Wavelet coorbit spaces beyond the irreducible setting

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***Abstract*—We study coorbit spaces associated to the quasi-regular representation of a semidirect product group $G = \mathbb{R}^d \rtimes H$ acting on $\mathrm{L}^2(\mathbb{R}^d)$. Unlike for well-known cases such as shearlets or wavelets based on rotation and scalar dilation, we do not assume the associated representation to be irreducible. For this reason, the standard coorbit theory, as introduced by Feichtinger and Gröchenig, is no longer applicable. Instead, we employ the theory developed by Christensen and Olafsson.**

We introduce conditions on the dilation group for which the more general setting allows to define coorbit spaces, and to recover most features of the general theory. The price one has to pay for the additional generality is a more restrictive set of analyzing vectors.

We thus obtain a unified approach to coorbit spaces associated to the irreducibly acting matrix groups, as well as to anisotropic Besov spaces, which are precisely the coorbit spaces associated to the cyclic dilation groups generated by expansive matrices, and various other examples.

I. Introduction

Fix a closed matrix group $H < \mathrm{GL}(d, \mathbb{R})$, the so-called **dilation group**. We let $G = \mathbb{R}^d \rtimes H$, which is the group of affine mappings generated by H and all translations. Elements of G are denoted by pairs $(x, h) \in \mathbb{R}^d \times H$, and the product of two group elements is given by $(x, h)(y, g) = (x+hy, hg)$. The left Haar measure of G is given by $\mathrm{d}(x, h) = |\det(h)|^{-1} \mathrm{d}x\, \mathrm{d}h$, with $\mathrm{d}h$ denoting left Haar measure on H.

G acts unitarily on $\mathrm{L}^2(\mathbb{R}^d)$ by the **quasi-regular representation** defined by

$$[\pi(x, h)f](y) = |\det(h)|^{-1/2} \cdot f\left(h^{-1}(y - x)\right) \ . \quad (1)$$

Picking a suitable function $u \in \mathrm{L}^2(\mathbb{R}^d)$, we then define the continuous wavelet transform of a function $f \in \mathrm{L}^2(\mathbb{R}^d)$ as

$$\mathcal{W}_u f : G \ni (x, h) \mapsto \langle f, \pi(x, h)u\rangle \ .$$

The operator $\mathcal{W}_u : f \mapsto \mathcal{W}_u f$ clearly depends on the choice both of the wavelet and the dilation group H. We are mostly concerned with groups and wavelets that allow to reconstruct f from $\mathcal{W}_u f$. Hence we call u **admissible** whenever $\mathcal{W}_u$ is an isometry into $L^2(G)$. In this case, wavelet inversion can be formulated via the weak-sense formula

$$f = \int_H \int_{\mathbb{R}^d} \mathcal{W}_u f(x, h)\ \pi(x, h)u\ d(x, h) \ .$$

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This reconstruction formula shows that the system $(\pi(x, h)u)_{(x,h)\in G}$ obtained by dilating and translating the mother wavelet acts as a *continuous frame* of $\mathrm{L}^2(\mathbb{R}^d)$, since f can be recovered as a (continuous) superposition of these building blocks. A *coorbit theory* associated to this system can now be understood as a theory of *sparse vectors* with respect to the building blocks. We measure the wavelet coefficient decay by imposing a norm on $\mathcal{W}_u f$, i.e., we fix a Banach space Y of functions on G, and let

$$\|f\|_{Co^u Y} = \|\mathcal{W}_u f\|_Y \ ,$$

defining the associated space $Co^u Y$ as the space of all signals f for which this norm is finite. In order to be of interest, the theory should have the following crucial features:

- **Banach space property**: The space $Co^u Y$ is a well-defined Banach space. This typically requires to extend the wavelet transform from the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$ to a well-chosen space of distributions.
- **Consistency:** There exists a well-understood and accessible class of analyzing wavelets $\mathcal{A}$ (possibly depending on the space Y) such that $Co^{u_1} Y = Co^{u_2} Y$, with equivalent norms, for all $u_1, u_2 \in \mathcal{A}$.
- **Discretization:** For suitable choices of analyzing wavelets, the continuous wavelet system $(\pi(x, h)u)_{(x,h)\in G}$ can be replaced by a discrete subfamily $(\pi(x, h)u)_{(x,h)\in X}$ which is a *Banach frame*, and gives rise to *atomic decompositions*.

By the original coorbit theory of Feichtinger and Gröchenig [1], [2], these features are guaranteed in the case where the underlying representation is *irreducible and integrable*, and the wavelet coorbit theory associated to this setting has been elucidated in [3], [4], [5], [6], [7]. However, there exist interesting examples of representations and associated coorbit space constructions which do not belong to this class, and it is the chief purpose of this paper to introduce a common framework for these examples. This will comprise the irreducible settings, but also cases where the associated dilation group H is a one-parameter group or cyclic. In the latter case, the associated coorbit spaces are the anisotropic Besov spaces, as introduced by Bownik in [8]. We will also exhibit examples that are situated between these two extremes.

Our results are based on an extension of coorbit space theory due to Christensen and Olafsson [9], which allows to treat reducible representations, and also to drop the integrability requirement; see also [10] for a related approach. Previously investigated applications of [9] include Besov-type spaces on stratified groups [11], [12], on symmetric cones [13], as well as atomic decompositions in Bergmann spaces on the unit disk [14].

II. Admissible vectors with integrable reproducing kernels

Just as in the precursor paper [7], the existence and properties of coorbit spaces associated to a general dilation group H crucially depend on the *dual action* of H, defined by

$$H \times \mathbb{R}^d \ni (h, \xi) \mapsto h^T \xi \ .$$

All analyzing wavelets used in this paper will be required to be *admissible*, by which we mean that the wavelet transform $\mathcal{W}_u$ is a well-defined isometry of $\mathrm{L}^2(\mathbb{R}^d)$ into $\mathrm{L}^2(G)$. The following criterion is well-known [15].

Lemma II.1. *Let $u \in \mathrm{L}^2(\mathbb{R}^d)$. Then u is admissible iff*

$$\int_H \left|\widehat{u}(h^T \xi)\right|^2 dh = 1 \ \ (a.e.\ \xi \in \mathbb{R}^d)$$

The following definition describes the class of matrix groups that we will treat in this paper:

Definition II.2. *The dilation group H is called* integrably admissible *if there exists an open subset $\mathcal{O} \subset \mathbb{R}^d$, that is invariant under the dual action of H, with the following properties:*

(i) *$\mathbb{R}^d \setminus \mathcal{O}$ has measure zero.*
(ii) *$\mathcal{O}$ is invariant under the dual action, and the dual action of H on $\mathcal{O}$ is* proper*, i.e., for all compact subsets $C \subset \mathcal{O}$, the set*

$$\{(h, \xi) \in H \times \mathcal{O} : (h^T \xi, \xi) \in C \times C\} \subset H \times \mathcal{O}$$

is compact, as well.

(iii) *There exists a compact subset $C \subset \mathcal{O}$ such that $\mathcal{O} = H^T C$.*

We call the open set $\mathcal{O}$ the essential frequency support *of π.*

Remark II.3. *The compactness condition (iii) is equivalent to the fact that the orbit space $\mathcal{O}/H^T$ is compact, when endowed with the quotient topology. Note that properness of the action entails in particular that the stabilizer groups*

$$H_\xi = \{h \in H : h^T \xi = \xi\}$$

are compact, for all $\xi \in \mathcal{O}$.

Given compact sets $K_1, K_2 \subset \mathcal{O}$, we let

$$((K_1, K_2)) = \{h \in H : h^{-T} K_1 \cap K_2 \neq \emptyset\} \ .$$

If H is integrably admissible, this set will always be compact. In fact, it can be shown that H is integrably admissible with essential frequency support $\mathcal{O}$ iff there exists some relatively compact and open set C with $H^T C = \mathcal{O}$, and in addition, for every such set (equivalently: for some *such set) C, the set $((C, C))$ is relatively compact.*

For these facts, and an exploration of abelian integrably admissible dilation groups and their relationships to concepts in abstract harmonic analysis such as open compact subsets of the unitary dual, we refer to the recent paper [16].

Remark II.4. *The following list of examples demonstrates that the conditions on the dual action are rather mild.*

(a) *It is known [17] that the quasi-regular representation π is irreducible and square-integrable iff there is a single open orbit $\mathcal{O} = H^T \xi$ of the dual action, which in addition has compact associated stabilizers. Clearly, such an open orbit fulfills the conditions of Definition II.2, with $C = \{\xi\}$. The related coorbit spaces are the subject of the recent series of papers [3], [4], [5], [6], [7]. A full classification, up to conjugacy, of the dilation groups acting irreducibly in dimension three is given in the recent preprint [18]; already here the picture is rather complex.*

(b) *If H is a cyclic group, generated by an invertible matrix A, it is known by [19] that the action of H is integrably admissible iff either A or A^{-1} is* expansive*, i.e., only has eigenvalues of modulus > 1. Here, the essential frequency support is $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$. Up to suitable identification, the associated coorbit spaces turn out to be the* anisotropic Besov spaces *introduced by Bownik [8].*

(c) *There exist many intermediate cases between the two extremes listed above. A classification of the abelian integrably admissible matrix groups in dimension three can be found in [16, Proposition 18]. To give concrete two-dimensional examples, fix $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, and let*

$$H = \left\{ \begin{pmatrix} a & & \\ & a^\alpha b^\beta & \\ & & b \end{pmatrix} : a, b > 0 \right\} \ .$$

Then H is integrably admissible, but does not act irreducibly.

Coorbit theory crucially depends on the set of analyzing vectors, and on a so-called reservoir space containing the coorbit spaces. The following definition contains our choices for both.

Definition II.5. *For an open subset $\mathcal{O} \subset \mathbb{R}^d$, we let $\mathcal{D}_\mathcal{O}$ denote the space of compactly supported C^∞-functions on $\mathcal{O}$, endowed with the usual Fréchet topology [20]. The dual space of $\mathcal{D}_\mathcal{O}$ is denoted by $\mathcal{D}'_\mathcal{O}$. We let $\mathcal{S}_\mathcal{O} = \mathcal{F}^{-1}(\mathcal{D}_\mathcal{O})$, and topologize it with the preimage of the Fréchet topology on $\mathcal{D}_\mathcal{O}$. We let $\mathcal{S}'_\mathcal{O}$ denote the dual of $\mathcal{S}_\mathcal{O}$. Thus we may think of $\mathcal{S}'_\mathcal{O}$ as the inverse Fourier image of the distributions on $\mathcal{O}$ (in a formal rather than a literal sense).*

We will also need the semi-norms $\|f\|_{K,N} = \sup_{|\alpha| \leq N} \|(\partial^\alpha f) \cdot \mathbf{1}_K\|_\infty$, for $K \subset \mathbb{R}^d$ compact and $N \geq 0$. Here $\mathbf{1}_K$ denotes the indicator function of K.

Throughout the following, we use $\langle \cdot, \cdot \rangle : \mathcal{S}'_\mathcal{O} \times \mathcal{S}_\mathcal{O} \to \mathbb{C}$ to denote the sesquilinear duality pairing. This allows to *conveniently extend the definition of the wavelet transform: Given $\varphi \in \mathcal{S}'_{\mathcal{O}}$ and $u \in \mathcal{S}_{\mathcal{O}}$, we let*

$$\mathcal{W}_u\varphi(x,h) = \langle \varphi, \pi(x,h)u \rangle \ .$$

Lemma II.6. *Let H be integrably admissible, with essential frequency support $\mathcal{O}$. Let $u \in \mathcal{S}_{\mathcal{O}}$, and $K_1 = \mathrm{supp}(\widehat{u})$.*

(a) Let $v \in \mathcal{S}_{\mathcal{O}}$, and let $K_2 = \mathrm{supp}(\widehat{v})$. Then we have, for all $N \in \mathbb{N}$,

$$\begin{aligned} |\mathcal{W}_u v(x,h)| \quad &\leq \quad C_N(1+|x|)^{-N}(1+\|h\|)^N \|\widehat{u}\|_{K_1,N} \\ & \qquad \|\widehat{v}\|_{K_2,N} \mathbf{1}_{((K_1,K_2))}(h) \ . \end{aligned}$$

with a global constant $C_N > 0$.

(b) Let φ denote an element of $\mathcal{S}'_{\mathcal{O}}$. Then there exist measurable functions $\alpha : H \to \mathbb{R}^+$ and $m : H \to \mathbb{N}_0$ that are bounded on compact sets, such that

$$|\mathcal{W}_u\varphi(x,h)| \leq \alpha(h)(1+|x|)^{m(h)}(1+\|h\|)^{m(h)}\|\widehat{u}\|_{K_1,m(h)}$$

Proof. The proof of part (a) follows by adapting the arguments proving [3, Theorem 3.7]. Note that for $h \notin ((K_1, K_2))$, the supports of $(\pi(x,h)u)^\wedge$ and $\widehat{v}$ are disjoint, hence the wavelet transform vanishes.

For part (b), [20, Theorem 6.6] implies that for every $K \subset \mathcal{O}$ compact there exists an $N(K) \in \mathbb{N}$ and $C(K) > 0$ such that, for all $v \in \mathcal{S}_{\mathcal{O}}$ with $\mathrm{supp}(\widehat{v}) \subset K$ we have

$$|\langle \varphi, v \rangle| \leq C(K)\|\widehat{v}\|_{K,N(K)} \ .$$

Applying this to $v = \pi(x,h)u$ yields, via analogous arguments as for (a),

$$|\mathcal{W}_u\varphi| \leq C(K)(1+|x|)^{N(K)}(1+\|h\|)^{N(K)}\|\widehat{u}\|_{K_1,N(K)}$$

whenever $K \supset \mathrm{supp}((\pi(x,h)u)^\wedge) = h^{-T}K_1$.

Hence, in order to define the auxiliary functions α and m, we pick an increasing covering $\mathcal{O} = \bigcup_{\ell \in \mathbb{N}} A_\ell$ consisting of open, relatively compact sets A_ℓ, and define, for $h \in H$, the index $\ell(h) = \inf\{n \in \mathbb{N} : h^{-T}K_1 \subset \overline{A}_n\}$. Then $\alpha(h) = C(\overline{A_{\ell(h)}})$ and $m(h) = N(\overline{A_{\ell(h)}})$ are as desired. □

Another obvious requirement of coorbit space theory is the availability of nice analyzing wavelets. This is provided by the following result [16, Theorem 10].

Theorem II.7. *Let H be integrably admissible, with essential frequency support $\mathcal{O}$. Then there exists an admissible vector $u \in \mathcal{S}_{\mathcal{O}}$.*

III. Well-definedness of coorbit spaces

We next define the coefficient spaces that we want to use for the introduction of coorbit spaces for our setting.

Definition III.1. *An* admissible weight *on G is a continuous function $v : G \to \mathbb{R}^+$ fulfilling the submultiplicativity condition $v((x_1,h_1)(x_2,h_2)) \leq Cv(x_1,h_1)v(x_2,h_2)$ and the growth estimate $v(x,h) \leq (1+|x|)^s v_0(h)$, with $s \geq 0$, and v_0 bounded on compact sets. Given such a v and $1 \leq p,q \leq \infty$, we define the associated* weighted mixed L^p-spaces *on the group G by*

$$L^{p,q}_v(G) = \left\{ F : G \to \mathbb{C} \ : \int_H \left(\int_{\mathbb{R}^d} |F(x,h)|^p v(x,h)^p dx \right)^{q/p} \frac{dh}{|\det(h)|} < \infty \right\} ,$$

with the obvious norm $\|\cdot\|_{L^{p,q}_v}$. Here $1 \leq p,q < \infty$; the definition for $p = \infty$ and/or $q = \infty$ uses the essential supremum instead.

The next proposition collects the properties of weighted mixed L^p-space that we will need in the following. For the proofs of (a) and (b), we refer to [3], the proof of (c) is standard.

Proposition III.2. *Let $v : G \to \mathbb{R}^+$ denote an admissible submultiplicative weight, and $1 \leq \infty p, q \leq \infty$.*

(a) $Y = \mathrm{L}^{p,q}(G)$ is a solid Banach function space*, i..e., for all measurable $F_1, F_2 : G \to \mathbb{C}$ satisfying $|F_1| \leq |F_2|$ a.e., as well as $F_2 \in Y$, we have $F_1 \in Y$, with $\|F_1\|_Y \leq \|F_2\|_Y$. In addition, G acts strongly continuously on Y by left and right translations.*

(b) For every $1 \leq p,q \leq \infty$ and every admissible weight v, there exists an admissible weight $w : G \to \mathbb{R}^+$ such that we have the following Young estimate: For all $F_1 \in \mathrm{L}^{p,q}_v(G)$ and $F_2 \in \mathrm{L}^1_w(G)$, we have $\|F_1 * F_2\|_{L^{p,q}_v} \leq \|F_1\|_{L^{p,q}_v}\|F_2\|_{L^1_w}$

(c) Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ fulfill the conditions $1/p_1 + 1/p_2 = 1 = 1/q_1 + 1/q_2$ (with $1/\infty = 0$). Then we have the Hölder inequality

$$\left| \int_G F_1(x,h)F_2(x,h)d(x,h) \right| \leq \|F_1\|_{L^{p_1,q_1}_v}\|F_2\|_{L^{p_2,q_2}_{1/v}}$$

We can now verify the central assumptions of [9], thus making the results of that paper available to our setting:

Lemma III.3. *Let H be integrably admissible with essential frequency support $\mathcal{O}$. Pick an admissible weight v on G, and let $Y = \mathrm{L}^{p,q}_v(G)$, for $1 \leq p,q \leq \infty$. Let $u \in \mathcal{S}_{\mathcal{O}}$ denote an admissible vector. Then the following statements hold:*

(R1) For all $v \in \mathcal{S}_{\mathcal{O}}$: $\mathcal{W}_u v = \mathcal{W}_u v * \mathcal{W}_u u$

(R2) The mapping $Y \to \mathbb{C}$,
$F \mapsto \int_G F(x,h)\mathcal{W}_u u((x,h)^{-1})d(x,h)$ is continuous.

(R3) If $F = F * \mathcal{W}_u u \in Y$, then the mapping $\mathcal{S}_{\mathcal{O}} \ni v \mapsto \int_G F(x,h)\langle \pi(x,h)u, v\rangle d(x,h)$ is a well-defined element of $\mathcal{S}'_{\mathcal{O}}$.

(R4) The map $S'_{\mathcal{O}} \to \mathbb{C}$,

$$\phi \mapsto \int_G \langle \phi, \pi(x,h)u \rangle \langle \pi(x,h)u, u \rangle d(x,h) \, .$$

is weakly continuous.

Proof. Since u is admissible, we have the reproducing kernel relation $\mathcal{W}_u v = \mathcal{W}_u v * \mathcal{W}_u u$ for all $v \in \mathrm{L}^2(\mathbb{R}^d) \supset \mathcal{S}_{\mathcal{O}}$. This proves (R1). For (R2), first note that Lemma II.6 (a) entails that $\mathcal{W}_u u \in \mathrm{L}^{p',q'}_{1/v}(G)$, with p', q' the conjugate exponents to p, q.

Hence (R2) follows from the generalized Hölder inequality III.2(c). For (R3) observe that, again by Lemma II.6 (a), $\mathcal{W}_u : \mathcal{S}_{\mathcal{O}} \to \mathrm{L}^{p',q'}_{1/v}(G)$ is a bounded operator. Combining this with the Hölder estimate III.2(c) yields the desired statement.

Statement (R4) is proved once we have shown that

$$\int_G \langle \phi, \pi(x,h)u \rangle \langle \pi(x,h)u, u \rangle d(x,h) = \langle \phi, u \rangle \ .$$

To see this, consider the Fourier-side wavelet inversion formula

$$\widehat{u}(\xi) = \int_G \mathcal{W}_u u(x,h) \left(\pi(x,h)u\right)^\wedge (\xi) d(x,h) \ , \qquad (2)$$

with absolute convergence guaranteed by II.6(a). Since the dual action of H is proper, and $\widehat{u}$ is compactly supported, there exists a compact set $K \subset \mathcal{O}$ such that, for all $(x,h) \in G$,

$$\mathrm{supp}\left(\mathcal{W}_u u(x,h) \left(\pi(x,h)u\right)^\wedge\right) \subset K \ .$$

Now, given any $N \in \mathbb{N}$, the decay estimate from Lemma II.6(a) (and the fact that compact subsets of H have finite Haar measure) allows to establish that

$$G \ni (x,h) \mapsto \left(\mathcal{W}_u u(x,h) \pi(x,h) u\right)^\wedge$$

is a Bochner-integrable map into the Banach space $(C^N(K), \| \cdot \|_{K,N})$. By [20, Theorem 6.6] (and the Hahn-Banach theorem) $\widehat{\phi}$ induces a continuous linear functional on this space for N sufficiently large. Now [21, V.5, Corollary 2] allows to interchange Bochner integration and the evaluation of continuous linear functionals, which entails

$$\begin{aligned} \langle \phi, u \rangle &= \langle \widehat{\phi}, \widehat{u} \rangle = \int_G \langle \widehat{\phi}, (\mathcal{W}_u u(x,h) \pi(x,h) u)^\wedge \rangle d(x,h) \\ &= \int_G \langle \phi, \pi(x,h) u \rangle \langle \pi(x,h) u, u \rangle d(x,h) \ , \end{aligned}$$

as claimed. □

Now the following theorem is a direct application of [9, Theorem 2.3].

Theorem III.4. *Let H be integrably admissible, with essential frequency support $\mathcal{O}$. Let $u \in \mathcal{S}_{\mathcal{O}}$ be admissible. Then the coorbit space*

$$Co^u(\mathrm{L}^{p,q}_v) = \{ \phi \in \mathcal{S}'_{\mathcal{O}} : \mathcal{W}_u \phi \in \mathrm{L}^{p,q}_v(G) \} \ .$$

is a well-defined Banach space.

IV. Consistency

As the notation in Theorem III.4 reminds us, the definition of coorbit spaces is still allowed to depend on the choice of window. It is important to control this dependence; ideally, we would like to have a class of conveniently described analyzing windows such that the coorbit spaces are independent of the precise choice of window within this class.

It is known that the definition of such a class may depend on specific properties of the representation. For example, if the underlying group acts *irreducibly and integrably*, one can define a space of analyzing vectors as follows: For a suitable weight w, which depends on the coefficient space Y, the space

$$\mathcal{A}_w = \{ u \in \mathrm{L}^2(\mathbb{R}^d) : \mathcal{W}_u u \in \mathrm{L}^1_w(G) \} \ .$$

is *consistent* for the definition of CoY [1], [2]. As the example presented in [3, Section 2.1] illustrates, irreducibility is a crucial element here: The cited source constructs two vectors u_1, u_2 that are admissible for the dilation group $\mathbb{R}^+ \cdot \mathrm{Id}$ acting on $\mathbb{R}^2$, with $\mathcal{W}_{u_1} u_1 = \mathcal{W}_{u_2} u_2 \in \mathrm{L}^1_w(G)$, for every reasonable weight w on $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$, but $Co^{u_1}(\mathrm{L}^1) \neq Co^{u_2}(\mathrm{L}^1)$. Note that the dilation group is integrably admissible, hence this example is directly relevant to our discussion.

Nonetheless, picking admissible vectors in $\mathcal{S}_{\mathcal{O}}$ yields consistency:

Theorem IV.1. *Let H be integrably admissible, with essential frequency support $\mathcal{O}$. Given two admissible vectors $u_1, u_2 \in \mathcal{S}_{\mathcal{O}}$, we have $Co^{u_1}(\mathrm{L}^{p,q}_v) = Co^{u_2}(\mathrm{L}^{p,q}_v)$, with equivalent norms.*

Proof. We want to invoke [9, Theorem 2.7], and therefore need to check the following facts:

- For all $v \in \mathcal{S}_{\mathcal{O}}$ we have $\mathcal{W}_{u_i} v * \mathcal{W}_{u_j} u_i = c_{i,j} \mathcal{W}_{u_j} v$, with suitable scalars $c_{i,j}$.
- The map $Y \ni F \mapsto F * \mathcal{W}_{u_j} u_i \in Y$ is well-defined and continuous.
- The map $\mathcal{S}'_{\mathcal{O}} \ni \phi \mapsto \int_G \langle \phi, \pi(x) u_i \rangle \langle \pi(y) u_i, u_j \rangle dx \in \mathbb{C}$ is weakly continuous.

The first condition follows from the fact that the $\mathcal{W}_{u_i}$ are isometries $\mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(G)$ intertwining π with the left regular representation. The second condition follows by a Young estimate similar to the one used to prove (R3) in the proof of III.3. The third condition is proved in the same way as condition (R4) in the cited proof. □

V. Outlook

The chief purpose of this paper was to provide a unified view of diverse constructions of function spaces defined in terms of coefficient decay of generalized wavelet transforms. The main difference between the irreducible and non-irreducible settings became visible in connection with consistency. Nonetheless, Definition II.2 can be seen to provide the grounds for a satisfactory coorbit theory, with consistently defined spaces. There are open questions remaining.

- **Discretization:** The discretization methods for coorbit theory developed by Feichtinger and Gröchenig has been adapted to the nonirreducible setting by Christensen in [22], and we expect that it is applicable to the type of analyzing vectors that we study here. The argument establishing oscillation estimates in the proof of [3, Lemma 2.7] seems readily adaptable to the current setting, and then the results from [22] should be available to yield Banach frames and atomic decompositions.
- **Identification with known spaces:** As we pointed out above, the theory presented here applies to cyclic dilation groups, and the associated coorbit spaces are essentially

the anisotropic Besov spaces. The reason we use the qualifier *essentially* in the previous sentence has to do with the fact that the standard definition of anisotropic Besov spaces [8] describes them as subspaces of the space of *tempered distributions modulo polynomials*, whereas the construction presented here uses the reservoir space $\mathcal{S}'_{\mathcal{O}}$. Hence any statement that two different constructions of spaces of coorbit type coincide necessarily requires some sort of *identification* of certain objects. Making these identifications explicit can be remarkably cumbersome. (See e.g. the proof of [23, Theorem 5.6].)
A partial answer to this problem is provided by results from coorbit theory addressing the independence of the coorbit space construction of the precise choice of reservoir space; see [1, Theorem 4.2] for the original setting, and [9, Theorems 2.8 – 2.10] for the current one. An alternative, which often works when comparing two smoothness spaces $\mathcal{Y}$ and $\mathcal{Z}$, is available whenever both $\mathcal{Y}$ and $\mathcal{Z}$ have atomic decompositions with atoms contained in $\mathcal{S}_{\mathcal{O}}$, that converge in the respective norms of the smoothness spaces. Under these assumptions, $\mathcal{S}_{\mathcal{O}}$ is densely contained in both, and thus it suffices to check the equivalence of the norms $\|\cdot\|_{\mathcal{Y}}$ and$\|\cdot\|_{\mathcal{Z}}$ on $\mathcal{S}_{\mathcal{O}}$ to conclude that there exists a unique isomorphism $\mathcal{Y} \to \mathcal{Z}$ that extends the identity map on $\mathcal{S}_{\mathcal{O}}$.

- **Alternative descriptions as decomposition spaces:** We expect that the spaces constructed here are further examples of the *decomposition space* family introduced by Feichtinger and Gröbner [24]. The wavelet coorbit spaces associated to irreducible group actions, as well as the anisotropic Besov spaces have decomposition space descriptions (see [5] and [23], respectively), and it seems plausible to expect this to hold for the coorbit spaces of the kind introduced here. In fact, the definition of decomposition spaces used in [5], [25] refers to the same reservoir $\mathcal{S}_{\mathcal{O}}$ employed in this paper, hence the proof may actually be simpler than the one in [5].
The decomposition space view would be beneficial for a further comprehensive understanding of the different spaces. There exists an elaborate embedding theory [25], [26], [27] that is particularly suited to providing an understanding of the dependence of the coorbit spaces on the underlying group, and their relationship to classical smoothness spaces such as Sobolev spaces.

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