



Hilbert  
(Blumenthal)  
Modular  
Forms  
for  
 $\mathbb{Q}(\sqrt{5})$ ,  $\mathbb{Q}(\sqrt{13})$   
and  $\mathbb{Q}(\sqrt{17})$ .

- 1.) Introduction
- 2.) Eisenstein series and theta series
- 3.) Borcherds products
- 4.) Generators and relations for  $\hat{M}^5$ ,  $\hat{M}^{13}$  and  $\hat{M}^{17}$ .

# 1. Introduction

- $p$  prime,  $p \equiv 1 \pmod{4}$ , especially  $p \in \{5, 13, 17\}$ ,

- $\mathcal{K} = \mathbb{Q}(\sqrt{p})$  with integers  $\mathfrak{o} := \mathbb{Z} + \frac{1+\sqrt{p}}{2}\mathbb{Z}$ .

$$\bar{\lambda} := \lambda_1 - \lambda_2 \sqrt{p} \quad (\lambda = \lambda_1 + \lambda_2 \sqrt{p} \in \mathcal{K}, \lambda_1, \lambda_2 \in \mathbb{Q})$$

$$N(\lambda) = \lambda \bar{\lambda} = \lambda_1^2 - p \lambda_2^2 \quad (\text{norm})$$

$$S(\lambda) = \lambda + \bar{\lambda} \quad (\text{trace})$$

$$\varepsilon_0 = \min\{x \in \mathfrak{o}^*; x > 1\} \quad (\text{fundamental unit})$$

Then  $\mathfrak{o}^* = \pm \varepsilon_0^{\mathbb{Z}}$ .

- $\Gamma = \mathrm{SL}(2, \mathfrak{o})$  operates on  $\mathbb{H}^2 = \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}^2$ :

$$\gamma\tau = \left( \frac{a\tau_1 + b}{c\tau_1 + d}, \frac{\bar{a}\tau_2 + \bar{b}}{\bar{c}\tau_2 + \bar{d}} \right),$$

where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and  $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$ .

- This motivates  $S(\lambda\tau) := \lambda\tau_1 + \bar{\lambda}\tau_2$  and

$$N(c\tau + d)^k := (c\tau_1 + d)^k(\bar{c}\tau_2 + \bar{d})^k, \quad (z^k := \exp(k \ln z))$$

- $\Gamma = \langle J, T, T_w \rangle$ , where  $w = \frac{1}{2}(1 + \sqrt{p})$  and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T_w := \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}.$$

- $\widehat{\Gamma} = \langle \sigma, \Gamma \rangle$  with reflection  $\sigma : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ ,  $(\tau_1, \tau_2) \mapsto (\tau_2, \tau_1)$ .

**Definition (Hilbert Modular Form).** A Hilbert modular form  $f$  (HMF) of weight  $k \in \mathbb{Q}$  with multiplier system (m.s.)  $\mu : \Gamma \rightarrow \mathbb{C}^*$  is a holomorphic function  $\mathbb{H}^2 \rightarrow \mathbb{C}$  satisfying

$$f(\gamma\tau) = \mu(\gamma) N(c\tau + d)^k f(\tau) \quad \forall \tau \in \mathbb{H}^2, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We denote the corresponding vector space by  $M_k^p(\mu)$ . In case  $f(\sigma\tau) = \mu(\sigma)f(\tau)$  for all  $\tau \in \mathbb{H}^2$ , we call  $f$  an extended HMF ( $f \in \widehat{M}_k^p(\mu)$ , where we extend  $\mu$  to a map  $\widehat{\Gamma} \rightarrow \mathbb{C}^*$ ).

**Remark.** We obtain from Gundlach (1985):

$$f \in M_k^5(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0 \text{ and } \mu \equiv 1,$$

$$f \in M_k^{13}(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0, \mu(J) = \mu(T)^3 = \mu(T_w)^3 = 1,$$

$$f \in M_k^{17}(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0/2, \mu^2(T) = \mu(T_w)^4 = (-1)^{2k} \text{ and} \\ \mu(J) = \mu(T)^3.$$

$\mu$  is unique by the given conditions. If  $k \in \mathbb{Z}$ , then  $\mu$  is a character.

## 2. Eisenstein series and theta series

**Definition (Eisenstein series).** Let  $r \in \mathbb{N}$  and define the  $2r^{\text{th}}$  Eisenstein series  $E_{2r} : \mathbb{H}^2 \rightarrow \mathbb{C}$  by

$$E_{2r}(\tau) := \sum_{\substack{M \in \Gamma_\infty \setminus \Gamma \\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} N(c\tau + d)^{-2r},$$

where  $\Gamma_\infty$  denotes the subgroup  $\left\langle -E, T, T_w, D := \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix} \right\rangle$  of  $\Gamma$  fixing  $\infty = (\infty, \infty)$ .

$E_{2r}$  is an extended Hilbert modular form of weight  $2r$  with trivial multiplier system. ( $E_{2r} \in \hat{M}_{2r}^p(1)$ ).

**Definition ( $\Theta$ -series).** Define the Siegel halfspace

$$\mathcal{H}_2 := \{Z := X + iY \in M_2(\mathbb{C}); Z = Z^{\text{tr}}, Y > 0\}.$$

For  $m = (m', m'') \in \{0, 1\}^{2+2}$  with  $(m')^{\text{tr}} m'' \in 2\mathbb{Z}$  and all  $Z \in \mathcal{H}_2$  define

$$\Theta_m(Z) := \sum_{g \in \mathbb{Z}^2} e^{\pi i ((g+m'/2)^{\text{tr}} Z (g+m'/2) + (g+m'/2)^{\text{tr}} m'')}.$$

There are exactly 10 such Theta series. Denote their product by  $\Theta$ .

**Theorem and Definition (Hammond).** Write  $p = u^2 + v^2$  (for  $\equiv 1 \pmod{4}$ ) where  $v$  is even,  $u, v \in \mathbb{Z}$  and define  $\omega := \frac{1}{2}(u + \sqrt{p})$ . Then

$$\epsilon : \mathbb{H}^2 \rightarrow \mathcal{H}_2, \quad \tau \mapsto \begin{pmatrix} s\left(\frac{\omega}{\sqrt{p}}\tau\right) & s\left(\frac{v}{2\sqrt{p}}\tau\right) \\ s\left(\frac{v}{2\sqrt{p}}\tau\right) & s\left(\frac{\omega}{\sqrt{p}}\tau\right) \end{pmatrix}$$

induces a map between modular forms.

**Remark.** One easily checks that the images of theta products are invariant under  $\sigma$ .

**Lemma (Hammond, 1966).** There are three algebraically independent HMFs, namely two Eisenstein series  $E_4$ ,  $E_6$  and a theta product  $\Theta \circ \epsilon$ .

**Example.** In case  $\mathcal{K} = \mathbb{Q}(\sqrt{17})$  Hermann (1981) introduces the (extended) Hilbert modular form  $\eta_2$  of weight  $\frac{3}{2}$  with multiplier system  $\mu_{17}$  ( $\mu_{17}(J) = -i$ ,  $\mu_{17}(T) = i$  and  $\mu_{17}(T_w) = e^{5\pi i/4}$ ):

$$\begin{aligned}\eta_2 := & \theta_{1100}\theta_{0011}\theta_{0000} + \theta_{1100}\theta_{0010}\theta_{0001} \\ & + \theta_{1001}\theta_{0110}\theta_{0000} - \theta_{1001}\theta_{0100}\theta_{0010} \\ & + \theta_{1000}\theta_{0100}\theta_{0011} - \theta_{1000}\theta_{0110}\theta_{0001}\end{aligned}$$

where  $\theta_m := \Theta_m \circ \epsilon$ . Its restriction to the diagonal is given by  $8\eta^6$ , respectively  $\eta_2(z, z) = 8\eta^6(z)$  for all  $z \in \mathbb{H}$ .

### 3. Borcherds products

- $A_k(p)$  : vector space of nearly hol. modular forms (meromorphic in cusps)  $\mathbb{H} \rightarrow \mathbb{C}$  of weight  $k$  for the group

$$\Gamma_0(p) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}); c \equiv 0 \pmod{p} \right\}$$

with character  $\chi_p$  induced by the Legendre symbol  $\left(\frac{\cdot}{p}\right)$ .

- Define (where  $q = e^{2\pi i\tau}$ )

$$A_k^+(p) = \left\{ \sum_{n \in \mathbb{Z}} a(n)q^n \in A_k(p); a(n) = 0 \text{ if } \chi_p(n) = -1 \right\}$$

- $S_k(p)$ : subspace of cusp forms in  $A_k(p)$ .
- $S_k^+(p)$ : subspace of cusp forms in  $A_k^+(p)$ .

**Definition.** *principal part* of  $f = \sum_n a_n q^n \in A_k^+(p)$ :  $\sum_{n<0} a_n q^n$ .

**Lemma (cp. Bruinier, Bundschuh (2003)).** *There is a modular form in  $A_0^+(p)$  with prescribed principal part  $\sum_{n<0} a(n)q^n$  iff*

$$\begin{aligned} \forall n < 0, \chi_p(n) = -1 : \quad & a(n) = 0 \quad \text{and} \\ \forall \sum_{m>0} b(m)q^m \in S_2^+(p) : \quad & \sum_{n<0} s(n)a(n)b(-n) = 0 \end{aligned}$$

where  $s(n) = 2$  if  $p|n$  and  $s(n) = 1$  otherwise.

**Lemma (Hecke (1940)).** *If  $p \equiv 1 \pmod{4}$ , then*

$$\dim S_2(p) = 2 \left\lfloor \frac{p-5}{24} \right\rfloor \quad (= 0 \text{ iff } p \leq 17)$$

**Remark.** *For  $p \in \{5, 13, 17\}$  there is a basis*

$$\{f_n = s(n)^{-1}q^{-n} + O(1) \mid \chi_p(n) \geq 0\} \text{ of } A_0^+(p).$$

**Theorem (Borcherds, Bruinier (2003)).**

For  $f = \sum_{n \in \mathbb{Z}} a(n)q^n \in A_0^+(p)$  with  $s(n)a(n) \in \mathbb{Z}$  for all  $n < 0$  there is a meromorphic  $\Psi : \mathbb{H}^2 \rightarrow \mathbb{C}$ , a Weyl chamber  $W \subset \mathbb{H}^2$  and  $\rho_W \in \mathcal{K}$ , such that

$$\Psi(\tau) = e^{2\pi i S(\rho_W \tau)} \prod_{\substack{\mu \in \frac{1}{\sqrt{p}}\mathfrak{o} \\ (\mu, W) > 0}} (1 - e^{2\pi i S(\mu \tau)})^{s(p\mu\bar{\mu})a(p\mu\bar{\mu})}$$

for all  $\tau \in W$  with  $\operatorname{Im}(\tau_1)\operatorname{Im}(\tau_2) > |\min\{n; a(n) \neq 0\}|/p$ .

- The Fourier expansion of  $\Psi$  can be calculated.
- $\Psi$  is an extended Hilbert modular form for  $\widehat{\Gamma}$ , if  $a(n) \geq 0$  for all  $n < 0$ .
- Its divisor depends only on the principal part of  $f$ .
- The weight of  $\Psi$  and its multiplier system are known.

**Definition.** We define

- $\hat{M}^p$ : Ring of extended HMFs.
- $M^p(1) := \sum_k M_k^p(1)$ .
- $\varphi : \mathbb{H} \rightarrow \mathbb{D} := \{\tau \in \mathbb{H}^2; \tau_1 = \tau_2\}, z \mapsto (z, z)$ .
- $\mathbb{D}_{\varepsilon_0} := \{\tau \in \mathbb{H}^2; \tau_1 = \varepsilon_0^2 \tau_2\}$ .

We have  $M_0^p(1) = \mathbb{C}$ ,  $M_0^p(\mu) = \{0\}$  for all  $\mu \not\equiv 1$  and  $M_k^p(\mu) = \{0\}$  for all  $k < 0$  and multiplier systems  $\mu$ .

**Lemma.**  $f \in M_k^p(\mu) \Rightarrow f \circ \varphi$  is an elliptic modular form of weight  $2k$  with character  $\mu|_{SL(2, \mathbb{Z})}$ . If  $f$  is a cusp form, then  $f \circ \varphi$  is a cusp form.

**Definition.** For  $f : \mathbb{H}^2 \rightarrow \mathbb{C}$  define

$$f^\pm(\tau_1, \tau_2) = \frac{1}{2}(f(\tau_1, \tau_2) \pm f(\tau_2, \tau_1))$$

**Remark.** Then  $f = f^+ + f^-$  and  $f^-$  vanishes on  $\mathbb{D}$ . If  $f$  is a HMF with symmetric m.s. (i.e.  $\mu(\overline{M}) = \mu(M)$ , then  $f^+$  and  $f^-$  are HMFs of same weight and multiplier system.

**Remark.** If  $\mu$  is symmetric,  $f^+$  and  $f^-$  are extended HMFs ( $f^+(\sigma\tau) = f^+(\tau)$  and  $f^-(\sigma\tau) = -f^-(\tau)$  for all  $\tau \in \mathbb{H}^2$ ).

**Borcherds-Products:**

$f \in A_0^+(p)$	$\mapsto \Psi$	divisor	
$f_1 = q^{-1} + O(1)$	$\mapsto \Psi_1$	$\Gamma \cdot \mathbb{D}$	$f _{\mathbb{D}} \equiv 0 \Rightarrow \Psi_1 f$
$f_p = \frac{1}{2}q^{-p} + O(1)$	$\mapsto \Psi_p$	$\Gamma \cdot \mathbb{D}_{\varepsilon_0}$	$f _{\mathbb{D}_{\varepsilon_0}} \equiv 0 \Rightarrow \Psi_p f$
$f_j = \frac{1}{s(n)}q^{-j} + O(1)$	$\mapsto \Psi_j$		

## 4. Generators and Relations for $\widehat{M}^5$ , $\widehat{M}^{13}$ and $\widehat{M}^{17}$

**Remark.**  $\widehat{M}^p$  is the ring of all HMFs with symmetric multiplier systems  $\mu : \Gamma \rightarrow \mathbb{C}^*$ :

$$\widehat{M}^p = \sum_k \sum_{\mu: \Gamma \rightarrow \mathbb{C}^*, \mu(\overline{M}) = \mu(M)} M_k^p(\mu)$$

**Theorem (Gundlach, Resnikoff, ...).**

$M^5 = \mathbb{C}[X_2, X_5, X_6, X_{15}]/R_{30}$ , where

$X_2 = E_2, \quad X_5 = \Psi_1, \quad X_6 = E_6, \quad X_{15} = \Psi_5$  and

$$R_{30} : \quad \Psi_5^2 - \left( \frac{67}{25}E_6 - \frac{42}{25}E_2^3 \right) \left( \frac{67}{43200} (E_2^3 - E_6) \right)^4 = \Psi_1^2(\dots)$$

$f$	$E_2$	$\Psi_1$	$e_6 := \frac{67}{25}E_6 - \frac{42}{25}E_2^3$	$\Psi_5$
$f \circ \varphi$	$g_2$	0	$g_3^2$	$\Delta^2 g_3$
weight of $f$	2	5	6	15

*Proof by induction (weight k):*

$M_0^5(1) = \mathbb{C}$ ,  $M_{-k}^5(1) = \{0\}$  for all  $k > 0$ . In case  $\mathbb{Q}(\sqrt{5})$ :  $\mu \equiv 1$ .

Let  $f \in M_k^5(1)$ . Write  $D := \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix}$ .

$k$  odd:  $\tau = D(\tau_2, \tau_1)$  on  $\mathbb{D}_{\varepsilon_0}$ ,  $N(\varepsilon_0^{-1}) = -1$

$$\Rightarrow f^+|_{\mathbb{D}_{\varepsilon_0}} \stackrel{\mu \equiv 1}{=} -f^+|_{\mathbb{D}_{\varepsilon_0}} \quad \Rightarrow \quad \Psi_5|f^+ \text{ and } \Psi_1|f^-$$

$k$  even:  $f \circ \varphi$  is an elliptic mod. form of weight  $2k$  for  $SL(2, \mathbb{Z})$

$$\Rightarrow \text{there is a polynomial } q: f \circ \varphi - q(g_2, g_3^2) \equiv 0.$$

$$\Rightarrow f - q(E_2, e_6)|_{\mathbb{D}} = 0 \quad \Rightarrow \quad \Psi_1|f - q(E_2, e_6).$$

The relation immediately follows from the elliptic case.  $\square$

**Theorem.**  $\hat{M}^{13} = \mathbb{C}[\Psi_1, \frac{\Psi_4}{\Psi_1}, E_2, \Psi_{13}]/R_{14}$ , where

$$X_1 = \Psi_1, \quad X_2 = \frac{\Psi_4}{\Psi_1}, \quad Y_2 = E_2, \quad X_7 = \Psi_{13} \text{ and}$$

$$\begin{aligned} R_{14} : \Psi_{13}^2 - \left(\frac{\Psi_4}{2\Psi_1}\right)^4 \left(E_2^3 - 2^6 3^3 \left(\frac{\Psi_4}{2\Psi_1}\right)^3\right) &= -108\Psi_1^{12}\Psi_2 - \frac{27}{16}\Psi_1^{10}E_2^2 \\ &+ \frac{495}{8}\Psi_1^8\Psi_2^2E_2 - \frac{1459}{16}\Psi_1^6\Psi_2^4 + \frac{41}{8}\Psi_1^6\Psi_2E_2 - 512\Psi_1^6 \left(\frac{\Psi_4}{2\Psi_1}\right)^4 \\ &+ \frac{1}{16}\Psi_1^4E_2^5 - \frac{97}{4}\Psi_1^4\Psi_2^3E_2^2 - \frac{1}{8}\Psi_1^2\Psi_2^2E_2^4 - 144\Psi_1^2 \left(\frac{\Psi_4}{2\Psi_1}\right)^5 E_2 + \frac{189}{8}\Psi_1^2\Psi_2^5E_2 . \end{aligned}$$

$f$	$\Psi_1$	$\frac{\Psi_4}{2\Psi_1}$	$E_2$	$\Psi_{13}$
$f \circ \varphi$	0	$\eta^8$	$g_2$	$\eta^{16}g_3$
weight of $f$	1	2	2	7
multiplier $\mu$	$\mu_{13}$	$\mu_{13}$	1	$\mu_{13}^2$

*Proof:*

$$\mu_{13}(J) = 1, \quad \mu_{13}(T) = -\frac{1}{2} + \frac{1}{2}\sqrt{3}, \quad \mu_{13}(T_w) = -\frac{1}{2} - \frac{1}{2}\sqrt{3}.$$

$\mu$  is already determined by  $f \circ \varphi$ .

If  $k$  is odd, proceed as for  $\mathbb{Q}(\sqrt{5})$  ( $f^+(\tau) \in -e^{2\pi i \mathbb{Z}/3} f^+(\tau)$ ).

If  $k$  is even, there is a polynomial  $q$  satisfying  $f \circ \varphi = q(\eta^8, g_2)$ .

$\Rightarrow$

$$\Psi_1 | (f - q(\frac{\Psi_4}{2\Psi_1}, E_2))$$

Let  $q \neq 0$  be a polynomial,  $q\left(\Psi_1, \frac{\Psi_4}{2\Psi_1}, E_2, \Psi_{13}\right) \equiv 0$ .

Define  $r := q(0, \cdot, \cdot, \cdot)$ .

If  $r \equiv 0$  investigate  $q(X_1, X_2, X_3, X_4)/X_1$  instead of  $q$ .

Else  $r\left(\eta^8, g_2, \eta^{16}g_3\right) \equiv 0$ , so all elliptic relations are induced by

$$(\eta^{16}g_3)^2 - (\eta^8)^4g_3^2 = 0$$

A comparison of fourier expansions concludes the argument.

□

**Theorem.**  $\hat{M}^{17} = \mathbb{C}[X_{\frac{1}{2}}, X_2, X_{\frac{3}{2}}, Y_{\frac{3}{2}}, X_{\frac{9}{3}}]/\langle R_3, R_9 \rangle$ , where

$$X_{\frac{1}{2}} = \Psi_1, \quad X_2 = E_2, \quad X_{\frac{3}{2}} = -\Psi_2, \quad Y_{\frac{3}{2}} = \eta_2, \quad X_{\frac{9}{3}} = \Psi_{17}$$

$$R_3 : \quad \eta_2^2 - 64\Psi_2^2 = 16\Psi_1^2 E_2 \quad \text{and}$$

$$\begin{aligned} R_9 : \quad & \Psi_{17}^2 - \Psi_2^2 E_2^3 + 216\Psi_2^5 \eta_2 = -256\Psi_1^{18} \\ & - 176\Psi_1^{12} \Psi_2 \eta_2 - \frac{2671}{4096} \Psi_1^6 \eta_2^4 + \frac{103}{8} \Psi_1^4 E_2^2 \Psi_2 \eta_2 \\ & - \frac{87}{16} \Psi_1^{10} E_2^2 - \frac{99}{128} \Psi_1^2 E_2 \Psi_2 \eta_2 + \frac{1387}{128} \Psi_1^8 E_2 \eta_2^2. \end{aligned}$$

$f$	$\Psi_1$	$E_2$	$-\Psi_2$	$\eta_2/8$	$\Psi_{17}$
$f \circ \varphi$	0	$g_2$	$\eta^6$	$\eta^6$	$\eta^6 g_3$
weight of $f$	$\frac{1}{2}$	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$
multiplier $\mu$	$\mu_{17}$	1	$\mu_{17}^5$	$\mu_{17}$	$\mu_{17}^7$

**Problem 1:**  $\mu_{17}$  has order 8 and holds  $\mu_{17}^4|_{\mathrm{SL}(2, \mathbb{Z})} = 1$ , so we need two lifts of  $\eta^6$ . If  $\mu \not\equiv 1$ , there is  $\nu \in \mathfrak{o}$  s.t.  $\mu(T_\nu) = \mu\left(\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}\right) \neq 1$ . Then  $f(T_\nu\tau) = \mu(T_\nu)f(\tau) \neq f(\tau)$  holds as  $\tau$  and  $T_\nu\tau$  tend to  $\infty$ . Thus  $f$  and  $f \circ \varphi$  are cusp forms and  $\eta^6|f \circ \varphi$ .

**Problem 2:** For  $\hat{M}^5$  and  $\hat{M}^{13}$ , every symmetric HMF  $f^+$  of odd weight  $k$  is divisible by  $\Psi_p$ . Here:

$$\begin{aligned} f^+(\tau) &= f^+(D\bar{\tau}) = \mu(D) N(\varepsilon_0^{-1})^k f^+(\bar{\tau}) \\ &= e^{k\pi i} \mu(D) f^+(\tau), \quad (\bar{\tau} := (\tau_2, \tau_1)) \end{aligned}$$

for  $\tau \in \mathbb{D}_{\varepsilon_0}$ . Because of  $\mu_{17}^4(D) = 1$ , this property only depends on  $\mu|_{\mathrm{SL}(2, \mathbb{Z})}$ . We obtain:

Every symmetric HMF  $f$  for which  $f \circ \varphi$  is a multiple of  $g_3$  but not of  $g_3^2$ , is divisible by  $\Psi_{17}$ .

**Corollary.** We write  $X_4 = E_2^H$ ,  $X_6 = \Psi_1^3$ ,  $X_8 = \Psi_1^2 \frac{\Psi_4}{2\Psi_1}$ ,  $X_{10} = \Psi_1 \left( \frac{\Psi_4}{2\Psi_1} \right)^2$ ,  $X_{12} = \left( \frac{\Psi_4}{2\Psi_1} \right)^3$ ,  $X_{16} = \Psi_1 \Psi_{13}$  and  $X_{18} = \frac{\Psi_4}{2\Psi_1} \Psi_{13}$  and define the relations

$$R_{18} : X_{10}X_8 = X_{12}X_6 ,$$

$$R_{20} : X_{10}^2 = X_{12}X_8 ,$$

$$R_{24} : X_{16}X_8 = X_6X_{18} ,$$

$$\begin{aligned} R_{36} : X_{18}^2 &= X_{12}^2 X_4^3 - 1728 X_{12}^3 - 108 X_3 X_6^4 + \frac{1}{16} X_8^2 X_4^5 + \frac{41}{8} X_{12} X_6^2 X_4^3 - \frac{1459}{16} X_{12}^2 X_6^2 \\ &\quad + \frac{495}{8} X_{10}^2 X_6^2 X_4 - \frac{97}{4} X_8 X_4^2 X_{10}^2 - \frac{27}{16} X_{10} X_6^3 X_4^2 - \frac{1}{8} X_{10}^2 X_4^4 + \frac{189}{8} X_4 X_{12}^2 X_8 . \end{aligned}$$

Then

$$M^{13} = \mathbb{C}[X_4, X_6, X_8, X_{10}, X_{12}, X_{16}, X_{18}] / (R_{18}, R_{20}, R_{24}, R_{36}) .$$

	$X_4$	$X_{18}$	$X_{12}$	$X_{10}$	$X_{16}$	$X_8$	$X_6$
$f$	$E_2^H$	$\frac{\Psi_4}{2\Psi_1} \Psi_{13}$	$\left( \frac{\Psi_4}{2\Psi_1} \right)^3$	$\Psi_1 \left( \frac{\Psi_4}{2\Psi_1} \right)^2$	$\Psi_1 \Psi_{13}$	$\Psi_1^2 \frac{\Psi_4}{2\Psi_1}$	$\Psi_1^3$
$f \circ \delta$	$E_4$	$\Delta E_6$	$\Delta$	0	0	0	0
weight of $f$	4	18	12	10	16	8	6

**Corollary.** We write

$$\begin{aligned} X_2 &= E_2^H, \quad X_3 = -\Psi_2^3 \eta_2/8, \quad X_9 = \Psi_2^2 \Psi_{17} \eta_2/8, \quad X_5 = -\Psi_1 \Psi_2^3, \quad X_8 = \Psi_1 \Psi_2^2 \Psi_{17}, \\ X_4 &= -\Psi_1^2 \Psi_2 \eta_2/8, \quad X_7 = \Psi_1^2 \Psi_{17} \eta_2/8, \quad Y_3 = -\Psi_1^3 \Psi_2, \quad X_6 = \Psi_1^3 \Psi_{17}, \quad Y_5 = \Psi_1^7 \eta_2/8, \end{aligned}$$

and define the relations

$$\begin{aligned} R_9 : \quad X_4 X_5 &= X_3 X_6, & R_{10} : \quad Y_4 X_6 &= X_3^2 X_4, \\ R_{11} : \quad Y_5 X_6 &= X_3 X_4^2, & R_{12} : \quad X_4 X_8 &= X_5 X_7, \\ R'_{12} : \quad X_6 Y_6 &= X_5 X_7, & R_{13} : \quad X_6 X_7 &= X_9 X_4, \\ R_{14} : \quad X_5 X_9 &= X_6 X_8, \\ R_{18} : \quad X_9^2 &= X_3^2 (X_3 + X_2^3) - 256 X_4 Y_4^2 X_6 - 1408 X_3^2 X_4^3 - \frac{2671}{4} X_2 X_4^4 \\ &\quad - 2671 X_4^2 X_5^2 + \frac{2671}{4} X_2 X_3 X_4^2 X_5 - 103 X_2^2 X_4^2 X_6 - \frac{87}{16} X_2^2 X_4 Y_4 X_6 \\ &\quad + \frac{99}{128} X_2 X_4 X_6^2 + \frac{99}{512} X_2^2 X_4^2 X_6 + \frac{1387}{2} X_2 X_4^4. \end{aligned}$$

Then

$$M^{17}(1) = \mathbb{C}[X_2, X_3, X_4, Y_4, X_5, Y_5, X_6, Y_6, X_7, X_8, X_9]/(R_9, R_{10}, R_{11}, R_{12}, R'_{12}, R_{13}, R_{14}, R_{18})$$

	$X_2$	$X_6$	$X_9$	$X_5$	$X_8$
$f$	$E_2^H$	$-\Psi_2^3 \eta_2/8$	$\Psi_2^2 \Psi_{17} \eta_2/8$	$-\Psi_1 \Psi_2^3$	$\Psi_1 \Psi_2^2 \Psi_{17}$
$j = \text{zero order on } \mathbb{D}$	0	0	0	1	1
$f/\Psi_1^j \circ \delta$	$E_4$	$\Delta$	$\Delta E_6$	$\eta^{18}$	$\eta^{18} E_6$
weight of $f$	2	6	9	5	8

	$X_4$	$X_7$	$X_3$	$Y_6$	$Y_5$	$Y_4$
$f$	$-\Psi_1^2 \Psi_2 \eta_2/8$	$\Psi_1^2 \Psi_{17} \eta_2/8$	$-\Psi_1^3 \Psi_2$	$\Psi_1^3 \Psi_{17}$	$\Psi_1^7 \eta_2/8$	$\Psi_1^8$
$j$	2	2	3	3	7	8
$f/\Psi_1^j \circ \delta$	$\eta^{12}$	$\eta^{12} E_6$	$\eta^6$	$\eta^6 E_6$	$\eta^6$	1
weight of $f$	4	7	3	6	5	4