PROOF OF THE ALDER-ANDREWS CONJECTURE

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Abstract. Motivated by classical identities of Euler, Schur, and Rogers and Ramanujan, H. L. Alder investigated $q_d(n)$ and $Q_d(n)$, the number of partitions of n into d-distinct parts and into parts which are $\pm 1 \pmod{d+3}$, respectively. He conjectured that

$$q_d(n) \geq Q_d(n).$$

G. E. Andrews and A. J. Yee proved the conjecture for $d = 2^s - 1$ and also for $d \geq 32$. We complete the proof of Andrews's refinement of Alder's conjecture by determining effective asymptotic estimates for these partition functions (correcting and refining earlier work of G. Meinardus), thereby reducing the conjecture to a finite computation.

1. Introduction and Statement of Results

Recall the statements of Euler's identity and the first Rogers-Ramananujan identity, which are well-known in the theory of partitions:

$$\prod_{n=0}^{\infty} \frac{1}{1-q^{2n+1}} = \prod_{n=1}^{\infty}(1+q^n)$$

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Euler's identity states that the number of partitions into odd parts equals the number of partitions into distinct parts, and the first Rogers-Ramanujan identity tells us that the number of partitions into parts which are $\pm 1 \pmod 5$ equals the number of partitions into parts which are 2-distinct (a d-distinct partition is one where the difference between any two parts is at least d). Another related identity is a theorem of Schur which states that the partitions of n into parts which are $\pm 1 \pmod 6$ are in bijection with the partitions of n into 3-distinct parts where no consecutive multiples of 3 appear. In 1956, these three facts encouraged H.L. Alder to consider the partition functions

$$\begin{aligned} q_d(n) &:= p(n|d\text{-distinct parts}) \\ Q_d(n) &:= p(n|\text{parts} \pm 1 \pmod{d+3}) \\ \Delta_d(n) &:= q_d(n) - Q_d(n). \end{aligned}$$

He made the following conjecture:

Conjecture (Alder)**.** If $d, n \geq 1$, then

$$\Delta_d(n) \geq 0.$$

By the above discussion, the conjecture is true for $d \leq 3$, and the inequality can be replaced by an equality for $d = 1$ and 2. Large tables of values seem to suggest, however, that $q_d(n)$ and $Q_d(n)$ are rarely equal. Andrews [1] refined Alder's conjecture (see [3] for more information on this conjecture):

Conjecture (Alder-Andrews)**.** For $4 \leq d \leq 7$ and $n \geq 2d+9$, or $d \geq 8$ and $n \geq d+6$,

$$\Delta_d(n) > 0.$$

Remark: For any given d, there are only finitely many n not covered by the Alder-Andrews conjecture, and a simple argument shows that $\Delta_d(n) \geq 0$ for these n.

In essence, Alder's conjecture asks us to relate the coefficients of the two generating functions

$$\sum_{n=0}^{\infty} Q_d(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{n(d+3)-(d+2)})(1-q^{n(d+3)-1})}$$

and

$$\sum_{n=0}^{\infty} q_d(n)q^n = \sum_{n=0}^{\infty} \frac{q^{d\binom{n}{2}+n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Although the first generating function is essentially a weight 0 modular form, the second is generally not modular (except in the cases $d=1$ and 2). This is the root of the difficulty in proving Alder's conjecture, since the task is to relate Fourier coefficients of two functions which have different analytic properties.

However, there have been several significant advances toward proving Alder's conjecture. Using combinatorial methods, Andrews [1] proved that Alder's conjecture holds for all values of d which are of the form 2^s-1, $s \geq 4$. In addition, Yee ([9], [10]) proved that the conjecture holds for $d=7$ and for all $d \geq 32$. These results are of great importance because they resolve the conjecture except for $4 \leq d \leq 30, d \neq 7, 15$.

In addition, Andrews [1] deduced that

$$\lim_{n\to\infty} \Delta_d(n) = +\infty \qquad (1.1)$$

using powerful results of Meinardus ([6], [7]) which give asymptotic expressions for the coefficients $q_d(n)$ and $Q_d(n)$. Unfortunately, a mistake in [7] implies that one must argue further to establish (1.1). We correct the proof of Meinardus's main theorem (see the discussion after (3.12)) and show that the statement of the theorem remains unchanged. We first prove the following result, which can be made explicit:

Theorem 1.1. Let $d \geq 4$ and let $\alpha \in [0,1]$ be the root of $\alpha^d + \alpha - 1 = 0$. If

$$A := \frac{d}{2}\log^2\alpha + \sum_{r=1}^{\infty} \frac{\alpha^{rd}}{r^2},$$

then for every positive integer n we have

$$\Delta_d(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-3/4}\exp(2\sqrt{nA}) + \mathcal{E}_d(n),$$

where

$$\mathcal{E}_d(n) = O\left(n^{-\frac{5}{6}}\exp(2\sqrt{nA})\right).$$

Remark. The main term of $\Delta_d(n)$ is the same as the main term for $q_d(n)$ (cf. Theorem 3.1).

In the course of proving Theorem 1.1, we derive explicit approximations for $Q_d(n)$ and $q_d(n)$ (see Theorems 2.1 and 3.1, respectively). Using these results, we obtain the following:

Theorem 1.2. The Alder-Andrews Conjecture is true.

In order to prove Theorems 1.1 and 1.2, we consider $q_d(n)$ and $Q_d(n)$ independently and then compare the resulting effective estimates. Accordingly, in Section 2, we give explicit asymptotics for $Q_d(n)$, culminating in Theorem 2.1. Next, in Section 3, we laboriously make Meinardus's argument effective (and correct) in order to give an explicit asymptotic formula for $q_d(n)$ in Theorem 3.1. In Section 4 we use the results from Sections 2 and 3 to prove Theorems 1.1 and 1.2.

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2. Estimate of $Q_d(n)$ with explicit error bound

As before, let $Q_d(n)$ denote the number of partitions of n whose parts are $\pm 1 \pmod{d+3}$. From the work of Meinardus, we have that

$$Q_d(n) \sim \frac{(3d+9)^{-\frac{1}{4}}}{4\sin\left(\frac{\pi}{d+3}\right)} n^{-\frac{3}{4}} \exp\left(n^{\frac{1}{2}} \frac{2\pi}{\sqrt{3(d+3)}}\right).$$

In this formula, only the order of the error is known. We will bound the error explicitly, following closely the method of Meinardus [6] as it is presented by Andrews in Chapter 6 of [2]. This allows us to prove the following theorem:

Theorem 2.1. If $d \geq 4$ and n is a positive integer, then

$$Q_d(n) = \frac{(3d+9)^{-\frac{1}{4}}}{4\sin\left(\frac{\pi}{d+3}\right)} n^{-\frac{3}{4}} \exp\left(n^{\frac{1}{2}} \frac{2\pi}{\sqrt{3(d+3)}}\right) + R(n),$$

where $R(n)$ is an explicitly bounded function (see (2.13) at the end of this section).

Remark. An exact formula for $Q_d(n)$ is known due to the work of Subrahmanyasastri [8]. In addition, by using Maass-Poincaré series, Bringmann and Ono [5] obtained exact formulas in a much more general setting. However, we do not employ these results since the formulas are extremely complicated, and the proof of Theorems 1.1 and 1.2 do not require this level of precision.

2.1. Preliminary Facts.

Consider the generating function f associated to $Q_d(n)$,

$$f(\tau) := \prod_{\substack{n \equiv \pm 1 (d+3) \\ n \geq 0}} (1-q^n)^{-1} = 1 + \sum_{n=1}^{\infty} Q_d(n) q^n,$$

where $q = e^{-\tau}$ and $\Re(\tau) > 0$. Let $\tau = y + 2\pi i x$. We can then obtain a formula for $Q_d(n)$ by integrating $f(\tau)$ against $e^{n\tau}$. Consequently, we require a precise approximation of $f(\tau)$ so that we may properly make use of this integral formula. To do this, we need an additional function,

$$g(\tau) := \sum_{\substack{n \equiv \pm 1 (d+3) \\ n \geq 0}} q^n.$$

Lemma 2.2. If $\arg\tau > \frac{\pi}{4}$ and $|x| \leq \frac{1}{2}$, then

$$\Re(g(\tau)) - g(y) \leq -c_2 y^{-1},$$

where c_2 is an explicitly given constant depending only on d.

Proof. For notational convenience, we will consider the expression $-y\left(\Re(g(\tau)) - g(y)\right)$. Expanding, we find that

$$-y\left(\Re(g(\tau)) - g(y)\right) = S_1 + S_2 + S_3,$$

where

$$S_1 := \frac{\left(1-\cos(2\pi x)\right)\left(e^{(3d+8)y} - e^{(2d+5)y} - e^{(d+4)y} + e^{y}\right)}{\left(\frac{e^{(d+3)y}-1}{y}\right)\left(e^{(2d+6)y} - 2e^{(d+3)y}\cos(2\pi(d+3)x) + 1\right)},$$

$$S_2 := \frac{\left(1-\cos(2\pi (d+2) x)\right)\left(e^{(2d+7)y} - e^{(2d+5)y} - e^{(d+4)y} + e^{(d+2)y}\right)}{\left(\frac{e^{(d+3)y}-1}{y}\right)\left(e^{(2d+6)y} - 2e^{(d+3)y}\cos(2\pi(d+3)x) + 1\right)},$$

and

$$S_3 := \frac{\left(1-\cos(2\pi (d+3) x)\right)\left(2e^{(2d+5)y} + 2e^{(d+4)y}\right)}{\left(\frac{e^{(d+3)y}-1}{y}\right)\left(e^{(2d+6)y} - 2e^{(d+3)y}\cos(2\pi(d+3)x) + 1\right)}.$$

We note that when $y = 0$, each of S_1, S_2, and S_3 is defined. Namely, $S_1 = 0$, $S_2 = 0$, and $S_3 = \frac{2}{d+3}$.

Since these functions are even in x, we may assume $x \geq 0$. Further, the condition $\arg\tau > \frac{\pi}{4}$ implies that $y < 2\pi x$. To find c_2, we note that each $S_i \geq 0$ and so it suffices to bound one away from 0. We do this in three different cases.

Case 1: Suppose that $y \geq \frac{1}{2}$. Since $\frac{1}{2} > x > \frac{1}{2\pi} y$, it follows that $1-\cos(2\pi x) > 1 - \cos\frac{1}{2}$ and that S_1 is bounded away from 0. In particular,

$$S_1 \geq \frac{\pi\left(1-\cos\frac{1}{2}\right)\left(e^{\frac{3d+8}{2}} - e^{\frac{2d+5}{2}} - e^{\frac{d+4}{2}} + e^{\frac{1}{2}}\right)}{\left(e^{\pi(d+3)}-1\right)\left(e^{\pi(d+3)}+1\right)^2}. \tag{2.1}$$

Case 2: Suppose that $y < \frac{1}{2}$ and $\left|x - \frac{k}{d+3}\right| < \frac{y}{d+3}$ for some positive integer k. Although less obvious than in Case 1, S_1 will again be bounded away from 0:

$$S_1 \geq \frac{\pi\left(1-\cos\frac{\pi}{d+3}\right)}{e^{\pi(d+3)}-1} \frac{e^{(3d+8)y} - e^{(2d+5)y} - e^{(d+4)y} + e^{y}}{\left(e^{(d+3)y}-1\right)^2 + 8\pi^2 y^2 e^{(d+3)y}},$$

and so

$$S_1 \geq \frac{2\pi^3\left(1-\cos\frac{\pi}{d+3}\right)(d+2)(d+3)}{\left(e^{\pi(d+3)}-1\right)\left(\left(e^{(d+3)\pi}+1\right)^2 + 8\pi^4 e^{(d+3)\pi}\right)}. \tag{2.2}$$

Case 3: Suppose that $y < \frac{1}{2}$ and $\left|x - \frac{k}{d+3}\right| \geq \frac{y}{d+3}$ for some non-negative integer k. We additionally assume $\left|x - \frac{k}{d+3}\right| \leq \frac{1}{2(d+3)}$. This is permitted since every x is covered as we vary k. It will be S_3 that is bounded away from 0.

Let $u := 2\pi(d+3)\left|x - \frac{k}{d+3}\right|$ and note that $0 \le u \le \pi$, $y \le \frac{u}{2\pi}$, and $\cos u = \cos 2\pi(d+3)x$. Now, we have that

$$S_3 \ge \frac{4\pi}{e^{(d+3)\pi}-1} \frac{1-\cos u}{\left(e^{\frac{(d+3)u}{2\pi}}-1\right)^2 + 2e^{\frac{(d+3)u}{2\pi}}(1-\cos u)},$$

and a tedious analysis of the derivative of this function implies for $d \ge 4$ that

$$S_3 \ge \frac{8\pi}{(e^{(d+3)\pi}-1)\left(\left(e^{\frac{d+3}{2}}-1\right)^2 + 4e^{\frac{d+3}{2}}\right)}. \tag{2.3}$$

Obviously, we may take c_2 to be the minimum of the bounds (2.1), (2.2), and (2.3). □

Using Lemma 2.3, we now obtain an approximation for $f(\tau)$.

Lemma 2.3. If $|\arg \tau| \le \frac{\pi}{4}$ and $|x| \le \frac{1}{2}$, then

$$f(\tau) = \exp\left(\frac{\pi^2}{3(d+3)}\tau^{-1} + \log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right) + f_2(\tau)\right),$$

where $f_2(\tau) = O\left(y^{\frac{1}{2}}\right)$ is an explicitly bounded function. Furthermore, if $y \le y_{\max}$ is sufficiently small, $0 < \delta < \frac{2}{3}$, $0 < \varepsilon_1 < \frac{\delta}{2}$, $\beta := \frac{3}{2} - \frac{\delta}{4}$, and $y^\beta \le |x| \le \frac{1}{2}$, then there is a constant c_3 depending on d, ε_1 and δ such that

$$f(y + 2\pi i x) \le \exp\left(\frac{\pi^2}{3(d+3)}y^{-1} - c_3 y^{-\varepsilon_1}\right).$$

Remark. The precise statement of $y_{\max}$ being sufficiently small is given in the discussion following (2.4).

Proof. From page 91 of Andrews [2], we have that

$$\log f(\tau) = \tau^{-1}\frac{\pi^2}{3(d+3)} + \log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right) + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \tau^{-s}\Gamma(s)\zeta(s+1)D(s)ds,$$

where $D(s)$ is the Dirichlet series

$$D(s) := \sum_{\substack{n \equiv \pm 1 (d+3) \\ n \ge 0}} \frac{1}{n^s}$$

which converges for $\Re(s) > 1$. Writing

$$D(s) = (d+3)^{-s}\left(\zeta\left(s, \frac{1}{d+3}\right) + \zeta\left(s, \frac{d+2}{d+3}\right)\right),$$

where $\zeta(s,a)$ is the Hurwitz zeta function, we see that $D(s)$ can be analytically continued to the entire complex plane except for a pole of order 1 and residue $\frac{2}{d+3}$ at $s = 1$ (see, for example, page 255 of Apostol's book [4]).

We bound the integral by noting that $|D(s)| \leq |\zeta(s)|$, obtaining

$$\left| \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \tau^{-s}\Gamma(s)\zeta(s+1)D(s)ds \right| \leq \xi\sqrt{y},$$

where

$$\xi := \frac{\sqrt{2}}{2\pi}\int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\zeta\left(-\frac{1}{2}+it\right)\Gamma\left(-\frac{1}{2}+it\right)\right| dt.$$

The first statement of the lemma follows.

Remark. Numerical estimates show that $\xi < .224$.

We now turn to the second statement of the lemma, and again follow the method of Andrews [2]. We consider two cases: (1) $y^\beta \leq |x| \leq \frac{y}{2\pi}$ and (2) $\frac{y}{2\pi} \leq |x| \leq \frac{1}{2}$.

In the first case, we see that $|\arg \tau| \leq \frac{\pi}{4}$. Hence, we can apply the first statement of the lemma, yielding

$$\begin{aligned} \log|f(y+2\pi ix)| &\leq \frac{\pi^2 y^{-1}}{3(d+3)} + \frac{\pi^2 y^{-1}}{3(d+3)}\left(\left(1+4\pi^2x^2y^{-2}\right)^{-\frac{1}{2}} - 1\right) + \log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right) + \xi\sqrt{y} \\ &\leq \frac{\pi^2}{3(d+3)}y^{-1} - c_4 y^{-\frac{\delta}{2}}, \end{aligned}$$

where

$$c_4 := \frac{\pi^4}{3(d+3)}\left(2 - \frac{3}{2}y_{\max}^{1-\frac{\delta}{2}}\right) - \log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right) y_{\max}^{\frac{\delta}{2}} - \xi y_{\max}^{\frac{1+\delta}{2}}.$$

In Case 2, we have that

$$\log|f(y+2\pi ix)| = \log f(y) + \Re\left(g(\tau)\right) - g(y),$$

and using Lemma 2.2, we obtain

$$\log|f(y+2\pi ix)| \leq \frac{\pi^2}{3(d+3)}y^{-1} - c_5 y^{-1}$$

where

$$c_5 := c_2 - y_{\max}\log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right) - \xi y_{\max}^{\frac{3}{2}}. \tag{2.4}$$

We assume $y_{\max}$ is sufficiently small so as to make both c_4 and c_5 positive.

Hence, we may take

$$c_3 := \min\left(c_4\left(y_{\max}\right)^{\varepsilon_1 - \frac{\delta}{2}}, c_5\left(y_{\max}\right)^{\varepsilon_1 - 1}\right)$$

and the result follows. □

Remark. In the proof of Theorem 1.1, we need only bound $Q_d(n)$ since it is of lower order than $q_d(n)$. We shall ignore the restriction on $y_{\max}$ for convenience.

2.2. **Proof of Theorem 2.1.** From the Cauchy integral theorem, we have

$$\begin{aligned} Q_d(n) &= \frac{1}{2\pi i}\int_{\tau_0}^{\tau_0+2\pi i} f(\tau)\exp(n\tau)\,d\tau \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(y+2\pi i x)\exp(ny+2n\pi i x)\,dx. \end{aligned}$$

Applying the saddle point method, we take

$$y = n^{-\frac{1}{2}}\frac{\pi}{\sqrt{3(d+3)}}$$

and we let $m := ny$ for notational simplicity. Assuming the notation in Lemma 2.3, for $n \geq 6$, we have $y_{\max} \leq \left(\frac{1}{2\pi}\right)^{\frac{1}{\beta-1}}$, so that both cases in the proof of the second statement of Lemma 2.3 are nonvacuous. We have that

$$Q_d(n) = e^m \int_{-y^\beta}^{y^\beta} f(y+2\pi i x)\exp(2\pi i n x)\,dx + e^m R_1,$$

where

$$R_1 := \left(\int_{-\frac{1}{2}}^{-y^\beta} + \int_{y^\beta}^{\frac{1}{2}}\right) f(y+2\pi i x)\exp(2\pi i n x)\,dx.$$

By Lemma 2.3,

$$|R_1| \leq \exp\left[\frac{\pi^2}{3(d+3)}\left(\frac{m}{n}\right)^{-1} - c_3\left(\frac{m}{n}\right)^{-\varepsilon_1}\right],$$

so

$$|e^m R_1| \leq \exp\left(2m - c_3 m^{\varepsilon_1}\left(\frac{\pi^2}{3(d+3)}\right)^{-\varepsilon_1}\right). \tag{2.5}$$

Using Lemma 2.3, write

$$Q_d(n) = \exp\left(2m + \log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right)\right)\int_{-(m/n)^\beta}^{(m/n)^\beta}\exp(\varphi_1(x))\,dx + \exp(m)R_1,$$

where

$$\varphi_1(x) := m\left[\left(1+\frac{2\pi i x n}{m}\right)^{-1} - 1\right] + 2\pi i n x + g_1(x)$$

and

$$|g_1(x)| \leq \xi\sqrt{\frac{\pi^2}{3m(d+3)}}.$$

After making the change of variables $2\pi x = (m/n)\omega$, we obtain

$$Q_d(n) = \exp\left(2m + \log\frac{m}{n} + \log\left(\frac{1}{2\sin\frac{\pi}{d+3}}\right) - \log 2\pi\right) I + \exp(m)R_1,$$

where

$$I := \int_{-c_{10}m^{1-\beta}}^{c_{10}m^{1-\beta}} \exp(\varphi_2(\omega))\, d\omega,$$

$$c_{10} := 2\pi \left(\frac{\pi^2}{3(d+3)}\right)^{\beta-1},$$

and

$$\varphi_2(\omega) := m\left(\frac{1}{1+i\omega} - 1 + i\omega\right) + g_1(\omega).$$

We must now find an asymptotic expression for I. Write

$$I = \int_{-c_{10}m^{1-\beta}}^{c_{10}m^{1-\beta}} \exp\left(-m\omega^2\right)\, d\omega + R_2, \tag{2.6}$$

where

$$R_2 := \int_{-c_{10}m^{1-\beta}}^{c_{10}m^{1-\beta}} \exp\left(-m\omega^2\right)\left(\exp(\varphi_3(\omega)) - 1\right) d\omega,$$

with

$$\varphi_3(\omega) := m\left(\frac{1}{1+i\omega} - 1 + i\omega + \omega^2\right) + g_1(\omega).$$

Simplifying, we find that

$$|\varphi_3(\omega)| \le c_{10}^3 m^{\frac{3\delta-2}{4}} + \xi\sqrt{\frac{\pi^2}{3m(d+3)}}. \tag{2.7}$$

Substituting

$$m_{\min} = 2^{\frac{2-\delta}{4}}\pi^{\frac{10-\delta}{4}}(3(d+3))^{-1}$$

for m in (2.7), it follows that

$$|\varphi_3(\omega)| \le \frac{2^{\frac{44+8\delta-3\delta^2}{16}}\pi^{\frac{76+8\delta-3\delta^2}{16}}}{3(d+3)} + \xi(2\pi)^{\frac{\delta-2}{8}} =: \varphi_{3,\max}.$$

Thus, letting

$$c_6 := \frac{\exp(\varphi_{3,\max}) - 1}{\varphi_{3,\max}},$$

we have

$$|\exp(\varphi_3(\omega)) - 1| \le m^{-\frac{1}{2}+\frac{3\delta}{4}}\left(c_6 c_{10}^3 + \xi c_6 m_{\min}^{-\frac{3\delta}{4}}\sqrt{\frac{\pi^2}{3(d+3)}}\right) =: m^{-\frac{1}{2}+\frac{3\delta}{4}} c_7.$$

Hence, we conclude that

$$|R_2| \le 2c_{10}c_7 m^{\delta-1}. \tag{2.8}$$

Computing the integral in (2.6), we see that

$$\int_{-c_{10}m^{1-\beta}}^{-c_{10}m^{1-\beta}} \exp\left(-m\omega^2\right)\, d\omega = \left(\frac{\pi}{m}\right)^{\frac{1}{2}} + g_2(m), \tag{2.9}$$

where

$$|g_2(m)| \leq 2m^{-\frac{1}{2}} \exp\left(-c_{10} m^{\frac{\delta}{4}}\right). \tag{2.10}$$

Thus, we find that

$$I = \left(\frac{\pi}{m}\right)^{\frac{1}{2}} + g_2(m) + R_2. \tag{2.11}$$

Combining these results, we obtain the desired expression for $Q_d(n)$,

$$Q_d(n) = \left(\frac{n^{-\frac{3}{4}}\,(3(d+3))^{-\frac{1}{4}}}{4\sin\left(\frac{\pi}{d+3}\right)}\right) \exp\left(n^{\frac{1}{2}}\frac{2\pi}{\sqrt{3(d+3)}}\right) + R(n), \tag{2.12}$$

where

$$\begin{aligned} |R(n)| \;\leq\; & n^{-\frac{1}{4}}\left(\frac{\pi^{\frac{1}{2}}(3(d+3))^{-\frac{3}{4}}}{2\sin\left(\frac{\pi}{d+3}\right)}\right)\exp\left(n^{\frac{1}{2}}\frac{2\pi}{\sqrt{3(d+3)}} - n^{\frac{\delta}{8}}2\pi^{2-\frac{\delta}{4}}(3(d+3))^{-2+\frac{3\delta}{8}}\right) + \\ & n^{-1+\frac{\delta}{2}}\left(\frac{c_7\pi^{1+\frac{\delta}{2}}}{(3(d+3))^2\sin\left(\frac{\pi}{d+3}\right)}\right)\exp\left(n^{\frac{1}{2}}\frac{2\pi}{\sqrt{3(d+3)}}\right) + \\ & \exp\left(n^{\frac{1}{2}}\frac{2\pi}{\sqrt{3(d+3)}} - c_3 n^{\frac{\varepsilon_1}{2}}\left(\frac{\pi^2}{3(d+3)}\right)^{-\frac{3\varepsilon_1}{2}}\right). \end{aligned} \tag{2.13}$$

3. Estimate of $q_d(n)$ with explicit error bound

As before, let $q_d(n)$ denote the number of partitions of n with d-distinct parts (i.e., the difference between any two parts is at least d). Theorem 2 of [7] (with $k = m = 1$ and $\ell = d$) gives

$$q_d(n) \sim \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-3/4} \exp(2\sqrt{nA}),$$

where α and A depend only on d (see their definitions below in Theorem 3.1). We will bound the error explicitly, following closely the paper of Meinardus [7]. We make his calculations effective, and we obtain the following theorem.

Theorem 3.1. Let α be the unique real number in $[0, 1]$ satisfying $\alpha^d + \alpha - 1 = 0$, and let

$$A := \frac{d}{2}\log^2\alpha + \sum_{r=1}^{\infty}\frac{\alpha^{rd}}{r^2}.$$

If n is a positive integer, then

$$q_d(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-3/4} \exp(2\sqrt{nA}) + r_d(n)$$

where $|r_d(n)|$ can be bounded explicitly (see (3.34) at the end of this section).

3.1. **Preliminary Facts.** For fixed $d \geq 4$, we have the generating function

$$f(z) := \sum_{n=0}^{\infty} q_d(n) e^{-nz} \tag{3.1}$$

with $z = x + iy$. Hence, we obviously have that

$$q_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z) e^{nz}\, dy. \tag{3.2}$$

Therefore to estimate $q_d(n)$ we require strong approximations for $f(z)$.

Lemma 3.2. Assuming the notation above, for $|y| \leq x^{1+\varepsilon}$ and $x < \beta$, where

$$\beta := \min\left(-\frac{\pi}{\log \rho}\xi, \frac{2\alpha^{2-d}}{\pi d}, \frac{1}{2d} + \rho\left(\frac{1}{2} - \frac{\pi^2}{24}\right)\right)^{\frac{1}{\varepsilon}}$$

and $0 < \xi < 1$ is a constant, we have that

$$f(z) = \left(\alpha^{d-1}\left(d\alpha^{d-1} + 1\right)\right)^{-\frac{1}{2}} e^{\frac{A}{z}} (1 + f_{err}(z)),$$

where $f_{err}(z) = o(1)$ is an explicitly bounded function.

Lemma 3.3. Assuming the notation above, for $x < \beta$ and $x^{1+\varepsilon} < |y| \leq \pi$, we have that

$$|f(x+iy)| \leq \sqrt{\frac{2\pi}{dx}} e^{-\eta\rho x^{2\varepsilon-1}} (1 + f_2(\rho, x)) \exp\left(\frac{A}{x} + \frac{1-d}{2}\log\alpha + f_1(\rho, x)\right),$$

where f_1, f_2 are functions given in Lemma 3.4, and η is an explicitly given constant.

Remark. Although $\varepsilon = \frac{11}{24}$ in [7], we will benefit by varying ε in our work.

To prove Lemmas 3.2 and 3.3, we follow [7] and write, by the Cauchy Integral Theorem,

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} H(w, z)\Theta(w, z)\, \frac{dw}{w} \tag{3.3}$$

where $\mathcal{C}$ is a circle of radius $\rho := 1 - \alpha$ centered at the origin,

$$H(w, z) := \prod_{n=1}^{\infty} (1 - we^{-nz})^{-1}, \tag{3.4}$$

and

$$\Theta(w, z) := \sum_{n=-\infty}^{\infty} e^{-\frac{d}{2}n(n-1)z} w^{-n}. \tag{3.5}$$

Thus, we can obtain an effective estimate of $f(z)$ once we are armed with strong estimates for $H(w, z)$ and $\Theta(w, z)$.

Lemma 3.4. Let $\rho = \alpha^d = 1 - \alpha$ and suppose $w = \rho e^{i\varphi}$ with $-\pi \leq \varphi < \pi$. Then for $|y| \leq x^{1+\varepsilon}$ and $x < \beta$,

$$H(w, z) = \exp\left(\frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2} + \frac{1}{2}\log(1-w) + f_1(w, z)\right) \tag{3.6}$$

and

$$\Theta(w,z) = \sqrt{\frac{2\pi}{dz}} \exp\left(\frac{\log^2 w}{2dz} - \frac{1}{2}\log w\right)(1 + f_2(w,z)), \tag{3.7}$$

where, as $x \to 0$, $f_1(w,z) = O\left(x^{\frac{1}{2}}\right)$ and $f_2(w,z) = O\left(x + \exp\left[-\frac{c_0}{x}(\pi - |\varphi|) + c_1 x^{\varepsilon-1}\right]\right)$ are explicitly bounded functions.

Proof. First, (3.4) and the inverse Mellin transform yield

$$\log H(w,z) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} z^{-s}\Gamma(s)\zeta(s)D(s+1,w)\,ds, \tag{3.8}$$

where $\zeta(s)$ is the Riemann zeta function, $\Gamma(s)$ is the Gamma function, and

$$D(s,w) := \sum_{r\geq 1}\frac{w^r}{r^s},$$

which is defined as a function of s for all fixed w with $|w| < 1$.

Note that if $\theta_0 := \arctan x^\varepsilon$, then

$$|z^{1/2-it}| \leq |z|^{1/2}e^{\theta_0|t|} \leq \left(1 + x^{2\varepsilon}\right)^{\frac{1}{4}} x^{1/2}e^{\theta_0|t|}.$$

By changing the curve of integration and accounting for the poles at $s = 0$ and 1, we have

$$\log H(w,z) = \frac{1}{z}\sum_{r\geq 1}\frac{w^r}{r^2} + \frac{\log(1-w)}{2} + f_1(w,z),$$

where

$$\begin{aligned} |f_1(w,z)| &= \left|\frac{1}{2\pi i}\int_{-\infty}^{\infty} z^{1/2-it}\Gamma\left(-\frac{1}{2}+it\right)\zeta\left(-\frac{1}{2}+it\right)D\left(\frac{1}{2}+it,w\right)dt\right| \\ &\leq \left(1+x^{2\varepsilon}\right)^{\frac{1}{4}} 2^{-\frac{5}{2}}\pi^{-\frac{3}{2}}\zeta\left(\frac{3}{2}\right)\frac{\rho}{1-\rho}\frac{4}{\frac{\pi}{2}-\theta_0}x^{\frac{1}{2}} \\ &=: f_1(x) \end{aligned}$$

This proves the first statement of the lemma as $x^{2\varepsilon}$ and $\theta_0 = \arctan x^\varepsilon$ both tend toward 0 as $x \to 0$.

To prove the second statement, we use the transformation properties of theta functions to write

$$\Theta(w,z) = \sqrt{\frac{2\pi}{dz}}e^{\frac{(\log w - dz/2)^2}{2dz}}\sum_{\mu=-\infty}^{\infty} e^{\frac{-2\pi^2\mu^2}{dz} - \frac{2\pi i\mu}{dz}(\log w - dz/2)}. \tag{3.9}$$

The argument on page 295 of [7] completes the proof of the lemma, with

(3.10)

$$\begin{aligned} |f_2(w,z)| &\leq e^{\frac{d|z|}{8}}\left[e^{\frac{dx\sqrt{1+x^{2\varepsilon}}}{8}} - 1 + 2\frac{\exp\left(-\frac{4\pi^2(1-\xi)}{dx(1+x^{2\varepsilon})}\right)}{1-\exp\left(-\frac{2\pi^2(1-\xi)}{dx(1+x^{2\varepsilon})}\right)}\right] + 2\exp\left(\frac{2\pi(|\varphi|-\pi)}{dx(1+x^{2\varepsilon})} - \frac{2\pi\log\rho}{d}x^{\varepsilon-1} + \frac{d|z|}{8}\right) \\ &=: f_2(\varphi,z) = f_2^0(z) + f_2^\varphi(z)\exp\left(\frac{2\pi|\varphi|}{dx\left(1+x^{2\varepsilon}\right)}\right). \end{aligned}$$

□

We now prove Lemmas 3.2 and 3.3 using Lemma 3.4.

Proof of Lemma 3.2. Recall from (3.3) that

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} H(w,z)\Theta(w,z)\,\frac{dw}{w}.$$

Let $\varphi_0 = x^c$ with $\frac{3}{8} < c < \frac{1}{2}$. Then

$$f(z) = \frac{1}{2\pi i}\int_{\rho e^{-i\varphi_0}}^{\rho e^{i\varphi_0}} H(w,z)\Theta(w,z)\,\frac{dw}{w} + \frac{1}{2\pi i}\int_{\mathcal{B}} H(w,z)\Theta(w,z)\,\frac{dw}{w}, \tag{3.11}$$

where $\mathcal{B}$ is the circle $\mathcal{C}$ without the arc $\rho e^{-i\varphi_0}$ to $\rho e^{i\varphi_0}$.

We will first estimate the second integral in (3.11). We note here the error of Meinardus [7] in the bound of $\Theta(w,z)$ provided between equations (25) and (26). From Lemma 3.4, we have that

$$|\Theta(w,z)| \leq \sqrt{\frac{2\pi}{d|z|}}\rho^{-\frac{1}{2}}\exp\left(\frac{x\log^2\rho}{2d(x^2+y^2)} - \frac{\varphi^2 x}{2d(x^2+y^2)} + \frac{y\varphi\log\rho}{d(x^2+y^2)}\right)(1+|f_2(w,z)|)\,. \tag{3.12}$$

The term $\frac{y\varphi\log\rho}{d(x^2+y^2)}$ does not appear in [7] and tends to infinity if $y\varphi < 0$. This term arises from the main term of $\Theta(w,z)$, so its contribution cannot be ignored. Furthermore, it is $O\left(x^{\varepsilon-1}\right)$, and hence cannot be combined into the negative $O\left(x^{2c-1}\right)$ term arising from $\frac{\varphi^2 x}{2d(x^2+y^2)}$. However, the bound Meinardus claims on the product $|H(w,z)\Theta(w,z)|$ is correct. To see this, we need more than the bound

$$|H(w,z)| \leq H(\rho,x)$$

that was originally thought to be sufficient.

From Lemma 3.4, we have that

$$|H(w,z)| \leq \exp\left(|f_1(w,z)|\right)(1+\rho)^{\frac{1}{2}}\exp\left(\Re\left(\frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2}\right)\right), \tag{3.13}$$

and

$$\begin{aligned}
\Re\left(\frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2}\right) &= \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r\cos(r\varphi)}{r^2} + \frac{y}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r\sin(r\varphi)}{r^2} \\
&= \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2} + \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(\cos(r\varphi)-1) - \frac{y\varphi\log(1-\rho)}{x^2+y^2} \\
&\quad + \frac{y}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(\sin(r\varphi)-r\varphi).
\end{aligned}$$

Since $\rho = \alpha^d$ and $1-\rho = \alpha$, combining this with (3.13) and (3.12), we see that

$$\begin{aligned}
|H(w,z)\Theta(w,z)| \leq\ & \sqrt{\frac{2\pi}{d|z|}}\left(\frac{1+\rho}{\rho}\right)^{\frac{1}{2}}\exp\Bigg(|f_1(w,z)| + \frac{Ax}{x^2+y^2} - \frac{\varphi^2 x}{x^2+y^2} \\
& \cdot\left[\frac{1}{2d} + \frac{1}{\varphi^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(1-\cos(r\varphi)) - \frac{y}{\varphi^2 x}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(\sin(r\varphi)-r\varphi)\right]\Bigg) \\
& \cdot(1+|f_2(w,z)|).
\end{aligned}$$

Hence, as $x \to 0$ we recover Meinardus's bound on $|H(w,z)\Theta(w,z)|$.

Using the notation of Lemma 3.4,

(3.14)
$$\left|\int_{\mathcal{B}} H(w,z)\Theta(w,z)\frac{dw}{w}\right| \leq \sqrt{\frac{2\pi}{d|z|}}\left(\frac{1+\rho}{\rho}\right)^{\frac{1}{2}} \exp\left(f_1(x)+\frac{Ax}{x^2+y^2}\right)\Big[\left(1+f_2^0(z)\right) \cdot \int_{\mathcal{B}} e^{-\psi(\varphi,z)}d\varphi + f_2^{\varphi}(z)\int_{\mathcal{B}} \exp\left(-\psi(\varphi,z)+\frac{2\pi|\varphi|}{dx\left(1+x^{2\varepsilon}\right)}\right)d\varphi\Big],$$

where

$$\text{(3.15)}\quad \psi(\varphi,z) := \frac{\varphi^2 x}{2d(x^2+y^2)} + \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(1-\cos(r\varphi)) - \frac{y}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(\sin(r\varphi)-r\varphi).$$

We evaluate the two integrals of (3.14) separately.

For the integral
$$\int_{\mathcal{B}} \exp\left(-\psi(\varphi,z)\right)d\,\varphi,$$
we consider two parts based on the sign of $y\varphi$. Without loss of generality, we may assume that $y > 0$.

When $\varphi > 0$, $\sin(r\varphi) - r\varphi < 0$ for all r, and so $\psi(\varphi,z) > 0$. Then

$$\text{(3.16)}\quad \int_{\varphi_0}^{\pi} \exp\left(-\psi(\varphi,z)\right)d\,\varphi \leq \frac{1}{\psi_\varphi(\varphi_0,z)}\left[\exp\left(-\psi(\varphi_0,z)\right)-\exp\left(-\psi(\nu\varphi_0,z)\right)\right] + \frac{1}{\psi_\varphi(\nu\varphi_0,z)}\left[\exp\left(-\psi(\nu\varphi_0,z)\right)-\exp\left(-\psi(\pi,z)\right)\right],$$

where $\nu > 1$ is a constant.

When $\varphi < 0$, we note that $\sin(r\varphi) - r\varphi > 0$, and so

(3.17)
$$\begin{aligned}\psi(\varphi,z) &\geq \frac{\varphi^2 x}{2d(x^2+y^2)} + \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(1-\cos(r\varphi)) - \frac{x^{1+\varepsilon}}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\frac{r^3\varphi^3}{6}\right)\\ &= \frac{\varphi^2 x}{2d(x^2+y^2)} + \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}(1-\cos(r\varphi)) + \frac{\varphi^3 x^{1+\varepsilon}\alpha^{d-2}}{6(x^2+y^2)}\\ &=: \hat{\psi}(\varphi,z),\end{aligned}$$

whence

(3.18)
$$\int_{\varphi_0}^{\pi} \exp(-\psi(-\varphi,z))d\varphi \leq \frac{1}{\hat{\psi}_\varphi(-\varphi_0,z)}\left[\exp\left(-\hat{\psi}(-\varphi_0,z)\right)-\exp\left(-\hat{\psi}(-\nu\varphi_0,z)\right)\right] + + \frac{1}{\hat{\psi}_\varphi(-\nu\varphi_0,z)}\left[\exp\left(-\hat{\psi}(-\nu\varphi_0,z)\right)-\exp\left(-\hat{\psi}(-\pi,z)\right)\right].$$

Here we have used the fact that
$$\frac{\pi^2}{2}\alpha^{d-2}x^{\varepsilon} \leq \frac{\pi}{d}.$$

We now consider the second integral in (3.14). A weaker bound on $\psi(\varphi,z)$ suffices. In particular, we have
$$\psi(\varphi,z) \geq k\varphi^2,$$

where

$$k := \frac{x}{x^2+y^2}\left(\frac{1}{2d} - \frac{\pi\alpha^{d-2}}{6}\left|\frac{y}{x}\right| + \rho\left(\frac{1}{2} - \frac{\pi^2}{24}\right)\right),$$

which is positive since $x < \beta$.

Hence, we have that

$$\int_{\mathcal{B}} \exp\left(-\psi(\varphi,z) + \frac{2\pi|\varphi|}{dx\left(1+x^{2\varepsilon}\right)}\right) d\varphi \le \frac{dx\left(1+x^{2\varepsilon}\right)}{\pi - kdx\left(1+x^{2\varepsilon}\right)}\left[\exp\left(-k\pi^2 + \frac{2\pi^2}{dx\left(1+x^{2\varepsilon}\right)}\right) - \exp\left(-k\varphi_0^2 + \frac{2\pi\varphi_0}{dx\left(1+x^{2\varepsilon}\right)}\right)\right]. \tag{3.19}$$

Using (3.16), (3.18), and (3.19) in (3.14) gives an explicit bound for the second integral of (3.11), say $E_{\mathcal{B}}(z)$.

Following page 297 of [7], the first integral of (3.11) is given by

$$\begin{aligned} I &:= \frac{1}{2\pi i}\int_{\rho e^{-i\varphi_0}}^{\rho e^{i\varphi_0}} H(w,z)\Theta(w,z)\,\frac{dw}{w} \\ &= \frac{1}{\sqrt{2\pi dz}}\exp\left(\frac{A}{z} + \frac{1-d}{2}\log\alpha\right)\left(I_{\text{main}} + I_{\text{error}}\right), \end{aligned}$$

where

$$I_{\text{main}} := \int_{-\varphi_0}^{\varphi_0} \exp\left(-\frac{\varphi^2}{2dz}\left(d\alpha^{d-1}+1\right)\right) d\varphi \tag{3.20}$$

and

$$\begin{aligned} I_{\text{error}} := \int_{-\varphi_0}^{\varphi_0} &\left(\exp\left(\log\left(\frac{1-\rho e^{i\varphi}}{1-\rho}\right) + f_3(w,z) + f_1(w,z)\right)\left(1+f_2(w,z)\right) - 1\right) \\ &\cdot \exp\left(-\frac{\varphi^2}{2dz}\left(d\alpha^{d-1}+1\right)\right) d\varphi, \end{aligned} \tag{3.21}$$

where

$$|f_3(w,z)| \le \frac{\rho e}{6(1-\rho e)^2}\varphi^3.$$

Then we have

$$|I_{\text{error}}| \le 2\varphi_0\left(\frac{1-\rho\cos\varphi_0}{1-\rho}\exp\left(f_1(x) + \frac{\rho e}{6(1-\rho e)^2}\varphi_0^3\right)\left(1+f_2(\varphi_0,z)\right) - 1\right), \tag{3.22}$$

and

$$I_{\text{main}} = \sqrt{\frac{\pi z d}{d\alpha^{d-1}+1}} - 2\int_{\varphi_0}^{\infty} \exp\left(-\frac{\varphi^2}{2dz}\left(d\alpha^{d-1}+1\right)\right) d\varphi. \tag{3.23}$$

Hence, it follows that

$$I = \frac{\alpha^{\frac{1-d}{2}}}{\sqrt{d\alpha^{d-1}+1}}\exp\left(\frac{A}{z}\right) + \hat{E}_{\varphi_0}(w,z), \tag{3.24}$$

where

(3.25)

$$\begin{aligned}|\hat{E}_{\varphi_0}(w,z)| \leq\ & \frac{\alpha^{\frac{d-1}{2}}}{\sqrt{2\pi d|z|}} \exp\left(\frac{Ax}{x^2+y^2}\right)\left[\frac{(2d|z|)^{\frac{1}{2}}}{\varphi_0\left(d\alpha^{d-1}+1\right)^{\frac{1}{2}}}\exp\left(-\frac{\varphi_0^2 x\left(d\alpha^{d-1}+1\right)}{2d\left(x^2+y^2\right)}\right)\right.\\ & \left.+ 2\varphi_0\left(\frac{1-\rho\cos\varphi_0}{1-\rho}\exp\left(f_1(x)+\frac{\rho e}{6(1-\rho e)^2}\varphi_0^3\right)(1+f_2(\varphi_0,z))-1\right)\right]\\ =:\ & E_{\varphi_0}(z).\end{aligned}$$

Hence in conclusion we arrive at

$$|f_{err}(z)| \leq \left(E_{\varphi_0}(z)+E_{\mathcal{B}}(z)\right)\left(\alpha^{d-1}\left(d\alpha^{d-1}+1\right)\right)^{\frac{1}{2}}\exp\left(\frac{-Ax}{x^2+y^2}\right).$$

□

Proof of Lemma 3.3. In order to bound f for $x^{1+\varepsilon} < |y| \leq \pi$, note that

$$|\Theta(w,z)| \leq \Theta(\rho,x)$$

by (3.5). Also, (3.4) yields

$$\log|H(w,z)| = \Re\{\log H(w,z)\} \leq \log H(\rho,x) + \Re\left\{w\sum_{n\geq 1} e^{-nz}\right\} - \rho\sum_{n\geq 1} e^{-nx}.$$

On the other hand, we have that

$$\Re\left\{w\sum_{n\geq 1}e^{-nz}\right\}-\rho\sum_{n\geq 1}e^{-nx} \leq -\rho x^{2\varepsilon-1}\left(\beta^{1-2\varepsilon}e^{-\beta}\right)\left(\frac{1}{1-e^{-\beta}}-\frac{1}{\sqrt{1-2e^{-\beta}\cos\beta^{1+\varepsilon}+e^{-2\beta}}}\right).$$

To see this, note that

$$\Re\left\{w\sum_{n\geq 1}e^{-nz}\right\}-\rho\sum_{n\geq 1}e^{-nx} \leq -\rho e^{-x}\left(\frac{1}{1-e^{-x}}-\frac{1}{\sqrt{1-2e^{-x}\cos x^{1+\varepsilon}+e^{-2x}}}\right).$$

This then gives

$$\begin{aligned}\frac{\Re\left\{w\sum_{n\geq 1}e^{-nz}\right\}-\rho\sum_{n\geq 1}e^{-nx}}{-\rho x^{2\varepsilon-1}} &\geq x^{1-2\varepsilon}e^{-x}\left(\frac{1}{1-e^{-x}}-\frac{1}{\sqrt{1-2e^{-x}\cos x^{1+\varepsilon}+e^{-2x}}}\right)\\ &\geq \beta^{1-2\varepsilon}e^{-\beta}\left(\frac{1}{1-e^{-\beta}}-\frac{1}{\sqrt{1-2e^{-\beta}\cos\beta^{1+\varepsilon}+e^{-2\beta}}}\right)\\ &=: \eta.\end{aligned}$$

The statement of Lemma 3.3 now follows from (3.3) and Lemma 3.4. □

3.2. **Proof of Theorem 3.1.** From (3.2), it follows that

$$q_d(n) = I_1 + I_2, \tag{3.26}$$

where

$$I_1 := \frac{1}{2\pi}\int_{-x^{1+\varepsilon}}^{x^{1+\varepsilon}} f(z)e^{nz}\,dy$$

and
$$I_2 := \frac{1}{2\pi}\left(\int_{-\pi}^{-x^{1+\varepsilon}} + \int_{x^{1+\varepsilon}}^{\pi}\right) f(z)e^{nz}\, dy.$$
In this proof, we let
$$x = \sqrt{\frac{A}{n}}.$$
Following the idea of page 291 of [7], we split I_1 as
$$I_1 = \gamma e^{\frac{2A}{x}} \int_{-x^{1+\varepsilon}}^{x^{1+\varepsilon}} e^{-\frac{y^2 A}{x^3}}\, dy + E_2 + E_3, \tag{3.27}$$
where
$$\begin{aligned}\gamma &:= \frac{1}{2\pi\sqrt{\alpha^{d-1}\left(d\alpha^{d-1}+1\right)}},\\ E_2 &:= \gamma e^{\frac{2A}{x}} \int_{-x^{1+\varepsilon}}^{x^{1+\varepsilon}} e^{-\frac{y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1\right) dy,\end{aligned}$$
and
$$E_3 := \gamma e^{\frac{2A}{x}} \int_{-x^{1+\varepsilon}}^{x^{1+\varepsilon}} e^{\frac{-xy^2+iy^3}{x^2(x^2+y^2)}} f_{err}(z)\, dy.$$
The first integral in (3.27) can be written
$$\gamma e^{\frac{2A}{x}} \int_{-x^{1+\varepsilon}}^{x^{1+\varepsilon}} e^{-\frac{y^2 A}{x^3}}\, dy = \gamma e^{\frac{2A}{x}} \sqrt{\frac{\pi x^3}{A}} + E_1 \tag{3.28}$$
where
$$|E_1| \le \frac{\gamma}{A\sqrt{2}} x^{2-\varepsilon} e^{\frac{2A}{x} - Ax^{2\varepsilon-1}}. \tag{3.29}$$
For E_2, we further split the integral:
$$\begin{aligned}E_2 &= \gamma e^{\frac{2A}{x}} \int_{|y|\le x^{1+\varepsilon_2}} e^{-\frac{y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1\right) dy\\ &\quad + \gamma e^{\frac{2A}{x}} \int_{x^{1+\varepsilon_2}\le|y|\le x^{1+\varepsilon}} e^{-\frac{y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1\right) dy,\end{aligned}$$
with $\varepsilon_2 > \varepsilon$, $\varepsilon_2 > \frac{1}{3}$. Then
$$\left|\gamma e^{\frac{2A}{x}} \int_{|y|\le x^{1+\varepsilon_2}} e^{-\frac{y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1\right) dy\right| \le \gamma e^{\frac{2A}{x}} \left(\exp\left(Ax^{3\varepsilon_2-1}\right) - 1\right) \sqrt{\frac{\pi x^3}{A}} \tag{3.30}$$
and
(3.31)
$$\begin{aligned}\left|\gamma e^{\frac{2A}{x}} \int_{x^{1+\varepsilon_2}\le|y|\le x^{1+\varepsilon}} e^{-\frac{y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1\right) dy\right| &\le \gamma \exp\left(\tfrac{2A}{x} - \tfrac{Ax^{\varepsilon_2-2}}{1+x^{2\varepsilon}}\right) x^3\left(1+x^{2\varepsilon}\right)\\ &\quad + \tfrac{\gamma x^3}{A} \exp\left(\tfrac{2A}{x} - Ax^{\varepsilon_2-2}\right).\end{aligned}$$

Finally, for E_3, we have

$$|E_3| \le \gamma e^{\frac{2A}{x}} \left|f_{err}^{\max}\right| \left(\pi x^3 \left(1 + x^{2\varepsilon}\right)\right)^{\frac{1}{2}}. \tag{3.32}$$

To bound I_2, we apply Lemma 3.3 to find that

$$|I_2| \le \sqrt{\frac{2\pi}{dx}} e^{-\eta\rho x^{2\varepsilon-1}} (1 + f_2(\rho, x)) \exp\left(nx + \frac{A}{x} + \frac{1-d}{2}\log\alpha + f_1(\rho, x)\right). \tag{3.33}$$

Finally, we obtain

$$q_d(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-3/4} \exp(2\sqrt{nA}) + E_1 + E_2 + E_3 + I_2,$$

where $|E_1 + E_2 + E_3 + I_2|$ is bounded using the expressions in (3.29) - (3.33). The result follows with

$$|r_d(n)| \le |E_1| + |E_2| + |E_3| + |I_2|. \tag{3.34}$$

4. Proof of Alder's Conjecture

Using Theorems 2.1 and 3.1, we are now able to prove our main results.

Proof of Theorem 1.1. Applying the results of Sections 2 and 3, we have that

$$\begin{aligned} \Delta_d(n) &= q_d(n) - Q_d(n) \\ &= \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-3/4} \exp(2\sqrt{nA}) + \mathcal{E}_d(n), \end{aligned}$$

where

$$\mathcal{E}_d(n) = r_d(n) - Q_d(n). \tag{4.1}$$

As noted in the proof of Lemma 2.3, we relax the restriction on $y_{\max}$. Thus, Theorem 2.1 implies

$$Q_d(n) = O\left(\exp\left(\frac{2\pi}{\sqrt{3d+9}} n^{1/2} + c_0 n^{\frac{1}{6}}\right)\right),$$

where c_0 is some positive constant.

By Theorem 3.1,

$$|r_d(n)| \le |E_1| + |E_2| + |E_3| + |I_2|,$$

and a careful examination of each of these terms shows that

$$\begin{aligned} E_1 &= O\left(n^{-\frac{5}{6}} e^{2\sqrt{An}}\right), \\ E_2 &= O\left(n^{-\frac{3}{2}\varepsilon_2 - \frac{1}{4}} e^{2\sqrt{An}}\right), \\ E_3 &= O\left(n^{-\frac{15}{16}} e^{2\sqrt{An}}\right), \end{aligned}$$

and

$$I_2 = O\left(n^{\frac{1}{4}} e^{2\sqrt{An} - \eta\rho x^{2\varepsilon-1}}\right).$$

Hence, by choosing $\varepsilon_2 \ge \frac{7}{18}$, the result follows. □

Proof of Theorem 1.2. The works of Yee ([9],[10]) and Andrews [1] show that $\Delta_d(n) \geq 0$ when $d \geq 31$ and can be easily modified to show that the inequality is strict when $n \geq d+6$. For each remaining $4 \leq d \leq 30$, we use Theorems 2.1 and 3.1 to compute the smallest n such that our bounds imply $\Delta_d(n) > 0$. We denote this n by $\Omega(d)$, and a C++ program computed the values of $\Delta_d(n) \leq \Omega_d(n)$, which then confirmed the remaining cases of the Alder-Andrews Conjecture. As an example, we find that when $d = 30$, $\Omega(30) \leq 9.77 \cdot 10^6$. To get this, we take $\delta = 10^{-10}$ and $\varepsilon_1 = 5 \cdot 10^{-11}$ in Theorem 2.1 and, in Theorem 3.1, $\varepsilon = .16906$, $\varepsilon_2 = .499999$, $\xi = .99$, $c = .375000001$, and $\nu = 1$. Other d are similar, and all satisfy $\Omega(d) \leq \Omega(30)$.

□

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