\mathcal{L}_j \geq \mathcal{L}_1 \cong \mathcal{O}_X$ for $j = 2, \dots, r$.

(c) Apply (i) to the vector bundle $\mathcal{E}'$ on $X' = X \times_k k'$. $\square$

3.10 Corollary *Let X be a curve of genus 1 and index $i > 1$ over k. Let $\mathcal{E} \in \Omega(r, d)$. Let $d \geq r$ and $h^0(X, \mathcal{E}) > d$, or let $0 \leq d < r$ and $h^0(X, \mathcal{E}) > r$. The vector bundle $\mathcal{E}' := \mathcal{E} \otimes \mathcal{O}_{X'}$ decomposes over $X' := X \times_k k'$, for every field extension k'/k with $X(k') \neq \emptyset$.*

Let $\mathcal{E}$ be a vector bundle on the curve X of genus 1 with nontrivial global sections. There exists an extension

$$(E) \quad 0 \longrightarrow H^0(X, \mathcal{E}) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow 0$$

which corresponds to the identity on $H^0(X, \mathcal{E})$ (cf. [AEJ 1], p.1338 for the explicit construction). Define $m := h^0(X, \mathcal{E})$. If $\mathcal{E} \in \Omega(r', d)$ then (E) is nontrivial and therefore $\mathcal{O}_X^{(m)}$-complete. In particular, then $\deg \mathcal{E} = \deg \mathcal{M}$ and $h^0(X, \mathcal{E}) = h^0(X, \mathcal{M})$ by [At], Lemma 14, also $\operatorname{rank} \mathcal{M} = \operatorname{rank} \mathcal{E} + m$. By [At], Lemma 13*, the vector bundle $\mathcal{M}$ is uniquely determined up to isomorphism, since $h^1(X, \mathcal{E}^\vee) = m$. (By Serre-duality, $h^1(X, \mathcal{E}^\vee) = h^0(X, \mathcal{E}) = m$.)

3.11 Proposition *(i) Let $\mathcal{E} \in \overline{\Omega}(r', d)$ with $d \geq 0$, and in case $d = 0$ assume $m = h^0(X, \mathcal{E}) \neq 0$. Then there exists a vector bundle $\mathcal{M} \in \overline{\Omega}(r, d)$, which is unique up to isomorphism, given by the extension*

$$0 \longrightarrow H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $r = r' + m$ and

$$m = \begin{cases} d & \text{if} \quad d > 0, \\ 1 & \text{if} \quad d = 0. \end{cases}$$

(ii) Let $\mathcal{E} \in \Omega(r', d)$ with $d \geq 0$ and in case $d = 0$ assume $m = h^0(X, \mathcal{E}) \neq 0$. Then there exists a vector bundle $\mathcal{M} \in \Omega(r, d)$ which is uniquely determined up to isomorphism, given by the extension

$$0 \longrightarrow H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $r = r' + m$. If X has a k-rational point, then

$$m = d \quad \text{if} \quad d \geq r',$$
$$d \leq m \leq r' \quad \text{if} \quad 0 \leq d < r'.$$

In particular, $\mathcal{E}$ is not absolutely indecomposable if $d = 0$ and $m \neq 0, 1$, or if $0 < d < r'$ and $m \neq d$.

Proof In both cases (i) and (ii) there exists a vector bundle $\mathcal{M}$ of degree d and rank $r = r' + m$ which is uniquely determined up to isomorphism and given by the extension

$$(E) \quad 0 \longrightarrow H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Moreover, $h^0(X, \mathcal{E}) = m = h^0(X, \mathcal{M})$. By 3.8, $m = d$ if $d > 0$ and $m = 1$ if $d = 0$ for $\mathcal{E} \in \overline{\Omega}(r', d)$ in case (i). By 3.9, in case (ii) we conclude that for $X(k) \neq \emptyset$, $m = d$ if $d \geq r'$ and $d \leq m \leq r'$ if $0 \leq d < r'$.

It remains to show that $\mathcal{M} \in \overline{\Omega}(r, d)$ in (i), respectively $\mathcal{M} \in \Omega(r, d)$ in (ii): This is proved analogously as in [At], Lemma 16. or by directly applying [At], Lemma 16.

□

This generalizes [At], Lemma 16. Both in [T], Satz 6.2 and in [AEJ1], Proposition 4.1 (i), the following result was proved for a curve X of genus 1 and index 1, over a perfect base field k. These assumptions are not necessary. Imitating the induction on r from the proof of [At], Theorem 5(i) yields:

3.12 Theorem *There exists a vector bundle $\mathcal{F}_r \in \overline{\Omega}(r, 0)$ which is unique up to isomorphism, such that $H^0(X, \mathcal{F}_r) \neq 0$. There exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow 0.$$

Recall the following: Any k-homomorphism of fields $\sigma \colon \ell_1 \to \ell_2$ induces a morphism $X_2 \colon= X \times_k \ell_2 \to X \times_k \ell_1 =\colon X_1$. Let $\mathcal{E}_1$ be a vector bundle on $X \times_k \ell_1$, then ${}^\sigma\mathcal{E}_1 = \mathcal{E}_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_2}$ denotes the pullback of $\mathcal{E}_1$ to X_2. For $\ell_1 = \ell_2$ this pullback is a conjugate of $\mathcal{E}_1$.

Part (ii) of [At], Theorem 5 is generalized in [AEJ1], 4.1 (ii), for curves of genus 1 and index 1 over a perfect base field k, using a descent argument. This argument still works. Let E/k be the Jacobian of X/k. We obtain

3.13 Theorem *Let X be a curve of genus 1 over a perfect field k, with index i and period q. Let $\mathcal{E}$ be an absolutely indecomposable vector bundle of rank r and degree 0 on X. Additionally, let*

(i) X satisfy $H^0(Pic^0 \overline{X}) \cong Pic^0 X$, or

(ii) let r and i be coprime, or

(ii) let r and q be coprime.

Then there is a line bundle $\mathcal{S}$ of degree 0 on X which is unique up to isomorphism, such that $\mathcal{E} \cong \mathcal{S} \otimes \mathcal{F}_r$. In particular, $\mathcal{E}$ contains $\mathcal{S}$ as a subbundle, and $\det \mathcal{E} \cong \mathcal{S}^r$.

Proof Take $\mathcal{E} \in \overline{\Omega}(r, 0)$. By [At], Theorem 5(ii) it follows that $\overline{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}_{\overline{X}} \cong \mathcal{L} \otimes \mathcal{F}_r$ for a line bundle $\mathcal{L}$ of degree 0 on $\overline{X}$, which is unique up to isomorphism. Thus the isomorphism class of $\mathcal{L}$ is G-invariant. In case $H^0(\overline{X}, \mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$ by assumption (i), $\mathcal{L}$ is defined over X, and $\mathcal{L} \cong \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{X}}$ for some line bundle $\mathcal{S}$ on X implying the assertion. Now assume (ii). Obviously, $\det \overline{\mathcal{E}} \cong \mathcal{L}^r$ is a line bundle defined over X, but so is $\mathcal{L}^i$, since the cokernel of the injection $\mathrm{Pic} X \to H^0(\mathrm{Pic}(\overline{X}))$ is killed by the index i ([PoSch], 3.1). Let $\mathcal{L}^{-m}$ denote $\mathcal{L}^{\vee m}$, for any integer $m > 0$. This means that $\mathcal{L}$ is defined over X if there exists integers a, b such that $ai + br = 1$, in other words, if index and rank are coprime, as in (ii). If (iii) holds we apply the same argument as in (ii), using that the cokernel of the injection $\mathrm{Pic}^0 X \to H^0(\mathrm{Pic}^0\overline{X})$ is killed by the period of X ([PoSch], 3.2). $\square$

This shows that in the situation of 3.13, every absolutely indecomposable vector bundle of rank r and degree 0 has a determinant which is an r^{th}-root of unity.

At this point, it does not seem possible to directly adapt Atiyah's proof of Theorem 5(ii), since this would mean that we need a stronger version of 3.6, which actually excludes case (i) in case the vector bundle is absolutely indecomposable of rank r and degree ir. Then we could omit the hypothesis (i) in 3.13 and also that k needs to be perfect.

Note that the assumption in Theorem 3.13 holds in particular if X satisfies:

$(*)$ *For every finite Galois extension k'/k with Galois group $G' = \mathrm{Gal}(k'/k)$ there is an isomorphism*

$$\mathrm{Pic}^{G'}(X \times_k k') \cong \mathrm{Pic}(X).$$

This implies that ${}_2\mathrm{Pic} X = {}_2\mathrm{Pic} E$ if E/k is the Jacobian of X/k. In particular, $(*)$ is satisfied if $H^1(\mathrm{Princ}(X^{\mathrm{sep}})) = 0$ where $X^{\mathrm{sep}} = X \times_k k^{\mathrm{sep}}$. ($H^i(A)$ is used as an abbreviation for the cohomology group $H^i(G, A)$ where A is a G-group, $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.) Moreover, there is an exact sequence

$$0 \longrightarrow \mathrm{Pic} X \longrightarrow H^0(\mathrm{Pic}(X^{\mathrm{sep}})) \longrightarrow \mathrm{Br}\, k \longrightarrow \mathrm{Br}(X) \longrightarrow H^1(\mathrm{Pic} X^{\mathrm{sep}}) \longrightarrow H^3(k^{\mathrm{sep}\times})$$

with $\mathrm{Br}\, k = H^2(k^{\mathrm{sep}\times})$ the Brauer group of k, and $\mathrm{Br} X$ the cohomological Brauer group of X (the kernel of the canonical map $H^2(k^{\mathrm{sep}}(X)^x) \to H^2(\mathrm{Div}(X^{\mathrm{sep}}))$, since X is a curve).

The cokernel of the injection $\mathrm{Pic} X \to H^0(\mathrm{Pic} X^{\mathrm{sep}})$ is killed by the index i of X over k ([Po Sch], 3.1). Thus given any line bundle $\mathcal{L}$ on X^{sep} whose isomorphism class is G-invariant, but which is not defined over X, then $\mathcal{L}$ is an element of ${}_i\mathrm{Pic}(X^{\mathrm{sep}})$, since $\mathcal{L} \otimes \cdots \otimes \mathcal{L} \cong \mathcal{O}_{X^{\mathrm{sep}}}$ (i-times).

3.14 Example (i) ([T], p.93). The curve $X/\mathbb{Q}$ defined by $2y^2 = x^4 - 17$ has the Jacobian $E/\mathbb{Q}$ given by the equation $y^2 = x^3 + 17x$. Reichardt [R] proved that $X(\mathbb{Q}) = \emptyset$ (but $X(\mathbb{Q}_p) \neq \emptyset$ for all $\mathbb{Q}_p$). Here, $H^1(\mathrm{Princ}X^{\mathrm{sep}}) = 0$ and thus ${}_2\mathrm{Pic}X \cong \mathbb{Z}_2$ and $ind(X) = 2$.

(ii) ([Si], p.312). The curve $E/\mathbb{Q}$ given by the equation $y^2 = x(x-2)(x+2)$ is the Jacobian of the homogeneous spaces $C_d/\mathbb{Q}$ defined by

$$C_d/\mathbb{Q}\colon dw^2 = d^2 + z^4,$$

with $d \in \mathbb{Q}(S,2)$ and $\{\pm 1, \pm 2\}$ representatives for the cosets in $\mathbb{Q}(S,2)$. For $d < 0$, $C_d(\mathbb{R}) = \emptyset$, so

$$C_{-1}/\mathbb{R}\colon w^2 = -z^4 - 1$$

is a curve of genus 1 and index 2. Since $\mathrm{Br}(\mathbb{R}(x,y)/\mathbb{R}) = \{1, (-1,-1)_{\mathbb{R}}\}$ we know that ${}_2\mathrm{Pic}X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

3.15 Corollary *Let X be a curve of genus 1 over a perfect field k. Then*

$$\begin{aligned} {}_2\mathrm{Pic}X &\longrightarrow \{\mathcal{E} \in \overline{\Omega}(r,0) \mid \mathcal{E} \quad \text{a selfdual vector bundle on} \quad X\} \\ \mathcal{L} &\longmapsto \mathcal{L} \otimes \mathcal{F}_r \end{aligned}$$

is a bijective map, provided that the conditions (i), (ii) or (iii) from 3.13 are satisfied. In particular, for any rank r (resp., for any rank coprime to the index, or the period) there are at most 4 (and at least 1) isomorphism classes of selfdual absolutely indecomposable vector bundles on X.

The proof is straightforward, the case of a perfect base field k where $X(k) \neq \emptyset$ in 3.15 was already treated in [T], Satz 7.9.

3.16 Corollary ([At], Cor. 1,2, Lemma 17, 18) *Let X be a curve over a perfect field k such that $H^0(Pic^0\overline{X}) \cong Pic^0 X$, unless stated otherwise.*

(i) $\mathcal{F}_r$ is a selfdual vector bundle on X.

(ii) For all $s < r$ we have exact sequences

$$0 \longrightarrow \mathcal{F}_s \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-s} \longrightarrow 0.$$

(iii) $h^0(X, \mathcal{F}_r \otimes \mathcal{F}_s) = \min(r,s)$

(iv) For a line bundle $\mathcal{L}$ on X, $h^0(X, \mathcal{L} \otimes \mathcal{F}_r \otimes \mathcal{F}_s) = 0$ unless $h^0(X, \mathcal{L}) \neq 0$.

(v) $\mathcal{F}_r \otimes \mathcal{F}_s \cong \bigoplus_{j=1}^{\min(r,s)} \mathcal{F}_{r_j}$, where $\sum_j r_j = rs$.

Note that 3.16 (i), (iii) and (iv) do not need 3.13 in the proofs and are still valid for any curve of genus 1. However, until the end of this paragraph assume that k is perfect, and that the curve X satisfies $H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$.

Let $V(X)$ be the set of isomorphism classes of vector bundles on X. This is a free abelian semi-group with respect to $\oplus$, and can be embedded into a free abelian group $\hat{V}(X)$. By extending the tensor product operation $\otimes$ to $\hat{V}(X)$ we obtain a unital commutative ring with the unit element given by the class of $\mathcal{O}_X$. Let F be the subset of $V := V(X)$ generated by the classes f_r of the bundles $\mathcal{F}_r$, $r \geq 1$, with respect to $\oplus$, and $\hat{F}$ be the corresponding subgroup of $\hat{V} := \hat{V}(X)$. $\hat{F}$ is a subring of $\hat{V}$. It is a commutative ring with a unit element satisfying conditions (a) and (b) in [At], p. 436. The structure of $\hat{F}$ is not completely determined by (a) and (b). The following additional hypothesis is needed, which depends on the characteristic of the base field k of X:

$$(H_r) : f_r^2 = 1 + \sum_{j=2}^{r} f_{r_j}.$$

If $\hat{F}$ is a ring where (H_r) holds in addition to (a) and (b), for all $r \geq 1$, then $\hat{F}$ is uniquely determined up to isomorphism and its multiplicative structure is given by

$$f_r f_s = f_{r-s+1} + f_{r-s+3} + \cdots + f_{r+s-1} \qquad (r \geq s).$$

All this was already observed by Atiyah. We do reproduce the arguments here, since they are independent of the chosen base field. We only remark that $\mathcal{F}_r \otimes \mathcal{F}_r \cong \mathcal{O}_X \oplus \mathcal{G}$ for all r if char $k = 0$, and for all r coprime to char $k = p > 0$, where $\mathcal{G}_P$ is the subspace of trace zero endomorphisms of $\mathcal{E}nd(\mathcal{F}_r)_P$ for all $P \in X$ ([At], Corollary to Lemma 19). Hence (H_r) is true for all $r \geq 1$ if the characteristic of k is zero, and we obtain [At]. Theorem 8 for arbitrary curves of genus 1 over fields of characteristic zero:

3.17 Theorem. *Let X be a curve of genus 1 and index i over a field of characteristic zero such that $H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$. Let $\hat{F}$ denote the subring of $\hat{V}(X)$ generated by the isomorphism classes of the vector bundles $\mathcal{F}_r$, $r \geq 1$. Then $\hat{F}$ is a free abelian group generated by the classes of the $\mathcal{F}_r$, $r \geq 1$, and we have*

$$\mathcal{F}_r \otimes \mathcal{F}_s \cong \mathcal{F}_{r-s+1} \oplus \mathcal{F}_{r-s+3} \oplus \cdots \oplus \mathcal{F}_{r+s-1}$$

for $r \geq s$.

3.18 Theorem ([At], Theorem 9). *Let X be a curve of genus 1 and index i over a field of characteristic zero. Then $\mathcal{F}_r \cong S^{r-1}(\mathcal{F}_2)$ for $r \geq 1$, where S^m denotes the n-th symmetric product.*

This follows from the formula given in the theorem before, but also by a descent argument from Atiyah's result. Using the latter way of proof implies that the assumption

$H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$ is superfluous here. Otherwise the proofs are exactly as given in [At], since at no point in his arguments it becomes crucial that the base field of X is algebraically closed, or that the index of X needs to be 1.

4. Admissible pairs and absolutely indecomposable vector bundles of coprime rank and degree

Let h be the greatest common divisor of r and d. In [At], Theorem 6, a bijective map

$$\omega_{r,d}\colon \overline{\Omega}(h,0) \longrightarrow \overline{\Omega}(r,d)$$

is defined inductively, for k algebraically closed. By generalizing the inductive definition of the map $\omega_{r,d}$ given in the proof for a curve X over an arbitrary field k which has genus 1 and index $i \geq 1$, the following abstract algorithm is obtained:

Start with a pair (r,d) of positive integers, and a fixed integer i which divides d. Then repeatedly use the following two steps starting with $d_0 := d$, and $r = q_1 d_0 + r_1 (j \geq 0)$:

1. $d_{2j} = i q_{2j+2} r_{2j+1} + d_{2j+2}$ with $-\lfloor \frac{ir_{2j+1}-1}{2} \rfloor \leq d_{2j+2} \leq \lfloor \frac{ir_{2j+1}}{2} \rfloor$ where $\lfloor y \rfloor$ denotes the integral part of y. If $d_{2j+2} < 0$ continue with $-d_{2j+2}$ instead.
2. $r_{2j+1} = q_{2j+3} d_{2j+2} + r_{2j+3}$ with $0 < r_{2j+3} < d_{2j+2}$ provided that $d_{2j+2} \neq 0$, otherwise stop at the pair (r_{2j+1}, d_{2j+2}).

It is allowed to use an empty step where q_{2j+2} resp. q_{2j+3} is 0. At each step, r_{2j+1} and $|d_{2j}|$ are uniquely determined. The algorithm either stops at step 2. in which case $d_{2j} = 0$, or it goes on indefinitely. In any case we have $gcd(r_{2j+1}, d_{2j}) = gcd(r,d)$, so if the procedure stops with $d_{2j} = 0$, it stops at the pair $(h,0)$ with $h := gcd(r,d)$. One easily sees by induction that i/d_{2j} for all $j \geq 0$. Consequently, $r_{2j+1} \equiv r \mod i$ for all $j \geq 0$. Hence, if the procedure stops with $d_{2j} = 0$ we have $h = r_{2j-1} \equiv r \mod i$. Therefore, for $h \not\equiv r \mod i$ (which never happens for $i = 2$) the procedure will continue indefinitely. Moreover, if i/r then $d_{2j} \equiv d \mod i^2$. So if i/r but $i^2 \not/ \, d$ the procedure will not stop. More generally, it will not stop if $i \not/ \, \frac{d}{h}$.

Since $r_{2j+1} \leq |d_{2j}|$ and $|d_{2j+2}| \leq \lfloor \frac{ir_{2j+1}-1}{2} \rfloor$, the sequence $(r, d, r_1, |d_2|, r_3, |d_4|, \ldots)$ is non-increasing for $i \leq 2$.

A pair (r,d) of integers with $r \geq 1$ is called *admissible* with respect to i, if the algorithm described above stops at step 2. with the pair $(h,0)$ where h denotes the greatest common divisor of r and d.

Let $i = 2$ which is the easiest nontrivial case. Suppose $r_{2j+1} = |d_{2j}|$. Then $d_{2j} = 2q_{2j+2}r_{2j+1} + d_{2j+2}$ with the given bound on d_{j+2} implies $q_{2j+2} = 0$ and $|d_{2j+2}| = |d_{2j}|$.

We enter a loop, and thus stop at the pair $(r_{2j+1}, |d_{2j}|)$ with $r_{2j+1} = |d_{2j}|$. Suppose $|d_{2j+2}| = \lfloor\frac{ir_{2j+1}}{2}\rfloor = r_{2j+1}$. Then $r_{2j+1} = q_{2j+3}d_{2j+2} + r_{2j+3}$ with the given bound on r_{2j+3} implies $q_{2j+3} = 0$, hence $r_{2j+1} = r_{2j+3}$. Again we enter a loop, and stop at the pair $(r_{2j+1}, |d_{2j+2}|)$ with $r_{2j+1} = |d_{2j+2}|$.

4.1 Theorem *Let X be a curve of genus 1 and index i over k. Fix a line bundle $\mathcal{A}$ on X of degree i. For any admissible pair (r, d) with respect to i, $\mathcal{A}$ determines a unique bijection*

$$\omega_{r,d}: \overline{\Omega}_X(h, 0) \longrightarrow \overline{\Omega}_X(r, d)$$

with h the greatest common divisor of r and d, which is inductively defined as follows:
(i) $\omega_{r,0} = \mathrm{id}$,
(ii) $\omega_{r,d+ir}(\mathcal{E}) \cong \omega_{r,d}(\mathcal{E}) \otimes \mathcal{A}$,
(iii) for $0 < d < \min\{r, \lfloor\frac{ir-1}{2}\rfloor\}$ *there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{(d)} \longrightarrow \omega_{r,d}(\mathcal{E}) \longrightarrow \omega_{r-d,d}(\mathcal{E}) \longrightarrow 0,$$

(iv) for $d < 0$,

$$\omega_{r,d}(\mathcal{E}) \cong \omega_{r,-d}(\mathcal{E})^\vee.$$

Moreover,

$$\det \omega_{r,d}(\mathcal{E}) \cong \det \mathcal{E} \otimes \mathcal{A}^{d/i}$$

if the number of steps of type (iv) used to define $\omega_{r,d}$ is even, and

$$\det \omega_{r,d}(\mathcal{E}) \cong \det \mathcal{E}^\vee \otimes \mathcal{A}^{d/i}$$

otherwise.

Proof Fix a line bundle $\mathcal{A}$ on X of degree i. There is a bijection

$$\begin{aligned} \overline{\Omega}(r, d) &\longrightarrow \overline{\Omega}(r, d + ir) \\ \mathcal{E} &\longmapsto \mathcal{E} \otimes \mathcal{A}, \end{aligned}$$

or more generally, there exists a bijection between $\overline{\Omega}(r, d)$ and $\overline{\Omega}(r, e)$ for $d = c(ir) + e$ with $-\lfloor\frac{ir-1}{2}\rfloor \le e \le \lfloor\frac{ir}{2}\rfloor$. Let $0 < d < r$. By 3.8, for each $\mathcal{E} \in \overline{\Omega}(r, d)$ there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X^{(d)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

such that $\mathcal{G} \in \overline{\Omega}(r - d, d)$. Conversely, by 3.11 there is a vector bundle $\mathcal{E} \in \overline{\Omega}(r, d)$ uniquely determined up to isomorphism, which extends $\mathcal{G} \in \overline{\Omega}(r - d, d)$ by $\mathcal{O}_X^{(d)}$. Hence there is a bijection between $\overline{\Omega}(r,d)$ and $\overline{\Omega}(r-d,d)$, or more generally, between $\overline{\Omega}(r,d)$ and $\overline{\Omega}(b,d)$, for $r = ad+b$, $0 < b < d$. For $d < 0$ we use the canonical bijection $\overline{\Omega}(r,d) \to \overline{\Omega}(r,-d)$, $\mathcal{E} \mapsto \mathcal{E}^\vee$. Using the above "euclidean" algorithm on an admissible pair (r,d) with respect to i thus corresponds to constructing a bijection $\omega_{r,d}$ between $\overline{\Omega}(h,0)$ and $\overline{\Omega}(r,d)$ defined inductively by steps (i), (ii), (iii) and (iv). The formula for $\det \mathcal{E}$ is proved by a straightforward induction on r and d. $\square$

Let $\mathrm{Pic}^d X$ denote the set of line bundles on X of degree d. Let $\mathcal{L} \in \mathrm{Pic} X$. For a positive integer m, let $\mathcal{L}^{-m}$ denote the m-fold tensor product of the dual bundle of $\mathcal{L}$.

4.2 Theorem *Let X be a curve of genus 1 and index i over k. Let k be perfect and assume that $H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$, or that i and r are coprime in the following. Fix a point $P_0 \in X$ of degree i. For any admissible pair (r,d) with respect to i there is a canonical bijection (which depends on the choice of P_0) between the set of isomorphism classes $\overline{\Omega}(r,d)$ of absolutely indecomposable vector bundles of rank r and degree d on X and the set $Pic^0 X$ which identifies $\overline{\Omega}(r,d)$ and $Pic^0 X$ in such a way that the map*

$$\det\colon \overline{\Omega}(r,d) \longrightarrow \mathrm{Pic}^d X$$

corresponds to the map

$$\begin{aligned} G_1\colon \mathrm{Pic}^0 X &\longrightarrow \mathrm{Pic}^0 X \\ \mathcal{L} &\longmapsto \mathcal{L} \otimes \cdots \otimes \mathcal{L} \qquad (h-\text{times}) \end{aligned}$$

if the number of times step (iv) is used in the definition of $\omega_{r,d}$ is even, or to the map

$$\begin{aligned} G_2\colon \mathrm{Pic}^0 X &\longrightarrow \mathrm{Pic}^0 X \\ \mathcal{L} &\longmapsto \mathcal{L}^\vee \otimes \cdots \otimes \mathcal{L}^\vee \end{aligned}$$

if the number of times step (iv) is used in the definition of $\omega_{r,d}$ is odd. Again, h is the greatest common divisor of r and d.

Proof Define $\mathcal{A} := \mathcal{L}(P_0)$. By 3.13 and the above theorem there exists a bijective map

$$\mathrm{Pic}^0 X \longrightarrow \overline{\Omega}(h,0) \longrightarrow \overline{\Omega}(r,d).$$

Let us call this map $\omega = \tilde{\omega}_{r,d}$. Then $\det \tilde{\omega}_{r,d}(\mathcal{L}) \cong \mathcal{L}^h \otimes \mathcal{A}^{d/i}$ for any line bundle $\mathcal{L} \in \mathrm{Pic}^0 X$, or $\det \tilde{\omega}_{r,d}(\mathcal{L}) \cong \mathcal{L}^{-h} \otimes \mathcal{A}^{d/i}$, depending on the number of times step (iv) is used in the definition of $\omega_{r,d}$. Therefore the diagram made of the two maps ω and det,

$$\mathrm{Pic}^0 X \longrightarrow \overline{\Omega}(h,0) \longrightarrow \mathrm{Pic}^d X$$

and the maps G_1 or G_2 (depending on the situation) and $\tilde{\omega}_{1,d}$,

$$\mathrm{Pic}^0 X \longrightarrow \mathrm{Pic}^0 X \longrightarrow \mathrm{Pic}^d X$$

commutes. $\square$

Let E/k be the Jacobian of X/k. The special case where the curve has index 2 is easy to understand:

4.3 Corollary *Let X be a curve of genus 1 and index 2 over a perfect field k. Let h be the greatest common divisor of the integers $r > 0$ and d. If r is odd, there is a bijection*

$$\tilde{\omega}_{r,d}\colon \mathrm{Pic}^0 X \longrightarrow \overline{\Omega}(r,d).$$

4.4 Theorem *Let X be a curve of genus 1 and index i over a perfect field k such that $H^0(Pic^0\overline{X}) \cong Pic^0 X$. For any admissible pair (r,d) with respect to i there is a canonical bijection between the set $\overline{\Omega}(r,d)$ and the set $E(k)$ of k-rational points on E. Via this bijection, $\overline{\Omega}(r,d)$ and $E(k)$ are identified in such a way that the map*

$$\overline{\Omega}(r,d) \longrightarrow \mathrm{Pic}^d X,$$
$$\mathcal{E} \longmapsto \det \mathcal{E}$$

corresponds to the isogeny "multiplication by h"

$$[h]\colon E(k) \longrightarrow E(k),$$
$$P \longmapsto [h]P := P + \cdots + P \qquad (h - times),$$

or to the isogeny "multiplication by $-h$"

$$[-h]\colon E(k) \longrightarrow E(k),$$
$$P \longmapsto [-h]P := -P - \cdots - P \qquad (h - times),$$

where h is the greatest common divisor of r and d.

Proof The summation map sum $: \mathrm{Div}^0 X \to E^{\mathrm{sep}}$, $\sum n_i P_i \mapsto \sum [n_i](P_i - P_0)$ induces an isomorphism

$$H^0(X^{\mathrm{sep}}, \mathrm{Pic}^0(X^{\mathrm{sep}})) \cong E(k).$$

Define $\mathcal{A} := \mathcal{L}(P_0)$. Then $\mathcal{A}$ determines a bijection between $\mathrm{Pic}^0 X$ and $\overline{\Omega}(r,d)$ by 3.13 and 4.1 for any admissible pair (r,d) and the assertion follows from the assumption that $\mathrm{Pic}^0 X \cong H^0(X^{\mathrm{sep}}, \mathrm{Pic}^0(X^{\mathrm{sep}}))$. $\square$

An obvious question is when a given pair (r, d) is admissible with respect to i. Using a short MAPLE-program, it is very easy to check which (r, d) are admissible with respect to a given integer i. Plotting the admissible pairs (r, d) for small i, it is also immediately obvious, that the *slope* $\mu = \frac{d}{r}$ of a vector bundle of rank r and degree d decides whether it is admissible or not. This can also be verified by a straightforward calculation.

For $i = 1$ we obtain the usual euclidean algorithm applied to the pair (r, d), with the unusual "twist" that at some point negative "d-entries" are reversed into positive ones. Hence for $i = 1$, our inductive definition of $\omega_{r,d}$ is slightly different from the inductive definition of the map given in [At], Theorem 6. Atiyah did not use the bijective map $\overline{\Omega}(r, d) \to \overline{\Omega}(r, -d)$ and also assumed $0 < d < r$ w.l.o.g., instead of $-\lfloor \frac{r-1}{2} \rfloor \leq d \leq \lfloor \frac{r}{2} \rfloor$ which is used in the version here. In both versions, however, for $i = 1$ it is obvious that any pair (r, d) is "admissible". Thus we have reproved the classical result (for elliptic curves over algebraically closed fields) for elliptic curves with a rational point over any base field. That is, any fixed line bundle $\mathcal{A}$ on X of degree 1 determines a bijection

$$\omega_{r,d}\colon \overline{\Omega}_X(h, 0) \longrightarrow \overline{\Omega}_X(r, d),$$

for any curve X/k of genus 1 with a rational point, where $h = gcd(r, d)$. In particular, we obtain a bijection between the set $\overline{\Omega}_X(r, d)$ and the set $X(k)$ of k-rational points on X.

For any pair (r, d) with r and d coprime ($r > 0$), the map given by

$$\begin{aligned} \Omega_{\overline{X}}(r, d) &\longrightarrow \operatorname{Pic}^d \overline{X} \\ \mathcal{E} &\longmapsto \det \mathcal{E} \end{aligned}$$

is bijective ([At], Corollary to Theorem 7). An immediate consequence (proved by descent) is

4.5 Lemma *Suppose that k is perfect. Let r and d be coprime. Then any absolutely indecomposable vector bundle $\mathcal{E}$ on X of rank r and degree d is uniquely determined (up to isomorphism) by its determinant.*

That means the map $\overline{\Omega}_X(r, d) \to \operatorname{Pic}^d X$, $\mathcal{E} \mapsto \det \mathcal{E}$, is always injective provided that r and d are coprime and k is a perfect base field. Under certain stronger assumptions, the determinant induces a bijective map from $\overline{\Omega}_X(r, d)$ to $\operatorname{Pic}^d X$ in our more general setting as well: By descent we get for instance

4.6 Lemma *Let k be a perfect field, and X be a curve of genus 1 and index i over k satisfying $H^0(Pic^0 \overline{X}) \cong Pic^0 X$. Let (r, d) be an admissible pair such that r and d are coprime. Then*

$$\det\colon \overline{\Omega}_X(r, d) \longrightarrow \operatorname{Pic}^d X$$

is a bijective map.

Proof It remains to show surjectivity: Given $Q \in \mathrm{Pic}^d X$, we can find an element $\mathcal{G} \in \Omega_{\overline{X}}(r,d)$ such that $\det \mathcal{G} \cong \overline{Q}$. Since ${}^\sigma\overline{Q} \cong \overline{Q}$ for all $\sigma \in G$, we conclude that $\det({}^\sigma\mathcal{G}) \cong \det \mathcal{G}$ for all $\sigma \in G$, and hence ${}^\sigma\mathcal{G} \cong \mathcal{G}$ for all $\sigma \in G$ using the injectivity of the map det. Hence, by 4.11 below, $\mathcal{G}$ is defined over X, provided that (r,d) is an admissible pair. $\square$

With the same argument one can also prove that $\det\colon \overline{\Omega}(r,d) \to \mathrm{Pic}^d X$ is bijective, for an admissible pair (r,d) of coprime integers, in case the Brauer group $\mathrm{Br}(k)$ of the (perfect) base field is trivial (then ${}^\sigma\mathcal{G} \cong \mathcal{G}$ for all $\sigma \in G$ also implies that $\mathcal{G}$ is defined over X, since $\gamma(\mathcal{G}) \in \mathrm{Br}(k)$ is trivial, cf. [AEJ1]). A straightforward proof without descent shows that one can again drop the assumptions that $H^0(\overline{X}, \mathrm{Pic}^0 \overline{X}) \cong \mathrm{Pic}^0 X$ in 4.6, and one obtains the following as a direct consequence of Proposition 4.2:

4.7 Corollary *Let X be a curve of genus 1 and index i over a perfect field k. Let r and d be coprime, i.e. $h = 1$. Then the map*

$$\det\colon \overline{\Omega}(r,d) \longrightarrow \mathrm{Pic}^d X$$

is bijective, for any admissible pair (r,d) with respect to i. In particular, it is bijective for any pair (r,d), if X has index 1.

As in [At], for any admissible pair (r,d) with respect to i, denote a vector bundle on X which is isomorphic to $\omega_{r,d}(\mathcal{F}_h)$ by $\mathcal{E}_{\mathcal{A}}(r,d)$. In particular, that means that $\mathcal{E}_{\mathcal{A}}(h,0) \cong \mathcal{F}_h$.

4.8 Proposition *Let k be perfect. Let r and d be coprime, and let (r,d) be an admissible pair with respect to the index i of the curve X over k of genus 1.*
(i) $\mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L} \cong \mathcal{E}_{\mathcal{A}}(r,d)$ if and only if $\mathcal{L}^r \cong \mathcal{O}_X$, for any $\mathcal{L} \in \mathrm{Pic}^0 X$.
(ii) $\mathcal{E}_{\mathcal{A}}(r,d)^\vee \cong \mathcal{E}_{\mathcal{A}}(r,-d)$.
(iii) Suppose that $\mathcal{E} \in \overline{\Omega}(r,d)$ such that $\mathcal{E} \cong \omega_{r,d}(\mathcal{S} \otimes \mathcal{F}_h)$ with a line bundle $\mathcal{S}$ of degree 0 such that $\mathcal{S}$ is isomorphic to the r^{th} power of some line bundle on X. Then $\mathcal{E} \cong \mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}$ for a line bundle $\mathcal{L}$ on X of degree 0.

Proof (i) $\mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L} \cong \mathcal{E}_{\mathcal{A}}(r,d)$ if and only if $\det(\mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}) \cong \det \mathcal{E}_{\mathcal{A}}(r,d)$ by the above corollary. This however is equivalent to $\det \mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}^r \cong \det \mathcal{E}_{\mathcal{A}}(r,d)$, and we obtain the assertion.

(ii) $\det \mathcal{E}_{\mathcal{A}}(r,d)^\vee \cong (\det \mathcal{E} \otimes \mathcal{A}^{d/i})^\vee \cong \mathcal{A}^{-d/i} \cong \det \mathcal{E}_{\mathcal{A}}(r,-d)$.

(iii) By 4.1, $\det \mathcal{E} \cong \mathcal{S} \otimes \mathcal{A}^{d/i}$ or $\det \mathcal{E} \cong \mathcal{S}^\vee \otimes \mathcal{A}^{d/i}$. Furthermore, for any line bundle $\mathcal{L}$ on X of degree 0 it is known that $\det(\mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}) \cong \mathcal{A}^{d/i} \otimes \mathcal{L}^r$. Therefore it follows that $\mathcal{E} \cong \mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}$ or $\mathcal{E} \cong \mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}^\vee$ if $\mathcal{S} \cong \mathcal{L}^r$ (4.7). $\square$

Atiyah ([At], Corollary to Thm.7) thus proves that every vector bundle $\mathcal{E} \in \overline{\Omega}(r,d)$ of coprime rank and degree is isomorphic to a bundle $\mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}$, for a suitable line bundle

$\mathcal{L}$ of degree 0, provided that the base field k is algebraically closed. For curves of genus 1 over arbitrary base fields this description does not hold in general: Let k be a number field. By the Mordell-Weil Theorem (cf. for instance [Si], p.189), the group $E(k)$ is finitely generated, hence not divisible, and it contains elements which do not have r^{th} roots. In our setting, this means that there are vector bundles $\mathcal{E} \in \overline{\Omega}(r,d)$ of coprime rank and degree, which cannot be expressed as the tensor product of $\mathcal{E}_{\mathcal{A}}(r,d)$ with some suitable line bundle.

4.9 Corollary *Let $k = \mathbb{R}$. Suppose that $\mathcal{E}$ is an absolutely indecomposable vector bundle on X of odd rank and coprime rank and degree. Then $\mathcal{E} \cong \mathcal{E}_{\mathcal{A}}(r,d) \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on X of degree 0.*

Proof For $k = \mathbb{R}$, the index of X is either 1 or 2. Moreover, in case $i = 2$ any pair (r,d) with r odd is admissible (4.3). Also, any line bundle $\mathcal{L}$ of degree 0 has an r^{th} root ([SiT], p.42). □

4.10 Proposition *Let X be a curve of genus 1 and index i over k. Let $\mathcal{A}_0$ be a fixed line bundle on X of degree i. Let*

$$\omega := \omega_{r,d} \colon \overline{\Omega}_{X^{\text{sep}}}(h,0) \longrightarrow \overline{\Omega}_{X^{\text{sep}}}(r,d)$$

be the bijection defined in 4.1 which is determined by $\mathcal{A} := \mathcal{A}_0 \otimes \mathcal{O}_{X^{\text{sep}}}$, for any admissible pair (r,d) with respect to i, h the greatest common divisor of r and d. Let $\mathcal{E}$ be an indecomposable vector bundle on X^{sep} of rank h and degree 0.
(a) *$\omega({}^{\sigma}\mathcal{E}) \cong {}^{\sigma}\omega(\mathcal{E})$, for all $\sigma \in G = \mathrm{Gal}(k^{\text{sep}}/k)$.*
(b) *$\mathcal{E}$ is defined over X if and only if $\omega(\mathcal{E})$ is defined over X.*

This generalizes [T], 6.7 and 6.8 to curves of genus 1 with *arbitrary* index.

Proof Both (a) and (b) are proved by induction.

(a) We have $\omega_{r,0}({}^{\sigma}\mathcal{E}) = {}^{\sigma}\mathcal{E} \cong {}^{\sigma}\omega_{r,0}(\mathcal{E})$. Furthermore, if $\omega_{r,d}({}^{\sigma}\mathcal{E}) \cong {}^{\sigma}\omega_{r,d}(\mathcal{E})$ then ${}^{\sigma}(\omega_{r,d+ir}(\mathcal{E})) \cong \omega_{r,d}({}^{\sigma}\mathcal{E}) \otimes {}^{\sigma}\mathcal{A} \cong \omega_{r,d}({}^{\sigma}\mathcal{E}) \otimes \mathcal{A} \cong \omega_{r,d+ir}({}^{\sigma}\mathcal{E})$ since $\mathcal{A} \cong {}^{\sigma}\mathcal{A}$. For $0 < d < \min\{r, \lfloor \frac{ir-1}{2} \rfloor\} =: S_r$, assuming $\omega_{r-d,d}({}^{\sigma}\mathcal{E}) \cong {}^{\sigma}(\omega_{r-d,d}(\mathcal{E}))$ the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X^{\text{sep}}}^{(d)} \longrightarrow \omega_{r,d}(\mathcal{E}) \longrightarrow \omega_{r-d,d}(\mathcal{E}) \longrightarrow 0,$$

yields the short exact sequences

$$0 \longrightarrow \mathcal{O}_{X^{\text{sep}}}^{(d)} \longrightarrow {}^{\sigma}(\omega_{r,d}(\mathcal{E})) \longrightarrow \omega_{r-d,d}({}^{\sigma}\mathcal{E}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{X^{\text{sep}}}^{(d)} \longrightarrow \omega_{r,d}({}^{\sigma}\mathcal{E}) \longrightarrow \omega_{r-d,d}({}^{\sigma}\mathcal{E}) \longrightarrow 0.$$

It follows that ${}^{\sigma}(\omega_{r,d}(\mathcal{E})) \cong \omega_{r,d}({}^{\sigma}\mathcal{E})$. For $d < 0$, $\omega_{r,d}({}^{\sigma}\mathcal{E}) \cong \omega_{r,-d}({}^{\sigma}\mathcal{E})^{\vee} \cong {}^{\sigma}(\omega_{r,-d}(\mathcal{E})^{\vee}) \cong {}^{\sigma}(\omega_{r,d}(\mathcal{E}))$.

(b) A bundle $\mathcal{E}$ of rank h and degree 0 is defined over X if and only if $\omega_{r,0}(\mathcal{E}) = \mathcal{E}$ is defined over X. Since $\mathcal{A}$ is defined over X, we know that $\omega_{r,d+r}(\mathcal{E}) \cong \omega_{r,d}(\mathcal{E}) \otimes \mathcal{A}$ is defined over X if and only if $\omega_{r,d}(\mathcal{E})$ is defined over X, which by induction hypothesis is the case if and only if $\mathcal{E}$ is defined over X. For $0 < d < S_r$ consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X^{\text{sep}}}^{(d)} \longrightarrow \omega_{r,d}(\mathcal{E}) \longrightarrow \omega_{r-d,d}(\mathcal{E}) \longrightarrow 0.$$

By induction hypothesis, $\mathcal{E}$ is defined over X if and only if $\omega_{r-d,d}(\mathcal{E})$ is defined over X, that means $\omega_{r-d,d}(\mathcal{E}) \cong \mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{\text{sep}}}$ for a suitable vector bundle $\mathcal{E}_0$ on X. By 3.11 there is a unique absolutely indecomposable vector bundle $\mathcal{G}_0 \in \overline{\Omega}_X(r,d)$ such that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X^{(d)} \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

and obviously, $\mathcal{G}_0 \otimes \mathcal{G}_{X^{\text{sep}}} \cong \omega_{r,d}(\mathcal{E})$. Thus, $\omega_{r,d}(\mathcal{E})$ is defined over X. Conversely, if $\omega_{r,d}(\mathcal{E})$ is defined over X, then a similar argument using 3.8 implies that also $\omega_{r-d,d}(\mathcal{E})$ is defined over X and therefore $\mathcal{E}$ is, too. It remains to check what happens in (iv), however, this is straightforward, too. $\square$

For a proper scheme X over a perfect field k, [AEJ1], 3.7 gives a necessary and sufficient condition for an indecomposable vector bundle $\mathcal{E}$ on $\overline{X}$, which is G-invariant, to be defined over X: For any such $\mathcal{E}$ there exists a pure indecomposable vector bundle $\mathcal{M}$ on X, unique up to isomorphism, of type $\mathcal{E}$ ([AEJ1], 3.4). Define $\gamma(\mathcal{E})$ to be the class of the central simple k-algebra $D(\mathcal{M}) := \mathrm{End}(\mathcal{M})/rad(\mathrm{End}(\mathcal{M}))$ (where rad is the Jacobson radical) in the Brauer group $\mathrm{Br}(k)$ of k. The bundle $\mathcal{E}$ is defined over X if and only if $\gamma(\mathcal{E})$ is trivial. As remarked in [AEJ1], 3.8, this does not sound like a practical criterium, however, the authors proceed to point out a way to compute $\gamma(\mathcal{E})$ without first finding the vector bundle $\mathcal{M}$: They compute the cocycle which gives the class of $\gamma(\mathcal{E})$ in $\mathrm{Br}(k)$, thus generalizing ([T], 4.14), which only studied the situation for curves.

If X/k is an elliptic curve with a rational point, every absolutely indecomposable vector bundle $\mathcal{E}$ on $\overline{X}$ which is invariant under the action of G is already defined over X ([AEJ1], 4.2). This result is not true any more for the general situation where X/k is a curve of genus 1 and index i greater than 1. The following weaker version is obtained instead:

4.11 Theorem *Let X be a curve of genus 1 and index i over a perfect field k, satisfying $H^0(\mathrm{Pic}^0(X^{\text{sep}})) \cong \mathrm{Pic}^0 X$. Let (r,d) be an admissible pair with respect to i. Then any*

G-invariant isomorphism class of an indecomposable vector bundle $\mathcal{G} \in \Omega_{\overline{X}}(r,d)$ is defined over X. In particular, $\gamma(\mathcal{G})$ is trivial.

Thus, here there are classes of vector bundles on $\overline{X}$ where G-invariance automatically implies that the bundle is defined over X. We do not need to first compute $D(\mathcal{M})$ directly as above to check this.

Proof Let $\mathcal{G} \in \Omega_{\overline{X}}(r,d)$ such that ${}^{\sigma}\mathcal{G} \cong \mathcal{G}$ for all $\sigma \in G = \mathrm{Gal}(\overline{k}/k)$. There exists an $\mathcal{E} \in \Omega_{\overline{X}}(h,0)$ such that $\omega_{r,d}(\mathcal{E}) \cong \mathcal{G}$. This implies that

$$ {}^{\sigma}\mathcal{G} \cong {}^{\sigma}\omega_{r,d}(\mathcal{E}) \cong \omega_{r,d}({}^{\sigma}\mathcal{E}) \cong \omega_{r,d}(\mathcal{E}) \cong \mathcal{G}, $$

and hence that $\mathcal{E}$ has a G-invariant isomorphism class. By 3.13, $\mathcal{E} \cong \mathcal{L} \otimes \mathcal{F}_h$ for a suitable $\mathcal{L} \in \mathrm{Pic}^0\overline{X}$, so this G-invariance implies that also ${}^{\sigma}\mathcal{L} \cong \mathcal{L}$ for all $\sigma \in G$. Hence $\mathcal{L}$ and also $\mathcal{L} \otimes \mathcal{F}_h$ is defined over X, which is equivalent to $\mathcal{G}$ being defined over X by 4.10. □

Using our previous results we can investigate indecomposable vector bundles as well. We have to restrict ourselves to perfect base fields, since one of the main results we need has only been proved in this setting: Let ℓ be a maximal field contained in $D(\mathcal{M})$, where $\mathcal{M}$ is an indecomposable vector bundle on X. Then there is an absolutely indecomposable vector bundle $\mathcal{N}$ on $Y = X \times_k \ell$ such that $\mathcal{M} = \mathrm{tr}_{\ell/k}(\mathcal{N})$ provided that k is perfect. In particular, $\deg \mathcal{M} = [\ell\colon k] \deg \mathcal{N}$ and rank $\mathcal{M} = [\ell\colon k]$ rank $\mathcal{N}$ ([AEJ1], 1.4, 1.8). Thus, if $(r,d) = (\mathrm{rank}\,\mathcal{M}, \deg \mathcal{M})$ is an admissible pair with respect to i, so is $(r',d') = (\mathrm{rank}\,\mathcal{N}, \deg \mathcal{N})$, and vice versa. This will be used repeatedly. If X is any complete regular curve over a perfect field k, then the invariant γ defined above restricted to line bundles corresponds to the map $H^0(\mathrm{Pic}\overline{X}) \to \mathrm{Br}(k)$ in the well-known exact sequence

$$ 0 \longrightarrow \mathrm{Pic}X \longrightarrow H^0(\mathrm{Pic}\overline{X}) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(k(X)) $$

([AEJ1], 3.9). Assuming that X has a rational point, it is proved in [AEJ1], 3.11 that $D(\mathcal{M})$ is a field, for any indecomposable vector bundle $\mathcal{M}$ on X, which decomposes into line bundles over $\overline{X}$. If X even is an elliptic curve, then $D(\mathcal{M})$ is a field, for any indecomposable vector bundle $\mathcal{M}$ on X ([T], 6.10 or [AEJ1], 4.3). If X is a curve of genus 1 without a rational point, we use a similar argument as in the two results just mentioned:

4.12 Proposition *Let X be a curve of genus 1 and index i over a perfect field k. Let $\mathcal{M}$ be an indecomposable vector bundle on X. Then $D(\mathcal{M})$ is a field if one of the following holds:*

(i) $\overline{\mathcal{M}} \cong \mathcal{M} \otimes \mathcal{O}_{\overline{X}}$ decomposes into the direct sum of line bundles on $\overline{X}$, and
$H^0(\overline{X}, Pic\overline{X}) \cong PicX$.

(ii) $\mathrm{Br}(k) = 0$.

(iii) $\mathcal{M}$ *has rank* r *and degree* d *with* (r,d) *an admissible pair with respect to* i*, and* $H^0(Pic^0\overline{X}) \cong Pic^0 X$.

Proof First we follow the argument given in [AEJ1], 3.11: Let $k' = Z(\mathcal{M})$ be the center of $D(\mathcal{M})$ and put $X' := X \times_k k'$. Write $\mathcal{M} = \mathrm{tr}_{k'/k}(\mathcal{G})$ for some vector bundle $\mathcal{G}$ on X'. Then $\overline{\mathcal{G}}$ is a direct summand of $\overline{\mathcal{M}}$ and $D(\mathcal{M}) = D(\mathcal{G})$. Replace k by k' and $\mathcal{M}$ by $\mathcal{G}$ and assume that $D(\mathcal{M})$ is central and hence $\mathcal{M}$ pure of a certain type, say of type $\mathcal{N}$. Since $\overline{\mathcal{M}}$ is obviously G-invariant, so is $\mathcal{N}$. It remains to check if $\mathcal{N}$ is defined over X (resp. if $\gamma(\mathcal{N})$ is trivial in $\mathrm{Br}(k)$) because of this. In that case it would follow that $D(\mathcal{M})$ is a field.

(i) If $\overline{\mathcal{M}}$ decomposes into the direct sum of line bundles then $\mathcal{N} \in \mathrm{Pic}\overline{X}$ and is by assumption defined over X.

(ii) If $\mathrm{Br}(k) = 0$ then $\gamma(\mathcal{N})$ has to be trivial.

(iii) If $\mathcal{N} \in \overline{\Omega}_{\overline{X}}(r', d')$ with (r', d') an admissible pair with respect to i, then $\mathcal{N}$ is defined over X by 4.11. However, since (r, d) is admissible by assumption, so is (r', d'). $\square$

For a curve of genus 1 over a perfect field which has index 1, the naturality of Atiyah's classification implies that for an indecomposable vector bundle $\mathcal{E}$ on X written as $\mathcal{E} = \mathrm{tr}_{\ell/k}(\mathcal{N})$ with $\mathcal{N}$ an absolutely indecomposable vector bundle on $X = X \times_k \ell$, the field ℓ is the residue class field of the ℓ-rational point corresponding to $\mathcal{N}$ under Atiyah's original classification ([AEJ1], 4.4). More generally:

4.14 Proposition *Let* X *be a curve of genus 1 and index* i *over a perfect field* k *satisfying* $H^0(\mathrm{Pic}^0(\overline{X})) \cong \mathrm{Pic}^0 X$*. Let* $\mathcal{E}$ *be an indecomposable vector bundle on* X *and* $\mathcal{E} = \mathrm{tr}_{\ell/k}(\mathcal{N})$ *for an absolutely indecomposable vector bundle* $\mathcal{N}$ *on* $Y = X \times_k \ell$ *such that* $\mathcal{N} \in \overline{\Omega}_Y(r,d)$ *where* (r,d) *is admissible with respect to* $ind\, Y = i'$*. Then* $\ell \cong k(P)$ *where* P *is the (*ℓ*-rational) point of* E/k *corresponding to* $\mathcal{N}$ *under the classification theorem 4.4.*

Proof By 4.4, we know that there is a bijection ω between $\overline{\Omega}_Y(r,d)$ and $E(\ell)$, since $H^0(\mathrm{Pic}^0(\overline{Y})) \cong \mathrm{Pic}^0 Y$ by assumption. Let $P \in E$ be the ℓ-rational point corresponding to $\mathcal{N}$ under this bijection. By [AEJ1], 1.9, $\mathcal{N}$ is not defined over Y_0, for any proper separable subextension ℓ_0 of ℓ/k. Thus $k(P) \cong \ell$, since for $k(P) \cong k$ or $k(P) \cong \ell_0$ we would obtain the contradiction that $\mathcal{N}$ has to be defined over Y_0 already. $\square$

It is an obvious question whether non-admissible pairs (r,d) play a special role, or whether the method of proof presented here is just not powerful enough. Under some additional assumption on the curve there indeed are no *absolutely* indecomposable vector bundles over a curve of genus 1 with non-admissible rank and degree (r,d). The following theorem is due to a conversation with J.P. Serre at the TMR-conference in Duisburg in

September 2001. It generalizes Serre's idea how to construct a counterexample of an absolutely indecomposable bundle of rank and degree 2 for the index 2 case.

4.15 Theorem *Let X be a curve of genus 1 and index $i > 1$ over a perfect field k, satisfying $H^0(\mathrm{Pic}(\overline{X})) \cong \mathrm{Pic}X$.*

(i) There is no absolutely indecomposable vector bundle on X of rank r and degree nr, for any non-zero integer n which is not a multiple of the index i.

(ii) There is no absolutely indecomposable vector bundle on X of rank r and degree d, where the pair (r, d) is not admissible with respect to the index i.

Proof (i) Choose a line bundle $\mathcal{S}$ on $\overline{X}$ of degree n unequal to 0 such that n is not a multiple of i. By 3.13, there is a bijective map between the sets $\mathrm{Pic}^0\overline{X}$ and $\overline{\Omega}(r, nr)$ given by $\mathcal{L} \to \mathcal{F}_r \otimes \mathcal{L} \otimes \mathcal{S}$. Suppose there exists an absolutely indecomposable vector bundle $\mathcal{E}$ on X of rank r and degree nr. Then $\overline{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}_{\overline{X}} \cong \mathcal{F}_r \otimes (\mathcal{L} \otimes \mathcal{S})$ for a suitable line bundle $\mathcal{L}$ of degree 0, and $\mathcal{N} := \mathcal{L} \otimes \mathcal{S} \in \mathrm{Pic}\overline{X}$ has degree n. For every $\sigma \in G$, we have $\mathcal{F}_r \otimes^\sigma \mathcal{N} \cong \mathcal{F}_r \otimes \mathcal{N}$, hence

$$({}^\sigma\mathcal{N} \otimes \mathcal{N}^\vee) \otimes \mathcal{F}_r \cong \mathcal{F}_r.$$

However, $\deg({}^\sigma\mathcal{N} \otimes \mathcal{N}^\vee) = 0$ and thus ${}^\sigma\mathcal{N} \cong \mathcal{N}$ by uniqueness. By assumption, $H^0(\mathrm{Pic}\overline{X}) \cong \mathrm{Pic}X$, so $\mathcal{N}$ must be defined over X, a contradiction, since it has degree n.

(ii) This is proved by induction on r and d analogously to the one used for 4.1. The induction beginning is given by (i). The steps are as given in the algorithm at the beginning of section 4. First there is a bijection between $\overline{\Omega}(r, d)$ and $\overline{\Omega}(r, d + ir)$ given by $\mathcal{E} \to \mathcal{E} \otimes \mathcal{A}$ for a fixed line bundle $\mathcal{A}$ on X of degree i. Obviously, $\overline{\Omega}(r, d)$ is an empty set if and only if $\overline{\Omega}(r, d + ir)$ is empty. The same argument applies to the bijection between the sets $\overline{\Omega}(r, d)$ and $\overline{\Omega}(r - d, d)$, for every $0 < d < r$, and of course the trivial one between $\overline{\Omega}(r, -d)$. Using the "euclidean algorithm" described at the beginning of section 4 on a non-admissible pair (r, nr) inductively proves that $\overline{\Omega}(r, d)$ is an empty set, for each non-admissible pair with respect to the index of X. $\square$

Using descent, Tillmann classifies the indecomposable vector bundles on a curve of genus 1 and index 1 over a perfect base field. For any pair (r, d) of integers $(r > 0)$ she constructs a bijection α between the set of isomorphism classes $\Omega(r, d)$ of indecomposable vector bundles on X of rank r and degree d, and the set of points $\{P \in X \mid \deg P \text{ divides } r\}$, where for a bundle $\mathcal{E} \in \Omega(r, d)$ the field $D(\mathcal{E})$ is isomorphic to the residue class field $k(P)$ of the point P associated to $\mathcal{E}$ under this map. Her map extends the bijection between $\overline{\Omega}_{\overline{X}}(r, d)$ and $\overline{X}$ given by [At] canonically: For an absolutely indecomposable vector bundle $\mathcal{E} \in \Omega(r, d)$, $\alpha(\mathcal{E}) \otimes \mathcal{O}_{\overline{X}} \cong \overline{\alpha}(\overline{\mathcal{E}})$, where $\overline{\alpha}$ denotes Atiyah's bijection from [At], Thm. 7.

In our more general situation, there is a bijection between $\Omega(r,d)$, for an admissible pair (r,d) with respect to i, and all the points of the *Jacobian* E/k of X/k of degree dividing r.

4.16 Proposition *Let X be a curve of genus 1 and index i over a perfect field k such that $H^0(Pic^0\overline{X}) \cong Pic^0 X$. Fix a line bundle $\mathcal{A}$ on X of degree i. For any admissible pair (r,d) with respect to i there is a bijection*

$$\alpha_{r,d}\colon \Omega(r,d) \longrightarrow \{P \in E \mid \deg P \text{ divides } r\},$$

where $\alpha_{r,d}(\mathcal{E})$ is a k-rational point on E, for any absolutely indecomposable vector bundle $\mathcal{E}$ on X, since $D(\mathcal{E}) \cong k(P)$.

Proof Let $\mathcal{E} \in \Omega(r,d)$. Since (r,d) is an admissible pair with respect to the index of X, $D(\mathcal{E}) = k'$ is a field extension of k (4.12). Let G here denote the set of k'/k-conjugates in $\overline{k}$. By [AEJ1], 1.8 and 1.9, there exists an absolutely indecomposable vector bundle $\mathcal{N}$ on $X' := X \times_k k'$ such that $\mathcal{E} = \mathrm{tr}_{k'/k}(\mathcal{N})$, and such that $\mathcal{N}$ is not already defined on $X_0 := X \times_k k_0$, for some proper subextension k_0 of k'/k. Put $r' := \operatorname{rank} \mathcal{N}$ and $d' := \deg \mathcal{N}$, then (r',d') again is an admissible pair with respect to i. In particular, $d = [k' : k]d'$ and $r = [k' : k]r'$. Moreover, $\mathcal{E} \otimes \mathcal{O}_{\overline{X}} \cong \bigoplus {}^{\sigma}\overline{\mathcal{N}}$ where σ runs through all the k'/k-conjugates, and where ${}^{\sigma}\mathcal{N} \ncong {}^{\tau}\mathcal{N}$ for all $\sigma \neq \tau$ ([T], 4.12). By 4.14, $D(\mathcal{N}) \cong k' \cong D(\mathcal{M}) \cong k(P)$, where P is the k'-rational point of E corresponding to $\mathcal{N}$ under the canonical bijection in 4.4,

$$\tilde{\omega}_{r',d'}\colon E(k') \longrightarrow \overline{\Omega}_{X'}(r',d')$$

which is defined with the help of the line bundle $\mathcal{A}' := \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ as explained in 4.1. Define $\alpha_{r,d}(\mathcal{E}) := P$. The resulting map is well-defined, since the point $P \in E(k')$ is independent of the choice of the indecomposable summand of $\overline{\mathcal{E}}$: Points corresponding to two nonisomorphic summands ${}^{\sigma}\overline{\mathcal{N}}$ and ${}^{\sigma}\overline{\mathcal{N}}$ of $\overline{\mathcal{E}}$ are conjugated with respect to G (4.10 and [T], 4.12).

To show the surjectivity of $\alpha_{r,d}$, take a point $P \in E$ such that $\deg P$ divides r, and put $k' := k(P)$. Define $r' := \frac{r}{[k':k]}$ and $d' := \frac{d}{[k':k]}$. The pair (r',d') is admissible with respect to i, since it has the same slope as (r,d). Let $\mathcal{N}$ be the absolutely indecomposable vector bundle on $X' := X \times_k k'$ which corresponds to P under the canonical bijection (defined with the help of the line bundle $\mathcal{A}' = \mathcal{A} \otimes \mathcal{O}_{X'}$) from 4.4. Define $\mathcal{E} := \mathrm{tr}_{k'/k}(\mathcal{N})$. Then $\mathcal{E}$ has rank r and degree d, and we claim that it is indeed indecomposable. Since ${}^{\sigma}P \neq {}^{\tau}P$ ([T], 3.5.2) it follows that ${}^{\sigma}\overline{\mathcal{N}} \ncong {}^{\tau}\overline{\mathcal{N}}$ as well, for all k'/k-conjugates $\sigma \neq \tau$ (4.9). However, this already implies that $\mathcal{E}$ is indecomposable, and that $D(\mathcal{E}) = D(\mathcal{N})$ ([AEJ1], 3.13). Since (r,d) is an admissible pair with respect to i, we obtain $k' \cong k(P)$, and in particular that the degree of P divides r by 4.13, and that $D(\mathcal{M})$ is a field by 4.11, indeed, $D(\mathcal{M}) \cong k'$.

To check injectivity, assume that there are two vector bundles $\mathcal{E}_1, \mathcal{E}_2 \in \overline{\Omega}(r,d)$ such that $\alpha_{r,d}(\mathcal{E}_1) = \alpha_{r,d}(\mathcal{E}_2)$. Let $\mathcal{E}_i = \mathrm{tr}_{k_i/k}(\mathcal{N}_i)$ with absolutely indecomposable vector bundles $\mathcal{N}_i$ on $X_i := X \times_k k_i$. In particular, then $D(\mathcal{E}_i) = k_i$, for $i = 1,2$, and $D(\mathcal{E}_1) \cong D(\mathcal{N}_1) \cong k(P_1)$ as well as $D(\mathcal{E}_2) \cong D(\mathcal{N}_2) \cong k(P_2)$ by 4.13, where P_i is the k_i-rational point on E corresponding to $\mathcal{N}_i$ under the canonical bijection in 4.14 (defined with the help of the line bundle $\mathcal{A}_i := \mathcal{A} \otimes \mathcal{O}_{X_i}$), for $i = 1,2$. Now $\alpha_{r,d}(\mathcal{E}_1) = \alpha_{r,d}(\mathcal{E}_2)$ implies that P_1 and P_2 are conjugated with respect to G, hence $k_1 \cong k_2$. Now it follows easily that $r' := \operatorname{rank} \mathcal{N}_1 = \operatorname{rank} \mathcal{N}_2 = \frac{r}{[k_2:k]}$ and $d' := \deg \mathcal{N}_1 = \deg \mathcal{N}_2 = \frac{d}{[k_2:k]}$, and that $\mathcal{N}_1$ and $\mathcal{N}_2$ are conjugated with respect to G as well (4.10, [T], 4.12). $\square$

4.17 Corollary *Let X be a curve of genus 1 over a perfect field k such that $H^0(Pic^0\overline{X}) \cong Pic^0 X$. There is a bijection*

$$\alpha_{r,d}: \Omega(r,0) \longrightarrow \{P \in E \mid \deg P \text{ divides } r\}$$

where $\alpha_{r,d}(\mathcal{E})$ is a k-rational point on the Jacobian E/k, for any absolutely indecomposable vector bundle $\mathcal{E}$ on X.

5. Vector bundles of arbitrary admissible rank and degree

For a curve X of genus 1 over a field k and arbitrary index, we describe the structure of the absolutely indecomposable vector bundles of rank r and degree d on X, where r and d are coprime.

5.1 Lemma (cf. [At], Lemma 22) *Let $\mathcal{E} \in \overline{\Omega}(r,d)$ with r and d coprime, (r,d) an admissible pair with respect to the index i of X.*
(a) If k is perfect and char $k = p > 0$ is coprime to r, then $\mathcal{E}nd_X\mathcal{E} \cong \bigoplus_i \mathcal{L}_i \oplus \ldots$, where the $\mathcal{L}_i$ are line bundles on X satisfying $\mathcal{L}_i^r \cong \mathcal{O}_X$.
(b) If char $k = 0$, then $\mathcal{E}nd_{\overline{X}}\overline{\mathcal{E}}$ decomposes into the direct sum of all line bundles on $\overline{X}$ of order dividing r, and $\mathcal{E}nd_X\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \mathrm{tr}_{k_1/k}(\mathcal{N}_1) \oplus \cdots \oplus \mathrm{tr}_{k_s/k}(\mathcal{N}_s)$ is the Krull-Schmidt decomposition of the endomorphism algebra of $\mathcal{E}$, with the $\mathcal{L}_i$ all the line bundles on X satisfying $\mathcal{L}_i^r \cong \mathcal{O}_X$, and with k_i/k separable finite field extensions, $\mathcal{N}_i \in \mathrm{Pic} X_i$ for $X_i := X \times_k k_i$, and $\mathcal{N}_i^r \cong \mathcal{O}_{X_i}$. In particular, $k_i \cong k(P_i)$ where P_i is the k_i-rational point of the Jacobian E/k of X/k corresponding to $\mathcal{N}_1$ under the classification Theorem 4.4., in case $H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$.

Proof Each absolutely indecomposable vector bundle on X of coprime rank and degree is uniquely determined by its determinant by 4.5. Therefore, $\mathcal{E} \cong \mathcal{E} \otimes \mathcal{L}_i$, for each line bundle $\mathcal{L}_i \in \mathrm{Pic}X$ satisfying $\mathcal{L}_i^r \cong \mathcal{O}_X$. This implies that $\mathcal{E}^\vee \otimes \mathcal{E} \cong \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{L}_i$ and thus $\mathcal{E}nd_X\mathcal{E} \cong \mathcal{E}nd_X\mathcal{E} \otimes \mathcal{L}_i$. Since $\mathcal{E}nd_X\mathcal{E}$ contains $\mathcal{O}_X$ as a direct summand ([At], Lemma 19), $\mathcal{E}nd_X\mathcal{E} \cong \bigoplus_i \mathcal{L}_i \oplus \ldots$. If char $k = 0$ there are exactly r^2 line bundles over $\overline{X}$ of order dividing r, so $\mathcal{E}nd_{\overline{X}}\overline{\mathcal{E}}$ decomposes into the direct sum of these. Thus, we obtain the Krull-Schmidt decomposition

$$\mathcal{E}nd_X\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \mathrm{tr}_{k_1/k}(\mathcal{N}_1) \oplus \cdots \oplus \mathrm{tr}_{k_s/k}(\mathcal{N}_s),$$

where k_i/k is a separable finite field extension, $\mathcal{N}_i \in \mathrm{Pic}X_i$, $X_i := X \times_k k_i$ is not defined over some proper subextension of k_i/k and satisfies $\mathcal{N}_i^r \cong \mathcal{O}_{X_i}$. In particular, $k_i \cong k(P_i)$ with P_i the k_i-rational point of the Jacobian E corresponding to $\mathcal{N}_i$ under the classification Theorem 4.4., if $H^0(\mathrm{Pic}^0\overline{X}) \cong \mathrm{Pic}^0 X$. □

For the following, recall that for index $i = 2$, every coprime pair (r, d) is admissible by 4.3. The proof uses 3.17 and the above lemma and is analogous to the one in [At].

5.2 Proposition (cf. [At], Lemma 23) *Let char $k = 0$. Let (r, d) be an admissible pair with respect to $i = ind\, X$ such that r and d are coprime. Then the vector bundle $\mathcal{E}_\mathcal{A}(r, d) \otimes \mathcal{F}_h$ is absolutely indecomposable of rank rh and degree hd, for any $h \geq 1$. In particular,* $\det(\mathcal{E}_\mathcal{A}(r, d) \otimes \mathcal{F}_h) \cong \mathcal{A}^{\frac{dh}{i}}$.

5.3 Proposition (cf. [At], Lemma 24). *Let char $k = 0$. Let (r, d) be an admissible pair with respect to $i = ind\, X$ such that r and d are coprime. Then*

$$\mathcal{E}_\mathcal{A}(r, d) \otimes \mathcal{F}_h \cong \mathcal{E}_\mathcal{A}(rh, dh).$$

Like [At], Lemma 24, this is proved by double induction on r and h.

5.4 Corollary (cf. [At], Corollary 1). *Let char $k = 0$. Let (r, d) be an admissible pair with respect to $i = ind\, X$. Then $\mathcal{E}_\mathcal{A}(r, d) \otimes \mathcal{L} \cong \mathcal{E}_\mathcal{A}(r, d)$ for a line bundle $\mathcal{L}$ on X if and only if $\mathcal{L}^{\frac{r}{h}} \cong \mathcal{O}_X$, where h denotes the greatest common divisor of r and d.*

Proof Put $r' = \frac{r}{h}$ and $d' = \frac{d}{h}$. By the last proposition, $\mathcal{E}_\mathcal{A}(r, d) \cong \mathcal{E}_\mathcal{A}(r', d') \otimes \mathcal{F}_h$. If $\mathcal{L}^{r'} \cong \mathcal{O}_X$ then $\mathcal{E}_\mathcal{A}(r', d') \otimes \mathcal{L} \cong \mathcal{E}_\mathcal{A}(r', d')$ by 4.5, and $\mathcal{E}_\mathcal{A}(r, d) \otimes \mathcal{L} \cong \mathcal{E}_\mathcal{A}(r, d)$.

Now assume that $\mathcal{E}_\mathcal{A}(r, d) \otimes \mathcal{L} \cong \mathcal{E}_\mathcal{A}(r, d)$. Then $\mathcal{E}nd(\mathcal{E}_\mathcal{A}(r, d)) \otimes \mathcal{L} \cong \mathcal{E}nd(\mathcal{E}_\mathcal{A}(r, d))$ and by 5.1,

$$\begin{aligned}\mathcal{E}nd(\mathcal{E}_\mathcal{A}(r, d)) &\cong \mathcal{E}nd(\mathcal{E}_\mathcal{A}(r', d')) \otimes \mathcal{E}nd_{\overline{X}}\mathcal{F}_h \\ &\cong (\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \ldots) \otimes \left(\bigoplus_{j=1}^{h} \mathcal{F}_{2j-1}\right)\end{aligned}$$

with the $\mathcal{L}_i$ all the line bundles on X satisfying $\mathcal{L}_i^{r'} \cong \mathcal{O}_X$. Comparing the direct summands of $\mathcal{E}nd(\mathcal{E}_\mathcal{A}(r,d))$ and $\mathcal{E}nd(\mathcal{E}_\mathcal{A}(r,d)) \otimes \mathcal{L}$ which are line bundles we obtain $\mathcal{L} \cong \mathcal{L}_i$ for some i, hence $\mathcal{L}^{r'} \cong \mathcal{O}_X$. $\square$

5.5 Corollary (cf. [At], Corollary 2) *Let char $k = 0$. Then $\mathcal{E}_\mathcal{A}(r,d)^\vee \cong \mathcal{E}_\mathcal{A}(r,-d)$, for any admissible pair (r,d) with respect to $i = ind\, X$.*

Proof By 5.3, $\mathcal{E}_\mathcal{A}(r,d)^\vee \cong \mathcal{E}_\mathcal{A}(r',d')^\vee \otimes \mathcal{F}_h$ with h the greatest common divisor of r and d, and $r' = \frac{r}{h}$, $d' = \frac{d}{h}$. By 4.8, we know that $\mathcal{E}_\mathcal{A}(r',d')^\vee \cong \mathcal{E}_\mathcal{A}(r',-d')$ and obtain $\mathcal{E}_\mathcal{A}(r,d)^\vee \cong \mathcal{E}_\mathcal{A}(r',-d') \otimes \mathcal{F}_h \cong \mathcal{E}_\mathcal{A}(r,-d)$. $\square$

5.6 Proposition (cf. [At], Lemma 25) *Let $k = \mathbb{R}$. Let $r - d$ be odd, and let r and d be coprime (and hence (r,d) an admissible pair with respect to the index i of X if $i = 2$ by 4.3) with $0 < d < r$, and let $\mathcal{L} \in \mathrm{Pic}^0 X$. There exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{(dh)} \longrightarrow \mathcal{E}_\mathcal{A}(rh,dh) \otimes \mathcal{L} \longrightarrow \mathcal{E}_\mathcal{A}(rh-dh,dh) \otimes \mathcal{S} \longrightarrow 0$$

where $\mathcal{S}$ is any line bundle satisfying $\mathcal{S}^{r-d} \cong \mathcal{L}^r$.

Proof Let $h = 1$. Since $0 < d < r$, there exists a short exact sequence

$$0 \longrightarrow \mathcal{O}_X^{(d)} \longrightarrow \mathcal{E}_\mathcal{A}(r,d) \otimes \mathcal{L} \longrightarrow \mathcal{E}' \longrightarrow 0$$

with $\mathcal{E}' \in \overline{\Omega}(r-d,d)$ by 3.8. Moreover, $\mathcal{E}' \cong \mathcal{E}_\mathcal{A}(r-d,d) \otimes \mathcal{S}$ for some $\mathcal{S} \in \mathrm{Pic}^0 X$ because of $k = \mathbb{R}$ (4.9). The remaining assertion then is obvious, and the induction beginning is proved. Now let $h > 1$. The exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F}_h \longrightarrow \mathcal{F}_{h-1} \longrightarrow 0$$

tensored with $\mathcal{E}_\mathcal{A}(r,d)$ with r and d as in the assumption yields the exact sequence

$$0 \longrightarrow \mathcal{E}_\mathcal{A}(r,d) \longrightarrow \mathcal{E}_\mathcal{A}(r,d) \otimes \mathcal{F}_h \longrightarrow \mathcal{E}_\mathcal{A}(r,d) \otimes \mathcal{F}_{h-1} \longrightarrow 0.$$

As in 5.3, write this sequence as

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$$

with $\mathcal{E}_j \in \overline{\Omega}(r_j,d_j)$ and $r_2 = rh$, $r_3 = r(h-1)$, $d_2 = dh$, $d_3 = d(h-1)$. In particular, $0 < d_i < r_i$ as well. Now consider an analogous diagram as in 5.3. Tensor the middle row with $\mathcal{L}$. We obtain a new diagram of the same form in which $\mathcal{G}_1$ and $\mathcal{G}_3$ are replaced by $\mathcal{G}_1 \otimes \mathcal{S}$ and $\mathcal{G}_3 \otimes \mathcal{S}$ where $\mathcal{S}^{r-d} \cong \mathcal{L}^r$ holds. As in the original proof of Atiyah, we can thus conclude that the midde term in the new bottom row must be isomorphic to $\mathcal{G}_2 \otimes \mathcal{S}$, proving the assertion. $\square$

5.7 Lemma (cf. [At], Lemma 26) *Let $k = \mathbb{R}$. Let (r, d) be an admissible pair with respect to the index i of X such that $r - d$ is odd and $0 \leq d < r$, and consider $\mathcal{E} \in \overline{\Omega}(r, d)$. Then there exists a line bundle $\mathcal{L}$ of degree zero such that $\mathcal{E} \cong \mathcal{E}_{\mathcal{A}}(r, d) \otimes \mathcal{L}$.*

For $i = 2$, this result contains no more information than what we already know from 4.9, where arbitrary admissible pairs with respect to i of coprime rank and degree are investigated.

Proof The case that $d = 0$ is covered already in 3.13. We proceed again by induction on r, assuming that $0 < d < r$. There exists an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{(d)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0$$

with $\mathcal{E}' \in \overline{\Omega}(r - d, d)$ by 3.8. By induction hypothesis, there is a line bundle $\mathcal{S}$ of degree zero such that $\mathcal{E}' \cong \mathcal{E}_{\mathcal{A}}(r - d, d) \otimes \mathcal{S}$. Let $\mathcal{L} \in \operatorname{Pic}^0 X$ such that $\mathcal{L}^{r/h} \cong \mathcal{S}^{(r-d)/h}$ with h the greatest common divisor of r and d. By 5.6, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{(d)} \longrightarrow \mathcal{E}_{\mathcal{A}}(r, d) \otimes \mathcal{L} \longrightarrow \mathcal{E}' \longrightarrow 0$$

and we obtain $\mathcal{E} \cong \mathcal{E}_{\mathcal{A}}(r, d) \otimes \mathcal{L}$. $\square$

Since $\mathcal{E}_{\mathcal{A}}(r, d) \otimes \mathcal{A} \cong \mathcal{E}_{\mathcal{A}}(r, d + ir)$ w.l.o.g. we only have to obtain results on admissible pairs (r, d) with respect to i satisfying $0 < d < ir$, or admissible pairs satisfying $-\left[\frac{ir-1}{2}\right] \leq d < \left[\frac{ir}{2}\right]$. Moreover, if we choose the latter range, using the fact that $\mathcal{E}_{\mathcal{A}}(r, d)^{\vee} \cong \mathcal{E}_{\mathcal{A}}(r, -d)$ allows us to even restrict ourselves to admissible pairs (r, d) with $0 \leq d \leq \left[\frac{ir-1}{2}\right]$. This shows that for $i = 2$ the requirement $0 < d < r$ needed in 5.7 and 5.8 actually is no restriction as long as r is odd (and $d \neq 0$). It creates problems only when the rank r is even. In that case we always have $0 < d < r$ instead, or $d = 0$. As a consequence of these last observations we obtain a generalization of [At], Theorem 10:

5.8 Theorem *Let $k = \mathbb{R}$. Every vector bundle $\mathcal{E} \in \overline{\Omega}(r, d)$, for an admissible pair (r, d) with respect to i such that $r - d$ is odd and $0 < d < r$, is isomorphic to $\mathcal{L} \otimes \mathcal{E}_{\mathcal{A}}(r, d)$, for a suitable line bundle $\mathcal{L} \in \operatorname{Pic}^0 X$. Furthermore, $\mathcal{L} \otimes \mathcal{E}_{\mathcal{A}}(r, d) \cong \mathcal{E}_{\mathcal{A}}(r, d)$ for some $\mathcal{L} \in \operatorname{Pic}^0 X$ if and only if $\mathcal{L}^{r/h} \cong \mathcal{O}_X$ with h the greatest common divisor of r and d. If $\omega_{r,d}\colon \overline{\Omega}(h, 0) \to \overline{\Omega}(r, d)$ is the bijection described in 4.1, then*

$$\omega_{r,d}(\mathcal{L}^{r/h} \otimes \mathcal{F}_h) \cong \mathcal{L} \otimes \omega_{r,d}(\mathcal{F}_h) \cong \mathcal{L} \otimes \mathcal{E}_{\mathcal{A}}(r, d).$$

The assumptions that the base field equals $\mathbb{R}$ and that $r - d$ has to be odd etc. guarantee that every line bundle of degree zero indeed has an r^{th} root of unity which is needed in the proof of 4.9, 5.6, 5.7 and 5.8. Similar statements are true over other base fields, as long as we require that every line bundle of degree zero over X has an r^{th} root of unity, for some fixed (rank) r, or for any.

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