Hilbert (Blumenthal) Modular Forms

for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$.

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1.) Introduction to Hilbert Modular Forms

2.) Eisenstein series and theta series

3.) Borcherds products

4.) Generators and relations for M^5, M^{13} and M^{17}.

(http://www.matha.rwth-aachen.de/people/mayer/index.html)

- p prime, $p \equiv 1 \pmod 4$, especially $p \in \{5, 13, 17\}$,

- $\mathcal{K} = \mathbb{Q}(\sqrt{p})$ with integers $\mathfrak{o} := \mathbb{Z} + \frac{1+\sqrt{p}}{2}\mathbb{Z}$.

  $\overline{\lambda} := \lambda_1 - \lambda_2\sqrt{p}$ $\qquad (\lambda = \lambda_1 + \lambda_2\sqrt{p} \in \mathcal{K}, \lambda_1, \lambda_2 \in \mathbb{Q})$

  $\mathrm{N}(\lambda) = \lambda\overline{\lambda} = \lambda_1^2 - p\lambda_2^2$ (norm)

  $\mathrm{S}(\lambda) = \lambda + \overline{\lambda}$ (trace)

  $\varepsilon_0 = \min\{x \in \mathfrak{o}^*;\ x > 1\}$ (fundamental unit)

  Then $\mathfrak{o}^* = \pm\varepsilon_0^{\mathbb{Z}}$.

- $\Gamma = \mathrm{SL}(2, \mathfrak{o})$ operates on $\mathbb{H}^2 = \{z \in \mathbb{C};\ \mathrm{Im}\,(z) > 0\}^2$:

$$\gamma\tau = \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{\overline{a}\tau_2 + \overline{b}}{\overline{c}\tau_2 + \overline{d}}\right),$$

  where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$.

- This motivates $S(\lambda\tau) := \lambda\tau_1 + \overline{\lambda}\tau_2$ and

  $N(c\tau+d)^k := (c\tau_1+d)^k(\overline{c}\tau_2+\overline{d})^k, \qquad (z^k := \exp(k \ln z))$

- $\Gamma =< J, T, T_w >$, where $w = \frac{1}{2}(1 + \sqrt{p})$ and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T_w := \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}.$$

Definition (Hilbert Modular Form). A Hilbert modular form f (HMF) of weight $k \in \mathbb{Q}$ with multiplier system (m.s.) $\mu : \Gamma \to \mathbb{C}^*$ (just a map) is a holomorphic function $\mathbb{H}^2 \to \mathbb{C}$ satisfying

$$f(\gamma\tau) = \mu(\gamma)N(c\tau + d)^k f(\tau) \qquad \forall \tau \in \mathbb{H}^2, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

We denote the corresponding vector space by $M_k^p(\mu)$.

Remark. *We obtain from Gundlach (1985):*
$f \in M_k^5(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0$ *and* $\mu \equiv 1$,

$f \in M_k^{13}(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0$, $\mu(J) = \mu(T)^3 = \mu(T_w)^3 = 1$,

$f \in M_k^{17}(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0/2$, $\mu^2(T) = \mu(T_w)^4 = (-1)^{2k}$ *and* $\mu(J) = \mu(T)^3$.

μ *is unique by the given conditions. If* $k \in \mathbb{Z}$, *then* μ *is a character.*

Definition (Eisenstein series). Let $r \in \mathbb{N} \setminus \{0\}$ and define the $2r^{\text{th}}$ Eisenstein series $E_{2r} : \mathbb{H}^2 \to \mathbb{C}$ by

$$E_{2r}(\tau) := \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} N(c\tau + d)^{-2r},$$

where Γ_∞ denotes the subgroup $\left\langle -E, T, T_w, D := \left(\begin{smallmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{smallmatrix}\right) \right\rangle$ of Γ fixing $\infty = (\infty, \infty)$.

E_{2r} is a Hilbert modular form of weight $2r$ with trivial multiplier system. ($E_{2r} \in M_{2r}^p(1)$).

Definition (Θ-series). Define the Siegel halfspace

$$\mathcal{H}_2 := \{Z := X + iY \in M_2(\mathbb{C});\ Z = Z^{\mathrm{tr}}, Y > 0\}.$$

For $m = (m', m'') \in \{0,1\}^4$ with $(m')^{\mathrm{tr}} m'' \in 2\mathbb{Z}$ and all $Z \in \mathcal{H}_2$ define

$$\Theta_m(Z) := \sum_{g \in \mathbb{Z}^2} \mathbf{e}^{\pi i \left((g+m'/2)^{\mathrm{tr}} Z (g+m'/2) + (g+m'/2)^{\mathrm{tr}} m''\right)}.$$

There are exactly 10 such Theta series. Denote their product by Θ.

Write $p = u^2 + v^2$ ($\equiv 1 \pmod 4$) where v is even, $u, v \in \mathbb{Z}$ and define $\omega := \frac{1}{2}(u + \sqrt{p})$. Then

$$\epsilon : \mathbb{H}^2 \to \mathcal{H}_2, \quad \tau \mapsto \begin{pmatrix} \mathsf{S}\left(\frac{\omega}{\sqrt{p}}\tau\right) & \mathsf{S}\left(\frac{v}{2\sqrt{p}}\tau\right) \\ \mathsf{S}\left(\frac{v}{2\sqrt{p}}\tau\right) & \mathsf{S}\left(\frac{\overline{\omega}}{\sqrt{p}}\tau\right) \end{pmatrix}$$

induces a map between modular forms.

Lemma (Hammond, 1966). *There are three algebraicly independent HMFs, namely two Eisenstein series E_4, E_6 and a theta product $\Theta \circ \epsilon$.*

In case $\mathcal{K} = \mathbb{Q}(\sqrt{17})$ Hermann (1981) introduces the Hilbert modular form η_2 of weight $\frac{3}{2}$ with multiplier system μ_{17} ($\mu_{17}(J) = -i$, $\mu_{17}(T) = i$ and $\mu_{17}(T_w) = e^{5\pi i/4}$):

$$\begin{aligned}\eta_2 :=& \theta_{1100}\theta_{0011}\theta_{0000} + \theta_{1100}\theta_{0010}\theta_{0001} \\ &+ \theta_{1001}\theta_{0110}\theta_{0000} - \theta_{1001}\theta_{0100}\theta_{0010} \\ &+ \theta_{1000}\theta_{0100}\theta_{0011} - \theta_{1000}\theta_{0110}\theta_{0001}\end{aligned}$$

where $\theta_m := \Theta_m \circ \epsilon$.

$A_k(p)$: vector space of nearly hol. modular forms (meromorphic in cusps) $\mathbb{H} \to \mathbb{C}$ of weight k for the group

$$\Gamma_0(p) = \left\{ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z});\ c \equiv 0 \pmod{p} \right\}$$

with character χ_p induced by the Legendre symbol $\left(\frac{\cdot}{p}\right)$.

Define (where $q = e^{2\pi i \tau}$)

$$A_k^+(p) = \left\{ \sum_{n \in \mathbb{Z}} a(n) q^n \in A_k(p); a(n) = 0 \text{ if } \chi_p(n) = -1 \right\}$$

$S_k(p)$: subspace of cusp forms in $A_k(p)$.

$S_k^+(p)$: subspace of cusp forms in $A_k^+(p)$.

principal part of $f = \sum_n a_n q^n \in A_k^+(p)$: $\sum_{n<0} a_n q^n$.

Lemma. *There is a modular form in* $A_0^+(p)$ *with prescribed principal part* $\sum_{n<0} a(n)q^n$ *iff*

$$\forall n < 0: \qquad \chi_p(n) = -1 \Rightarrow a(n) = 0 \text{ and}$$

$$\forall \sum_{m>0} b(m)q^m \in S_2^+(p): \qquad \sum_{n<0} s(n)a(n)b(-n) = 0$$

where $s(n) = 2$ *if* $p|n$ *and* $s(n) = 1$ *otherwise.*

Lemma (Hecke (1940)). *If* $p \equiv 1 \pmod 4$, *then* $\dim S_2(p) = 2\left\lfloor \frac{p-5}{24} \right\rfloor$ *(*$= 0$ *iff* $p \leq 17$*)*

For $p \in \{5, 13, 17\}$ there is a basis $\{f_n = s(n)^{-1}q^{-n} + O(1)\}$ of $A_0^+(p)$.

Theorem (Borcherds, Bruinier (2003)).
For $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p)$ *with* $s(n)a(n) \in \mathbb{Z}$ *for all* $n < 0$ *there is a meromorphic* $\Psi : \mathbb{H}^2 \to \mathbb{C}$, *a Weyl chamber* $W \subset \mathbb{H}^2$ *and* $\rho_W \in \mathcal{K}$, *such that*

$$\Psi(\tau) = e^{2\pi i S(\rho_W \tau)} \prod_{\substack{\mu \in \frac{1}{\sqrt{p}}\mathfrak{o} \\ (\mu, W) > 0}} \left(1 - e^{2\pi i S(\mu\tau)}\right)^{s(p\mu\overline{\mu})a(p\mu\overline{\mu})}$$

for all $\tau \in W$ *with* $\operatorname{Im}(\tau_1)\operatorname{Im}(\tau_2) > |\min\{n; a(n) \neq 0\}|/p$.

- *The Fourier expansion of* Ψ *can be calculated.*
- Ψ *is a* holomorphic *Hilbert modular form for* Γ.
- *Its divisor depends only on the principal part of* f.
- *The weight of* Ψ *and its multiplier system are known.*

Definition. We define

- M^p: Ring of HMFs with symmetric m.s. μ:

$$\mu(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \mu(\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right))$$

- $M^p(1) := \sum_k M_k^p(1)$.
- $\varphi : \mathbb{H} \to \mathbb{D} := \{\tau \in \mathbb{H}^2;\ \tau_1 = \tau_2\}, z \mapsto (z, z)$.
- $\mathbb{D}_{\varepsilon_0} := \{\tau \in \mathbb{H}^2;\ \tau_1 = \varepsilon_0^2 \tau_2\}$.

We have $M_0^p(1) = \mathbb{C}$, $M_0^p(\mu) = \{0\}$ for all $\mu \not\equiv 1$ and $M_k^p(\mu) = \{0\}$ for all $k < 0$ and multiplier systems μ.

Lemma. *$f \in M_k^p(\mu) \Rightarrow f \circ \varphi$ is an elliptic modular form of weight $2k$ with character $\mu|_{SL(2,\mathbb{Z})}$. If f is a cusp form, then $f \circ \varphi$ is a cusp form.*

Definition. For $f : \mathbb{H}^2 \to \mathbb{C}$ define $f^{\pm}(\tau_1, \tau_2) = \frac{1}{2}(f(\tau_1, \tau_2) \pm f(\tau_2, \tau_1))$

Then $f = f^+ + f^-$ and f^- vanishes on $\mathbb{D}$. If f is a HMF with symmetric m.s., then f^+ and f^- are HMFs of same weight and multiplier system.

Borcherds-Products:

$f \in A_0^+(p)$	$\longmapsto \Psi$	divisor	
$f_1 = q^{-1} + O(1)$	$\longmapsto \Psi_1$	$\Gamma \cdot \mathbb{D}$	$f\|_{\mathbb{D}} \equiv 0 \Rightarrow \Psi_1 \| f$
$f_p = \frac{1}{2}q^{-p} + O(1)$	$\longmapsto \Psi_p$	$\Gamma \cdot \mathbb{D}_{\varepsilon_0}$	$f\|_{\mathbb{D}_{\varepsilon_0}} \equiv 0 \Rightarrow \Psi_p \| f$
$f_j = \frac{1}{s(n)}q^{-j} + O(1)$	$\longmapsto \Psi_j$		

Theorem (Gundlach, Resnikoff, ...). *M^5 is generated by E_2, Ψ_1, E_6 and Ψ_5 and all relations are induced by*

$$\Psi_5^2 - \left(\frac{67}{25}E_6 - \frac{42}{25}E_2^3\right)\left(\frac{67}{43200}\left(E_2^3 - E_6\right)\right)^4 = \Psi_1^2(\dots)$$

f	E_2	Ψ_1	$e_6 := \frac{67}{25}E_6 - \frac{42}{25}E_2^3$	Ψ_5
$f \circ \varphi$	g_2	0	g_3^2	$\Delta^2 g_3$
weight of f	2	5	6	15

Proof by induction (weight k):

$M_0^5(1) = \mathbb{C}$, $M_0^5(\mu) = M_{-k}^5(\mu) = \{0\}$ for all $\mu \not\equiv 1, k > 0$.
In case $\mathbb{Q}(\sqrt{5})$: $\mu \equiv 1$.

...

Let $f \in M_k^5(1)$. Write $D := \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix}$.

k is odd: $\tau = D(\tau_2, \tau_1)$ on $\mathbb{D}_{\varepsilon_0}$, $N(\varepsilon_0^{-1}) = -1$

$$\Rightarrow \quad f^+|_{\mathbb{D}_{\varepsilon_0}} \overset{\mu \equiv 1}{=} -f^+|_{\mathbb{D}_{\varepsilon_0}} \quad \Rightarrow \quad \Psi_5 | f^+ \text{ and } \Psi_1 | f^-$$

k is even: $f \circ \varphi$ is an elliptic mod. form of weight $2k$ for $SL(2, \mathbb{Z})$

$\Rightarrow$ there is a polynomial q: $f \circ \varphi - q(g_2, g_3^2) \equiv 0$.

$$\Rightarrow \quad f - q(E_2, e_6)|_{\mathbb{D}} = 0 \quad \Rightarrow \quad \Psi_1 | f - q(E_2, e_6).$$

The relation immediately follows from the elliptic case. $\square$

Theorem. *M^{13} is generated by Ψ_1, $\frac{\Psi_4}{\Psi_1}$, E_2 and Ψ_{13}.*

f	Ψ_1	$\frac{\Psi_4}{2\Psi_1}$	E_2	Ψ_{13}
$f \circ \varphi$	0	η^8	g_2	$\eta^{16} g_3$
weight of f	1	2	2	7
multiplier μ	μ_{13}	μ_{13}	1	μ_{13}^2

$\mu_{13}(J) = 1$, $\mu_{13}(T) = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, $\mu_{13}(T_w) = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$.

μ is already determined by $f \circ \varphi$.

If k is odd, proceed as for $\mathbb{Q}(\sqrt{5})$ ($f^+(\tau) \in -e^{2\pi i \mathbb{Z}/3} f^+(\tau)$).

If k is even, there is a polynomial q satisfying $f \circ \varphi = q(\eta^8, g_2)$. $\Rightarrow$ $\Psi_1 | (f - q(\frac{\Psi_4}{2\Psi_1}, E_2))$ $\square$

Lemma. *All relations of the generators for M^{13} are induced by*

$$\begin{aligned}\Psi_{13}^2 - \left(\frac{\Psi_4}{2\Psi_1}\right)^4 \left(E_2^3 - 2^6 3^3 \left(\frac{\Psi_4}{2\Psi_1}\right)^3\right) = \\ - 108\Psi_1^{12}\Psi_2 - \frac{27}{16}\Psi_1^{10}E_2^2 + \frac{495}{8}\Psi_1^8\Psi_2^2E_2 \\ - \frac{1459}{16}\Psi_1^6\Psi_2^4 + \frac{41}{8}\Psi_1^6\Psi_2E_2 - 512\Psi_1^6\left(\frac{\Psi_4}{2\Psi_1}\right)^4 \\ + \frac{1}{16}\Psi_1^4E_2^5 - \frac{97}{4}\Psi_1^4\Psi_2^3E_2^2 - \frac{1}{8}\Psi_1^2\Psi_2^2E_2^4 \\ - 144\Psi_1^2\left(\frac{\Psi_4}{2\Psi_1}\right)^5 E_2 + \frac{189}{8}\Psi_1^2\Psi_2^5E_2\,.\end{aligned}$$

$q \neq 0$ polynomial, $q\left(\Psi_1, \frac{\Psi_4}{2\Psi_1}, E_2, \Psi_{13}\right) \equiv 0$.

$r := q(0, \cdot, \cdot, \cdot)$.

If $r \equiv 0$ look at $q(X_1, X_2, X_3, X_4)/X_1$ instead of q.

Else $r\left(\eta^8, g_2, \eta^{16} g_3\right) \equiv 0$ holds, all elliptic relations are induced by

$$(\eta^{16} g_3)^2 - (\eta^8)^4 g_3^2 = 0$$

A comparison of fourier expansions concludes the argument. $\square$

Theorem. *M^{17} is generated by Ψ_1, E_2, $-\Psi_2$, η_2 and Ψ_{17}.*

f	Ψ_1	E_2	$-\Psi_2$	$\eta_2/8$	Ψ_{17}
$f \circ \varphi$	0	g_2	η^6	η^6	$\eta^6 g_3$
weight of f	$\frac{1}{2}$	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$
multiplier μ	μ_{17}	1	μ_{17}^5	μ_{17}	μ_{17}^7

The proof is analogous to the ones for M^5 and M^{13}.

Problem 1: μ_{17} has order 8 and holds $\mu_{17}^4|_{\mathrm{SL}(2,\mathbb{Z})} = 1$, so we need two lifts of η^6. If $\mu \not\equiv 1$, there is $\nu \in \mathfrak{o}$ s.t. $\mu(T_\nu) = \mu(\left(\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}\right)) \neq 1$. Then $f(T_\nu \tau) = \mu(T_\nu) f(\tau) \neq f(\tau)$ holds as τ and $T_\nu \tau$ tend to ∞. Thus f and $f \circ \varphi$ are cusp forms and $\eta^6 | f \circ \varphi$.

Problem 2: For M^5 and M^{13}, every symmetric HMF f^+ of odd weight k is divisible by Ψ_p. Here:

$$\begin{aligned} f^+(\tau) = f^+(D\overline{\tau}) &= \mu(D)N(\varepsilon_0^{-1})^k f^+(\overline{\tau}) \\ &= e^{k\pi i}\mu(D)f^+(\tau), \qquad (\overline{\tau} := (\tau_2, \tau_1)) \end{aligned}$$

for $\tau \in \mathbb{D}_{\varepsilon_0}$. Because of $\mu_{17}^4(D) = 1$, this property only depends on $\mu|_{\mathrm{SL}(2,\mathbb{Z})}$. We obtain:

Every symmetric HMF f for which $f \circ \varphi$ is a multiple of g_3 but not of g_3^2, is divisible by Ψ_{17}.

Lemma.

$$\eta_2^2 - 64\Psi_2^2 = 16\Psi_1^2 E_2$$

and

$$\begin{aligned}\Psi_{17}^2 - \Psi_2^2 E_2^3 + 216\Psi_2^5\eta_2 = -256\Psi_1^{18} \\ - 176\Psi_1^{12}\Psi_2\eta_2 - \frac{2671}{4096}\Psi_1^6\eta_2^4 + \frac{103}{8}\Psi_1^4 E_2^2\Psi_2\eta_2 \\ - \frac{87}{16}\Psi_1^{10}E_2^2 - \frac{99}{128}\Psi_1^2 E_2\Psi_2\eta_2 + \frac{1387}{128}\Psi_1^8 E_2\eta_2^2\end{aligned}$$

induce all relations of the generators for M^{17}.

Corollary. $M^{13}(1)$ *is generated by*

$$M_{13} := \left\{ E_2, \frac{\Psi_4}{2\Psi_1}\Psi_{13}, \left(\frac{\Psi_4}{2\Psi_1}\right)^3, \Psi_1 \left(\frac{\Psi_4}{2\Psi_1}\right)^2, \Psi_1\Psi_{13}, \Psi_1^2\frac{\Psi_4}{2\Psi_1}, \Psi_1^3 \right\}$$

Corollary. $M^{17}(1)$ *is generated by*

$$\begin{aligned} M_{17} = \{ & E_2, \eta_2^8, \eta_2^3\Psi_2, \eta_2\Psi_{17}, \eta_2^2\Psi_1\Psi_2, \\ & \Psi_1\Psi_{17}, \eta_2\Psi_1^2\Psi_2, \Psi_1^3\Psi_2, \\ & \eta_2^4\Psi_1^4, \eta_2\Psi_1^7, \Psi_1^8 \} \end{aligned}$$