Characterizing abelian admissible matrix groups

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Overview

- Admissible matrix groups and inversion formulae
- Calderon’s condition and necessary criteria
- Necessary and sufficient conditions for admissible matrix groups
- Abelian admissible matrix groups
- A criterion for groups with real spectrum
- Checking admissibility for discrete abelian groups with real spectrum

Setup

- $H < \mathrm{GL}(n, \mathbb{R})$ a Lie-subgroup
- $G = \mathbb{R}^n \rtimes H$, the affine group generated by H and translations
- π quasiregular representation on $\mathrm{L}^2(\mathbb{R}^n)$, defined by

$$\pi(x,h)f(y) = |\det(h)|^{-1/2} f(h^{-1}(y-x)) \ .$$

- Given $f, g \in \mathrm{L}^2(\mathbb{R}^n)$, we write

$$V_f g(x,h) = \langle g, \pi(x,h)f \rangle$$

Definition.

- $f \in \mathrm{L}^2(\mathbb{R}^n)$ is called admissible if $V_f : \mathrm{L}^2(\mathbb{R}^n) \hookrightarrow \mathrm{L}^2(G)$ isometrically.
- H is called admissible if there exists an admissible vector $f \in \mathrm{L}^2(\mathbb{R}^n)$.

Inversion formula and the Calderon condition

Chief purpose of admissible vectors: Expansion of arbitrary $g \in \mathrm{L}^2(\mathbb{R}^n)$ in the wavelet system $(\pi(x,h)f)_{(x,h)\in G}$.

f is admissible iff

$$\forall g \in \mathrm{L}^2(\mathbb{R}^n) : g = \int_G \langle g, \pi(x,h)f \rangle \, \pi(x,h)f \; d\mu_G(x,h)$$

in the weak sense.

Chief questions:

- How do you recognize admissible vectors?
- How do you recognize admissible matrix groups?

Literature

Samples from the literature:

- Grossmann/Morlet/Paul (‘86): $ax+b$-group
- Murenzi (‘90): Similitude group, $H=\mathbb{R}^+ \times SO(n)$
- Bernier/Taylor (‘94), HF (‘96,‘98): Admissible matrix groups, irreducible case
- Mallat/Zhong
- Larson/Schulz/Speegle/Taylor: Admissible one-parameter and cyclic groups
- HF/Mayer (‘01), Laugesen/Weaver/Weiss/Wilson (‘01): General admissible matrix groups

Calderon admissibility condition and necessary admissibility criteria

Lemma 1

$f \in \mathrm{L}^2(\mathbb{R}^n)$ is admissible iff

$$\int_H |\widehat{f}(h^T\xi)| dh = 1 \text{ (a.e. } \xi \in \mathbb{R}^n \text{ .}$$

Theorem 2.

Suppose that H is admissible.

- H is closed.
- G is nonunimodular. Equivalently, $\det|_H \neq \Delta_H$.
- For almost every $\xi \in \mathbb{R}^n$, the stabilizer

  $$H_\xi = \{h \in H : h^T\xi = \xi\}$$

  is compact.

Sufficient criteria

Theorem 3. (HF/M,LWWW)
Suppose that H fulfills all necessary criteria from Theorem 2, and additionally, that, for almost every ξ, the orbit $H^T\xi \subset \mathbb{R}^n$ is locally closed. Then H is admissible.

Remarks:

- $H^T\xi$ is locally closed iff for some $\epsilon > 0$, the ϵ-stabilizer

$$H_{\xi,\epsilon} = \{h \in \mathbb{H} : |h^T\xi - \xi| \leq \epsilon\}$$

  is compact.
- Open question: Is the sufficient condition from Theorem 3 necessary?

Discrete admissible groups

Theorem 4. Assume that H is discrete. Then H is admissible iff

1. there exists $h \in H$ with $\det(h) \neq 1$; and
2. there exist $\Omega \subset \mathbb{R}^n$ Borel, H^T-invariant, with $|\mathbb{R}^n \setminus \Omega| = 0$, and $C \subset \Omega$ Borel, meeting each orbit in Ω in a single point.

Sketch of proof:

1. Step: H discrete $\Rightarrow$ H countable $\Rightarrow$ H^T acts freely a.e.
2. Step: If Condition 2. is met, there exists $C \subset \Omega$ Borel, meeting a.e. orbit in a single point. W.l.o.g. H^T acts freely on Ω, hence

$$\sum_{h \in H} \chi_C(h^T \xi) = 1 \ , \ (\text{ a.e. } \xi \in \mathbb{R}^n).$$

Using Condition 1., we can additionally ensure $|C| < \infty$. But then, $\chi_C \in \mathrm{L}^2(\mathbb{R}^n)$, and $\chi^\vee$ is admissible.

Discrete admissible groups, converse direction

Conversely, assume that f is admissible, and let

$$\Omega = \{\xi \in \mathbb{R}^n : \sum_{h \in H} |f(h^T \xi)|^2 = 1 \ .$$

We pick $\xi_{\mathcal{O}}$ from each orbit $\mathcal{O} \subset \Omega$ according to the following rule:

- Let $F_{\mathcal{O}}$ be the set of all $\xi \in \mathcal{O}$ such that

$$|\widehat{f}(\xi)| = \max\{|\widehat{f}(\omega)| : \omega \in \mathcal{O}\} \ .$$

- $F_{\mathcal{O}}$ is finite, and $\bigcup_{\mathcal{O} \subset \Omega} F_{\mathcal{O}}$ is Borel.
- Using a fixed Borel embedding $\Theta : \mathbb{R}^n \to [0,1]$, let

$$\xi_{\mathcal{O}} = \arg\max\{|\Theta(\omega)| \ : \ \omega \in F_{\mathcal{O}}\} \ .$$

Characterizing admissible matrix groups

Theorem 5. H is admissible iff, additionally to the necessary conditions from Theorem 2., it meets the following equivalent conditions:

- The Lebesgue measure on $\mathbb{R}^n$ decomposes into measures supported by the H^T-orbits.
- There exists $\Omega \subset \mathbb{R}^n$ Borel, H-invariant, with $|\mathbb{R}^n \setminus \Omega| = 0$, and such that the orbit space Ω/H^T is standard.
- There exists $\Omega \subset \mathbb{R}^n$ Borel, H-invariant, with $|\mathbb{R}^n \setminus \Omega| = 0$ and a Borel cross-section $\Omega/H^T \to \Omega$.
- There exists $\Omega \subset \mathbb{R}^n$ Borel, H-invariant, with $|\mathbb{R}^n \setminus \Omega| = 0$ and an analytic transversal $C \subset \Omega$ of the orbits in Ω.

Still open: Do these conditions imply that almost every point has a compact ϵ-stabilizer?

Abelian admissible groups

We now restrict attention to abelian matrix groups. The following results are known in this context:

- (HF, '98): There is a natural bijection

$$\{H < \mathrm{GL}(n,\mathbb{R}) : H \text{ abelian, admissible, } \dim(H) = n\} \text{ mod. conjugacy}$$
$$\updownarrow$$
$$\{\mathcal{A} \text{ commutative, associative algebra with unit, } \dim(\mathcal{A}) = n\}$$

  In particular, there are infinitely many conjugacy classes for $n > 6$.

- (Larson/Schulz/Speegle/Taylor 2006): One-parameter groups are admissible iff they are not contained in $\mathrm{SL}(n,\mathbb{R})$.

Not much known beyond that. Best possible

A useful equivalence relation

Lemma 6. Let $H < H' < \mathrm{GL}(n,\mathbb{R})$, with H'/H compact. Then H is admissible iff H' is.

Definition. For two subgroups $H, H' < \mathrm{GL}(n,\mathbb{R})$, we write $H \prec H'$ if $H \subset H'$ and H'/H is compact.
We let $\sim$ denote the equivalence relation generated by $\prec$, i.e., the symmetric, transitive hull of $\prec$.

Corollary 7. If $H \sim H'$, then H is admissible iff H' is admissible.

Theorem 8. Let $H < \mathrm{GL}(n,\mathbb{R})$ be abelian. Then there exists $H_c < \mathrm{GL}(n,\mathbb{R})$ simply connected, abelian with $H \sim H_c$.
If all elements of H have real spectrum, there exists $H_c \sim H$ such that all its elements have real spectrum.

Admissibility for simply connected groups with real spectrum

Theorem 9. Let $H < \mathrm{GL}(n, \mathbb{R})$ be abelian, closed, connected and simply connected. Let $\mathfrak{h} \subset \mathrm{gl}(n, \mathbb{R})$ denote the Lie algebra of H. Then H is admissible iff there exists $\xi \in \mathbb{R}^n$ such that the following condition holds:

$$(R_\xi) \quad \text{the map } \mathfrak{h} \ni X \mapsto X^T \xi \in \mathbb{R}^n \text{ has rank } \dim(H) .$$

In particular, $\dim(H) \leq n$.

Sketch of proof:

Step 1: The condition is necessary: Condition (R_ξ) is equivalent to saying that the stabilizer H_ξ is discrete. In particular, if (R_ξ) is violated, then H_ξ is a nontrivial subgroup of $H \equiv \mathbb{R}^d$, thus noncompact. If this holds for all ξ, then H is not admissible.

Step 2: Let $\Omega : \{\xi \in \mathbb{R}^n : (R_\xi) \text{ holds } \}$. Then Ω is Borel, H-invariant. Moreover, if $\Omega \neq \emptyset$, then $|\mathbb{R}^n \setminus \Omega| = 0$.

Proof of admissibility cont'd

Step 3: There exists an invariant, conull open subset Ω with the following properties:

- H^T acts freely on Ω.
- For all $\xi \in \Omega$: The canonical map $H \to H^T\xi$ is a homeomorphism. This implies the existence of a Borel transversal for Ω.

For this proof of Step 3, we employ joint block diagonalization of the elements of H: With respect to a suitable basis, each $h \in H$ has the form

$$h = \begin{pmatrix} h_1 & 0 & \ldots & 0 \\ 0 & h_2 & 0 \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \ldots & \ldots & h_\ell \end{pmatrix}$$

where each h_k is lower triangular, with unique eigenvalue. The number ℓ of different blocks, as well as the block sizes, are independent of h.

One block case

Assume that $\ell = 1$. Then $H = \exp(\mathbb{R}X_1) \cdot H_u$, where $X_1 \in \mathfrak{h}$ has nonvanishing diagonal, and H_u consists of unipotent matrices.

By the Chevalley-Rosenlicht Theorem, the H_u^T-orbits are closed and simply connected.

We pick

$$\Omega = \{\xi \in \mathbb{R}^n : \xi_1 \neq 0\,, (R_\xi) \text{ holds } \}\ .$$

Let $\xi \in \Omega$, and assume that $h \in H_\xi$. Then $h = e^s n(s) h_u$, for suitable unipotent matrices $n(s), h_u$. Since e^s is the unique eigenvalue of h, it follows that $s = 0$, and $n(s)$ is the identity matrix. But then, since H_u^T also acts locally freely at ξ, it follows that h_u is the identity, and H_ξ is trivial.

Using that H_u^T-orbits are homeomorphic to H_u, and the action of $\exp(\mathbb{R}X_1)$ on the first variable: All H^T-orbits in Ω are homeomorphic to H.

Induction step

Here, we can decompose H into

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & \sigma(h_1)h_2 \end{pmatrix}, h_1 \in H_1, h_2 \in H_2,$$

where

- $H_1 < \mathrm{GL}(m_1, \mathbb{R})$ is abelian.
- H_2 consists of lower triangular matrices with single eigenvalues.
- $\sigma : H_1 \to \mathrm{GL}(m_2, \mathbb{R})$ is a continuous homomorphism satisfying $\sigma(h_1)h_2 = h_2\sigma(h_1)$, for all $h_1 \in H_1, h_2 \in H_2$.
- Moreover, $\sigma(h_1)$ is lower triangular with single eigenvalue.

Finishing the induction step

Let Ω_1 be chosen according to the induction hypothesis, let be Ω_2 chosen by the procedure for the one block case.

Hence, for $h_1 \in H_1$: $\sigma(h_1)^T$ is upper triagonal, and thus $\sigma(h_1)^T \Omega_2 = \Omega_2$. Accordingly,

$$\Omega = \Omega_1 \times \Omega_2 \text{ is } H^T\text{- invariant, open, conull .}$$

For $(\xi_1, \xi_2) \in \Omega$, the quotient map

$$H \ni h \mapsto (h_1^T(\xi_1), h_2^T(\sigma(h_1)^T(\xi_2))) \in H^T \xi$$

has a continuous inverse, because the H_i-quotient maps have continuous inverses. In particular, H acts freely on Ω.

Deciding admissibility for discrete abelian matrix groups

Assume, we are given pairwise commuting matrices $A_1, \ldots, A_k$, all with real spectrum. We want to know: Is $H = \langle A_i : i = 1, \ldots, m \rangle$ admissible? This can be done by the following steps:

- Check, whether $|\det(A_i)| \neq 1$, for some i.
- Check, whether H is discrete.
- Compute simply connected and connected $H_c \sim H$ with real spectrum. Determine, whether H_c acts locally freely.

Is H closed?

First, block diagonalize $A_1, \ldots, A_k$ jointly (see above).

We may assume that all spectra are positive. (Replace A_i by A_i^2, if necessary. We then pass to a subgroup of finite index.)

Compute matrix logarithms $B_1, \ldots, B_k$: For each block, the logarithm is determined from the logarithm of the diagonal entry, and the (finite!) power series for the logarithm of the nilpotent part.

H is discrete iff $\dim(B_1, \ldots, B_k) = \dim_{\mathbb{Z}}(B_1, \ldots, B_k)$.

Computing H_c and check local freeness

If H is discrete, then $\mathfrak{h}_c = \mathrm{span}(B_1, \ldots, B_k)$ is the Lie algebra of a closed, simply connected group H_c with real spectrum, and $H_c \sim H$.

If $d = \dim(\mathfrak{h}_c) > n$, then H is not admissible.
Otherwise, let $C_1, \ldots, C_d$ be any basis of $\mathfrak{h}_c$. Given $\xi \in \mathbb{R}^n$, let $M_\xi : \mathbb{R}^d \to \mathbb{R}^n$,

$$M_\xi(s_1, \ldots, s_d) = \sum_{i=1}^{d} s_i C_i^T \xi \ .$$

For $J \subset \{1, \ldots, n\}$ with cardinality d, let $\Phi_J(\xi)$ denote the associated subdeterminant of M_ξ.

Φ_J is a polynomial, and it vanishes identically only if its coefficients are zero. But the coefficients are computable using (e.g.) the Leibniz formula.

Closing remarks

- Conjecture: H is admissible if G is nonunimodular and exponential.
- If H is closed, abelian, with $\dim(H) \leq 2$, then H is admissible iff $\det|_H \neq \Delta_H$.
- Interesting phenomenon: Let H_c be closed, abelian, connected, simply connected, with Borel transversal for the orbit. Let $H < H_c$ be a discrete subgroup. Then the following are equivalent:
  - H_c has at least one trivial fixed group.
  - There exists a Borel transversal for the H-orbits.