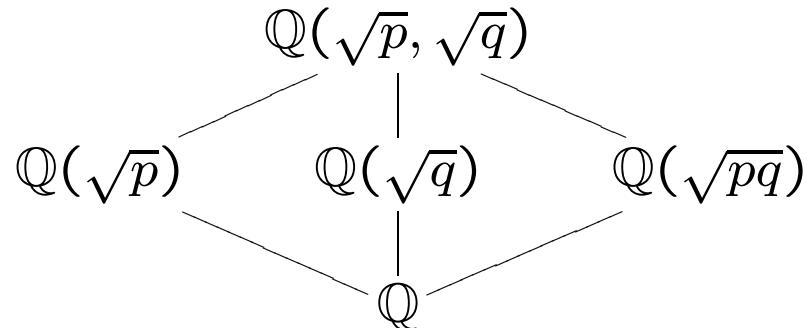




Hilbert modular forms for biquadratic number fields



in the special case
 $\mathbb{Q}(\sqrt{5}, \sqrt{13})$.

- 1.) Introduction and Notation
- 2.) Constructing Hilbert modular forms for biquadratic number fields
- 3.) Restriction of Hilbert modular forms

- Algebraic number field $\mathcal{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, $(\alpha_j \in \overline{\mathbb{Q}})$,
- Ring of integers $\mathfrak{o}_{\mathcal{K}}$ of \mathcal{K} ,
- Field automorphisms $\rho_1 : a \mapsto a^{(1)}, \dots, \rho_m : a \mapsto a^{(m)}$ with $\rho_1 = \text{id}$,
- Norm $N(\lambda) = \prod_{j=1}^m \lambda^{(j)} \in \mathbb{Q}$, $(\lambda \in \mathcal{K})$
- Trace $S(\lambda) = \sum_{j=1}^m \lambda^{(j)}$, $(\lambda \in \mathcal{K})$

- $\Gamma = \mathrm{SL}(2, \mathfrak{o}_{\mathcal{K}})$ operates on $\mathbb{H}^m = \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}^m$ by

$$\gamma\tau = \left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \frac{a^{(m)}\tau_m + b^{(m)}}{c^{(m)}\tau_m + d^{(m)}} \right),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau = (\tau_j)_{1 \leq j \leq m} \in \mathbb{H}^m$.

- General principle for Hilbert modular forms: The k -th field automorphism and the k -th upper half plane in \mathbb{H}^m belong together.
- This motivates $S(\lambda\tau) := \sum_{j=1}^m \lambda^{(j)}\tau_j$ and

$$\mathsf{N}(c\tau + d)^k := \prod_{j=1}^m (c^{(j)}\tau_j + d^{(j)})^k, \quad (z^k := \exp(k \ln z))$$

Definition (Hilbert Modular Form). A Hilbert modular form (HMF) f of weight $k \in \mathbb{Q}$ with multiplier system (m.s.) $\mu : \Gamma \rightarrow \mathbb{C}^*$ is a holomorphic function $\mathbb{H}^m \rightarrow \mathbb{C}$ satisfying

$$f(\gamma\tau) = \mu(\gamma) N(c\tau + d)^k f(\tau) \quad \forall \tau \in \mathbb{H}^m, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We denote the corresponding vector space by $M_k(\mathcal{K}, \mu)$.

Remark. We will only consider „symmetric“ multiplier systems μ , e.g. μ holding

$$\mu(\rho_j(M)) = \mu(M)$$

for all $M \in \Gamma$ and $1 \leq j \leq m$, where $\rho_j(M)$ is the matrix derived from M by elementwise application of the field automorphisms ρ_j . Denote the space of HMF with multiplier system μ by $M(\mathcal{K}, \mu)$ and write $M(\mathcal{K}) = \cup_{\mu} M(\mathcal{K}, \mu)$.

Definition (Eisenstein series). Let $k \in \mathbb{N}$ even and define the k^{th} Eisenstein series $E_k : \mathbb{H}^m \rightarrow \mathbb{C}$ by

$$E_k(\tau) := \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} N(c\tau + d)^{-k},$$

where Γ_∞ denotes the subgroup of Γ fixing $\infty = (\infty, \dots, \infty)$.

E_k is a symmetric modular form (invariant under permutations of $\tau \mapsto (\tau_{\pi(1)}, \dots, \tau_{\pi(m)})$), since Γ_∞ is invariant under the field automorphisms ρ_j .

Lemma (Siegel, 1969). Let $k \in \mathbb{N}$ even, then the k^{th} Eisenstein series $E_k : \mathbb{H}^m \rightarrow \mathbb{C}$ holds

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(\mathfrak{o}, k)(k+1)!} d^{\frac{1}{-k}} \sum_{\mathfrak{d}^{-1}|\nu>>0} \sigma_{k-1}(\nu) e^{2\pi i S(\nu\tau)}$$

where \mathfrak{d} is the fundamental ideal (in case of $\mathbb{Q}(\sqrt{5}, \sqrt{13})$ this is $(\sqrt{65})$), d is the discriminant,

$$\zeta(\mathfrak{o}, k) = \sum_{\text{principal ideals } (\mu)} N(\mu^{-k}) \text{ and}$$

$$\sigma_{k-1}(\nu) = \sum_{\substack{\text{principal ideals } \alpha, \\ \mathfrak{d}^{-1}|(\alpha)|\nu}} \text{sign}(N(\alpha^k)) N((\alpha)\mathfrak{d})^{k-1}$$

Trick: $z \mapsto E_k(z, \dots, z)$ is an elliptic modular form of weight $4k$, so we can calculate $\frac{(2\pi i)^k}{\zeta(\mathfrak{o}, k)(k+1)!} d^{\frac{1}{-k}}$ indirectly.

Test on integers: Given $\alpha = \alpha_1 + \alpha_2\sqrt{5} + \alpha_3\sqrt{13} + \alpha_4\sqrt{65} \in \mathcal{K}_{5,13} := \mathbb{Q}(\sqrt{5}, \sqrt{13})$, the multiplication $\mathcal{K}_{5,13} \rightarrow \mathcal{K}_{5,13}$, $x \mapsto \alpha \cdot x$ is a \mathbb{Q} -linear map with characteristic polynomial $\chi_\alpha(x) = x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$, where

$$c_1 = 169\alpha_3^4 + 4225\alpha_4^4 - 130\alpha_4^2\alpha_1^2 - 10\alpha_1^2\alpha_2^2 - 650\alpha_4^2\alpha_2^2 - 26\alpha_3^2\alpha_1^2 - 130\alpha_3^2\alpha_2^2 - 1690\alpha_3^2\alpha_4^2 + 520\alpha_1\alpha_4\alpha_3\alpha_2 + 25\alpha_2^4 + \alpha_1^4$$

$$c_2 = 260\alpha_4^2\alpha_1 + 52\alpha_3^2\alpha_1 + 20\alpha_1\alpha_2^2 - 4\alpha_1^3 - 520\alpha_4\alpha_3\alpha_2$$

$$c_3 = -26\alpha_3^2 - 130\alpha_4^2 - 10\alpha_2^2 + 6\alpha_1^2,$$

$$c_4 = -4\alpha_1.$$

We have $\alpha \in \mathfrak{o} \Leftrightarrow \chi_\alpha(x) \in \mathbb{Z}[x]$. Therefore

$$\mathfrak{o}_{\mathcal{K}_{5,13}} = \mathbb{Z} + \frac{1 + \sqrt{5}}{2}\mathbb{Z} + \frac{1 + \sqrt{13}}{2}\mathbb{Z} + \frac{1 + \sqrt{5} + \sqrt{13} + \sqrt{65}}{4}\mathbb{Z}$$

Definition (Θ -series). Define the Siegel halfspace

$$\mathcal{H}_m := \{Z := X + iY \in M_m(\mathbb{C}); Z = Z^{\text{tr}}, Y > 0\}.$$

For $k = (k', k'') \in \{0, 1\}^{m+m}$ with $(k')^{\text{tr}} k'' \in 2\mathbb{Z}$ and all $Z \in \mathcal{H}_m$ define

$$\Theta_k(Z) := \sum_{g \in \mathbb{Z}^m} e^{\pi i ((g+k'/2)^{\text{tr}} Z (g+k'/2) + (g+k'/2)^{\text{tr}} k'')}.$$

- Quadratic case: 10 non-identically vanishing theta-series.
- Biquadratic case: 136 non-vanishing theta-series.

Theorem and Definition (Hammond). Write $p = u^2 + v^2$ (for $\equiv 1 \pmod{4}$) where v is even, $u, v \in \mathbb{Z}$ and define $\omega := \frac{1}{2}(u + \sqrt{p})$. Then

$$\epsilon : \mathbb{H}^2 \rightarrow \mathcal{H}_2, \quad \tau \mapsto A_p \cdot \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \cdot A_p,$$

with

$$A_p := \begin{pmatrix} \sqrt{\frac{\omega\sqrt{p}}{p}} & \sqrt{\rho_2\left(\frac{\omega\sqrt{p}}{p}\right)} \\ \sqrt{\rho_2\left(\frac{\omega\sqrt{p}}{p}\right)} & -\sqrt{\frac{\omega\sqrt{p}}{p}} \end{pmatrix}$$

induces a map between modular forms. ($A_p^{-1} = A_p^{\text{tr}} = A_p$)

Remark. One easily checks that the images of theta products are symmetric.

Lemma. *In the case of $\mathcal{K}_{5,13}$ the map*

$$\epsilon : \mathbb{H}^4 \rightarrow \mathcal{H}_4, \quad \tau \mapsto A_5 \otimes A_{13} \cdot \begin{pmatrix} \tau_1 & & & \\ & \tau_2 & & \\ & & \tau_3 & \\ & & & \tau_4 \end{pmatrix} \cdot A_5 \otimes A_{13},$$

induces a map between modular forms.

Notation Write

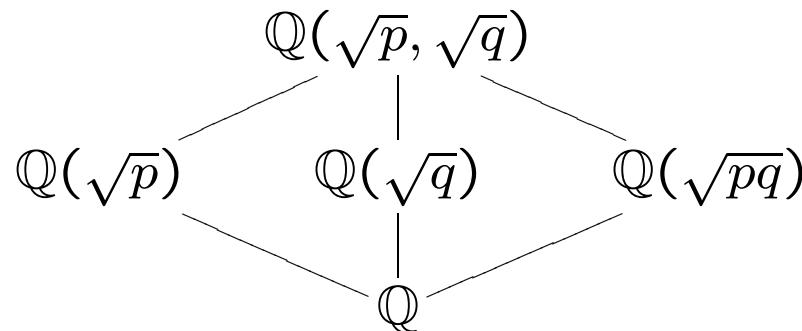
$$\begin{aligned} g_1 &= \exp(\pi i(\tau_1 + \tau_2 + \tau_3 + \tau_4)/2), \\ g_2 &= \exp(\pi i(\tau_1 - \tau_2 + \tau_3 - \tau_4)/(2\sqrt{5})), \\ g_3 &= \exp(\pi i(\tau_1 + \tau_2 - \tau_3 - \tau_4)/(2\sqrt{13})) \text{ and} \\ g_4 &= \exp(\pi i(\tau_1 - \tau_2 - \tau_3 + \tau_4)/(2\sqrt{65})). \end{aligned}$$

Example. *Theta series:*

$$\begin{aligned}
 (\Theta_{(0,0,0,0),(0,0,0,0)} \circ \epsilon)(\tau) = & 1 + 2 \left(\frac{g_3^{12}}{g_2^4 g_4^{12}} + g_2^4 g_3^{12} g_4^{12} + \frac{g_2^4}{g_3^{12} g_4^{12}} + \frac{g_4^{12}}{g_2^4 g_3^{12}} \right) g_1^4 \\
 & + (2g_4^{56} + 2g_4^{-56} + 2g_2^8 g_3^{-16} g_4^{-16} + 2g_2^{-16} g_3^{-24} g_4^{48} + 2g_2^8 g_3^{16} g_4^{16} \\
 & + 2g_4^{-8} + 2g_2^{-8} g_3^{-16} g_4^{16} + 2g_2^{16} g_3^{-24} g_4^{-48} + 2g_2^{-16} g_3^{24} g_4^{-48} + 2g_4^8 \\
 & + 2g_2^{-8} g_3^{16} g_4^{-16} + 2g_2^{-16} g_3^{24} g_4^{48}) g_1^8 + O(g_1^{12})
 \end{aligned}$$

Example. *Eisenstein series:*

$$E_2 = 1 + 120g_1 \left(\frac{g_2}{g_3 g_4} + g_2 g_3 g_4 + \frac{g_3}{g_2 g_4} + \frac{g_4}{g_2 g_3} \right) + O(g_1^2)$$



For p and q in $\mathbb{Z}_{\geq 2}$, $p \neq q$, write $\mathcal{K}_{p,q} = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. Enumerate the field automorphisms ρ_j in the following way:

	$\text{sign}(\rho_1)$	$\text{sign}(\rho_2)$	$\text{sign}(\rho_3)$	$\text{sign}(\rho_4)$
\sqrt{p}	+	-	+	-
\sqrt{q}	+	+	-	-
\sqrt{pq}	+	-	-	+

Lemma. Let $f : \mathbb{H}^4 \rightarrow \mathbb{C}$ be a HMF of weight $k \in \mathbb{Q}$ with multiplier system μ for $\mathcal{K}_{p,q}$. Then the functions

$$f_1 : \mathbb{H}^2 \rightarrow \mathbb{C}, f_1(\tau) = f(\tau_1, \tau_2, \tau_1, \tau_2)$$

$$f_2 : \mathbb{H}^2 \rightarrow \mathbb{C}, f_2(\tau) = f(\tau_1, \tau_1, \tau_2, \tau_2)$$

$$f_3 : \mathbb{H}^2 \rightarrow \mathbb{C}, f_3(\tau) = f(\tau_1, \tau_2, \tau_2, \tau_1)$$

are Hilbert modular forms of weight $2k$, more precisely

f_1 is a HMF for $\mathbb{Q}(\sqrt{p})$ with multiplier system $\mu|_{\mathrm{SL}(2, \mathfrak{o}_{\sqrt{p}})}$,

f_2 is a HMF for $\mathbb{Q}(\sqrt{q})$ with multiplier system $\mu|_{\mathrm{SL}(2, \mathfrak{o}_{\sqrt{q}})}$ and

f_3 is a HMF for $\mathbb{Q}(\sqrt{pq})$ with multiplier system $\mu|_{\mathrm{SL}(2, \mathfrak{o}_{\sqrt{pq}})}$.

Lemma. If $f \in M(\mathcal{K}_{p,q})$ is a HMF for $\mathcal{K}_{p,q}$ with Fourier expansion

$$f(\tau) = \sum_{\alpha} c(\alpha) e^{S(\alpha)}, \quad (\mu \equiv 1 \Rightarrow \alpha \in \sqrt{pq}^{-1} \mathfrak{o}_{\mathcal{K}_{p,q}})$$

then the Fourier expansions of f_1 , f_2 and f_3 are given by

$$f_1(\tau) = \sum_{\beta} \left(\sum_{\substack{\alpha \\ \alpha_1 + \alpha_3 = \beta_1 \\ \alpha_2 + \alpha_4 = \beta_2}} c(\alpha) \right) e^{S(\beta\tau)},$$

$$f_2(\tau) = \sum_{\beta} \left(\sum_{\substack{\alpha \\ \alpha_1 + \alpha_2 = \beta_1 \\ \alpha_3 + \alpha_4 = \beta_2}} c(\alpha) \right) e^{S(\beta\tau)},$$

$$f_3(\tau) = \sum_{\beta} \left(\sum_{\substack{\alpha \\ \alpha_1 + \alpha_4 = \beta_1 \\ \alpha_2 + \alpha_3 = \beta_2}} c(\alpha) \right) e^{S(\beta\tau)}.$$

Additionally $f_1(\infty, \infty) = f_2(\infty, \infty) = f_3(\infty, \infty) = f(\infty, \infty, \infty, \infty)$.

Remark. $M(\mathcal{K}_p)$ is the ring of all HMFs with symmetric multiplier systems $\mu : \Gamma \rightarrow \mathbb{C}^*$:

$$\widehat{M}^p = \sum_k \sum_{\mu : \Gamma \rightarrow \mathbb{C}^*, \mu(M^{(2)}) = \mu(M)} M_k^p(\mu)$$

Theorem (Gundlach, Resnikoff, ...).

$M(\mathcal{K}_5) = M(\mathcal{K}_5, 1) = \mathbb{C}[X_2, X_5, X_6, X_{15}]/R_{30}$, where

$X_2 = E_2, \quad X_5 = \Psi_1, \quad X_6 = E_6, \quad X_{15} = \Psi_5$ and

$$R_{30} : \quad \Psi_5^2 - \left(\frac{67}{25}E_6 - \frac{42}{25}E_2^3 \right) \left(\frac{67}{43200} (E_2^3 - E_6) \right)^4 = \Psi_1^2(\dots)$$

f	E_2	Ψ_1	$e_6 := \frac{67}{25}E_6 - \frac{42}{25}E_2^3$	Ψ_5
$f \circ \varphi$	g_2	0	g_3^2	$\Delta^2 g_3$
weight of f	2	5	6	15

(The Ψ_j are Borcherds products as well as theta series.

Theorem (M.). $M(\mathcal{K}_{13}) = \mathbb{C}[\Psi_1, \frac{\Psi_4}{\Psi_1}, E_2, \Psi_{13}]/R_{14}$, where

$$X_1 = \Psi_1, \quad X_2 = \frac{\Psi_4}{\Psi_1}, \quad Y_2 = E_2, \quad X_7 = \Psi_{13} \text{ and}$$

$$\begin{aligned} R_{14} : \Psi_{13}^2 - \left(\frac{\Psi_4}{2\Psi_1} \right)^4 \left(E_2^3 - 2^6 3^3 \left(\frac{\Psi_4}{2\Psi_1} \right)^3 \right) &= -108\Psi_1^{12}\Psi_2 - \frac{27}{16}\Psi_1^{10}E_2^2 \\ &+ \frac{495}{8}\Psi_1^8\Psi_2^2E_2 - \frac{1459}{16}\Psi_1^6\Psi_2^4 + \frac{41}{8}\Psi_1^6\Psi_2E_2 - 512\Psi_1^6 \left(\frac{\Psi_4}{2\Psi_1} \right)^4 \\ &+ \frac{1}{16}\Psi_1^4E_2^5 - \frac{97}{4}\Psi_1^4\Psi_2^3E_2^2 - \frac{1}{8}\Psi_1^2\Psi_2^2E_2^4 - 144\Psi_1^2 \left(\frac{\Psi_4}{2\Psi_1} \right)^5 E_2 + \frac{189}{8}\Psi_1^2\Psi_2^5E_2 . \end{aligned}$$

f	Ψ_1	$\frac{\Psi_4}{2\Psi_1}$	E_2	Ψ_{13}
$f \circ \varphi$	0	η^8	g_2	$\eta^{16}g_3$
weight of f	1	2	2	7
multiplier μ	μ_{13}	μ_{13}	1	μ_{13}^2

Corollary. We write $X_4 = E_2^H$, $X_6 = \Psi_1^3$, $X_8 = \Psi_1^2 \frac{\Psi_4}{2\Psi_1}$, $X_{10} = \Psi_1 \left(\frac{\Psi_4}{2\Psi_1} \right)^2$, $X_{12} = \left(\frac{\Psi_4}{2\Psi_1} \right)^3$, $X_{16} = \Psi_1 \Psi_{13}$ and $X_{18} = \frac{\Psi_4}{2\Psi_1} \Psi_{13}$ and define the relations

$$R_{18} : X_{10}X_8 = X_{12}X_6 ,$$

$$R_{20} : X_{10}^2 = X_{12}X_8 ,$$

$$R_{24} : X_{16}X_8 = X_6X_{18} ,$$

$$\begin{aligned} R_{36} : X_{18}^2 &= X_{12}^2 X_4^3 - 1728 X_{12}^3 - 108 X_3 X_6^4 + \frac{1}{16} X_8^2 X_4^5 + \frac{41}{8} X_{12} X_6^2 X_4^3 - \frac{1459}{16} X_{12}^2 X_6^2 \\ &\quad + \frac{495}{8} X_{10}^2 X_6^2 X_4 - \frac{97}{4} X_8 X_4^2 X_{10}^2 - \frac{27}{16} X_{10} X_6^3 X_4^2 - \frac{1}{8} X_{10}^2 X_4^4 + \frac{189}{8} X_4 X_{12}^2 X_8 . \end{aligned}$$

Then

$$M(\mathcal{K}_{13}, 1) = \mathbb{C}[X_4, X_6, X_8, X_{10}, X_{12}, X_{16}, X_{18}] / (R_{18}, R_{20}, R_{24}, R_{36}) .$$

	X_4	X_{18}	X_{12}	X_{10}	X_{16}	X_8	X_6
f	E_2^H	$\frac{\Psi_4}{2\Psi_1} \Psi_{13}$	$\left(\frac{\Psi_4}{2\Psi_1} \right)^3$	$\Psi_1 \left(\frac{\Psi_4}{2\Psi_1} \right)^2$	$\Psi_1 \Psi_{13}$	$\Psi_1^2 \frac{\Psi_4}{2\Psi_1}$	Ψ_1^3
$f \circ \delta$	E_4	ΔE_6	Δ	0	0	0	0
weight of f	4	18	12	10	16	8	6

19 symmetric HMFs of even weight for \mathcal{K}_{65} S. Mayer

Theorem (Hermann, 1983). *The Ring of HMFs of even weight for \mathcal{K}_{65} is given by*

$$\mathbb{C}[A_2, \hat{A}_2, B_2, \hat{B}_2, A_4, \hat{A}_4, B_4, A_6, \hat{A}_6] / (R_1, \dots, R_7)$$

- Eisenstein series and theta series can be calculated.
- Restrictions to 3 different spaces of HMFs for quadratic number fields are known, so it remains to classify the space of HMFs vanishing on these three sets.
- Perhaps this suffices to calculate the ring of symmetric HMFs for $\mathcal{K}_{5,13}$.

Thank you very much for your Attention