Hilbert Modular Forms

for the fields
$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$

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The image on the cover shows a memorial plaque inside the RWTH Aachen installed to commemorate the life of Ludwig Otto Blumenthal.

To my family

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Introduction

Prolog about Blumenthal's life

It is now over a hundred years that David Hilbert gave his sketches on a new type of modular functions to his doctoral student Ludwig Otto Blumenthal, who made them the foundation of his Habilitation *"Über Modulfunktionen von mehreren Veränderlichen"* (on modular functions of several variables). Blumenthal developed the theory of nowadays Hilbert Blumenthal modular forms in three important directions: he investigated the existence of a fundamental domain, introduced Poincaré series and proved two theorems of Weierstraß about the maximal number of algebraically independent modular functions (cf. [Bl03]). Later on he published a treatment of theta functions ([Bl04b]) built upon the more detailed part of Hilbert's notes.

It took some time before further results were obtained, since on the one hand algebraic geometry and the theory of complex functions had to evolve further (cf. [Ge88, p. 4]), on the other hand politics was directing almost all scientific efforts towards military purpose. The first world war was forthcoming and Blumenthal, who was by the time professor at the Aachen University of Technology (RWTH), became the head of some military weather stations (*"Feldwetterwarte"*) and in 1918 worked in the construction of aircrafts, from which arose his paper [Bl18] in 1918 (cf. [BV06, p. 7]). Returning to Aachen he continued mathematical work as well as he started occasionally to work on some historical topics like, for example, his biography of Hilbert [Bl22] (cf. [Be58, p. 390] and [BV06, p. 25 et seqq.]).

Blumenthal did not only publish in several mathematical fields, he also was managing editor of the *"Mathematische Annalen"* from 1906 to 1938, appointed editor of the *"Jahresberichte der Deutschen Mathematiker-Vereinigung"* (DMV) from 1924 to 1933 and he wrote English and French abstracts for the *"Zeitschrift für Angewandte Mathematik und Mechanik"* (ZAMM) from 1933 to 1938 (cf. [BV06, p.14 et seqq.]). Both the resignment from his work at the DMV in 1933 and the end of his work for the *"Mathematische Annalen"* and the ZAMM were neither accidently nor voluntary. He had to leave because of his Jewish ancestors, in 1938 the state banned him from his profession.

Blumenthal was denunciated by students of being a communist and was arrested on April, 27th in 1933, an error which was corrected 2 weeks later. But he was nevertheless suspended from his lectures and was removed from office on September, the 22nd. The formal reason was his

Figure 1: Ludwig Otto Blumenthal (1876–1944)

membership in the *"Deutsche Liga für Menschenrechte"*, the second reason most probably his classification as *"100% Jude"* (100% Jew) by German administration. So he was a victim of the antisemitism of the Nazi movement, even if he was a Lutheran, converted at the age of 18 (cf. [Fe03, p. 4]).

Blumenthal's son was at the time a student in Aachen and could not possibly continue his studies at the RWTH with everyone knowing his father had been removed from office. He emigrated to Great Britain and could continue his studies there. Since his sister, Blumenthals's daughter, studied not in Aachen but in Cologne, she could finish her Ph.D. before she, too, emigrated to Great Britain. Their parents also tried to leave Germany but in vain. The many applications Otto Blumenthal wrote for jobs abroad were all rejected (cf. [Fe03, p. 6 et seqq.]) and the support he got from individual people could not help him. Amongst others there were Paul Roentgen, Rector of the RWTH and Felix Rötscher, Pro-Rector, who attempted to keep the member of their faculty at his position at the RWTH (cf. [BV06, p. 9]), J. Hadamard, C. Caratheodory and T. Kármán, who tried in vain to find university positions for Blumenthal outside Germany so as to enable Blumenthal to emigrate (cf. [BV06, p. 11]), his mentor David Hilbert, at this time too old to help his former student, then Hecke, Behnke and van der Waerden, who forced Springer, the publisher of the "mathematische Annalen", not to release Blumenthal unless they should stop publishing the journal (cf. [BV06, p.15]). But the Nazi movement and state grew more and more insidious and powerful so the little help Blumenthal got could not prevent his and his wife's suffering and their later death.

In the beginning, the German government distinguished between those "Jews" who had fought in the first world war, like for example Blumenthal, and those who had not, but nevertheless Otto Blumenthal's situation got constantly worse. He started looking for jobs outside Germany, but at this time too many mostly young scientists emigrated. Only after a long search, in 1939 Blumenthal, who then was 62 years old, got a work permit in Delft, Netherlands. Hence he could emigrate there, having to leave all of his wealth but his furniture and books behind. He knew that he would have to live on welfare, since the work permit did not include an employment nor much hope for it. Only eleven months later he was back under the observation of the German administration, since German troops invaded the Netherlands and his refuge became a prison. (cf. [Fe03, p. 7 et seqq.]).

Since 1933 Blumenthal's life consisted of continued and growing discrimination by the state as well as by some students and colleagues (cf. [Fe03, p. 7]). In 1942, Blumenthal and his wife Mali had to leave on train for the concentration camp Westerbork and only could return due to the intervention of a Dutch reverend (cf. [Fe03, p. 16], [Th06, p. 89 et seqq.]). Afterwards they were forced to move several times from one lodging to another and were deported to the concentration camp Vught in 1943 and from there to Westerbork. Mali died there shortly after an inhuman treatment, which afflicted her until her death. Luckily, her husband knew not of it and assumed that she remembered her own or her children's youth, when she repeated "Nein, nein" (no, no) right before her death (cf. [Fe03, p. 16 et seqq.], [Th06, p.90, 91]).

Blumenthal new that his sister had been deported to the concentration camp Theresienstadt (Terezin) in 1942, so he tried to get transfered there, too. On January 20^{th}, 1944, he arrived there, where he was shocked to hear that his sister had died in July the year ago. His spirit rose a little bit after he met a Czech who still knew him from a talk in Prague, when times had been better. The Czech belonged to the “independent” administration of the concentration camp and took care of Blumenthal as good as he could. So Blumenthal became one of the protected persons in Theresienstadt. (cf. [Fe03, p. 18, 19])

But protected in Theresienstadt only meant that he got a quarter free of rats and bed bugs, got some more food than before and could be prevented from being send to Auschwitz. The Czech even managed to get him the permission to give lectures, fooling the SS into the belief that this was important for the water supply of the city. In Theresienstadt Blumenthal learned his 10^{th} language, after German, English, French, Russian, Italian, Bulgarian, Dutch, Latin and Greek now Czech, even if it was now much harder for him to learn a new language than it used to be (cf. [Fe03, p. 8, 19]). Blumenthal soon got ill, he had to stop all further activities. He survived the long and severe illness but not for long (cf. [Fe03, p. 18-19]). On November the 12^{th} he died after three days of unconsciousness. An autopsy revealed that he had old-age tuberculosis and cerebrospinal fluid (cf. [BV06, p.14]). *“But the deaths of perhaps 85% of the 870 „privileged“ inmates within two or three years makes clear that life at Terezin was very harsh, presumably in terms of nutrition, hygiene, clothing and warmth”* ([BV06, p.14]).

Blumenthal’s contribution to Hilbert (Blumenthal) modular forms

An accurate description of Blumenthal’s work is given in van der Geer’s book ([Ge88, p. 4]): *“[...] Blumenthal did the first pioneering work in a program outlined by Hilbert with the aim of creating a theory of modular functions of several variables that should be just as important in number theory and geometry as the theory of modular functions of one variable was at the beginning of this [20th] century. Since no general theory of complex spaces was available this was by no means an easy task. Blumenthal had at his disposal a manuscript by Hilbert from 1893/94 on the action of the modular group $\Gamma_{\mathcal{K}}$ of a totally real field $\mathcal{K}$ of degree n over $\mathbb{Q}$ on the product $\mathbb{H}^n$ of n upper half planes. According to Blumenthal it gave a sketchy description of general properties such as properly discontinuous action and fundamental domain but it contained precise information on the construction of modular functions by means of theta functions. Blumenthal gave a detailed account of the function theory involved but his construction of a fundamental domain had a flaw: he obtained a fundamental domain with only one cusp as in the case of the classical modular group. This mistake was corrected many years later by Maaß who showed that the number of cusps equals the class number h of $\mathcal{K}$.”*

How come this flaw? Both Hilbert and Blumenthal seemingly took for granted the existence of just one cusp as in the elliptic case. They overlooked that Blumenthal used the wrong group in his proof of the shape of the fundamental domain (in [Bl03] and [Bl04a]).

His work consists of the following three parts: First he investigates the fundamental domain of $\mathbb{H}^n/\,\mathrm{GL}(2,\mathfrak{o})$ for totally real number fi elds $\mathcal{K}$ of degree n with ring $\mathfrak{o}$ of integers, where $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\,(z) > 0\}$ is the upper half plane. Therefore he proves the discontinuous operation of the group $\mathrm{GL}(2,\mathfrak{o})$ on $\mathbb{H}^n$ and investigates the fixed points of the elements of $\mathrm{GL}(2,\mathfrak{o})$ on $\mathbb{H}$ and on its boundary. Then Blumenthal constructs a fundamental domain in three steps in which the group is enlarged successively. The product $\mathbb{H}^n$ of n upper halfplanes modulo the subgroup of translations in $\mathrm{GL}(2,\mathfrak{o})$ is the product of a parallelepiped (real parts) and a half space $\mathbb{R}^n_{>0}$ (imaginary parts), since translations in $\mathrm{GL}(2,\mathfrak{o})$ fix imaginary parts and the group of translations operates discretely on the real parts. The space $\mathbb{H}^n$ modulo the affine transformations, i.e. the transformations fixing the point at infinity, is the product of a cone (for the imaginary parts) and a parallelepiped (for the real parts). To see this, we first use matrices of the type $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right)$ In the third step, the whole group is investigated. Blumenthal shows using a theorem of Minkowski, that there is a constant $C=(C_1,\dots,C_n)$ with $C_j>0$ for all $1\leq j\leq n$, such that for every $\tau\in\mathbb{H}^n$ there is an element $M=\left(\begin{smallmatrix}1 & 0\\ \gamma & \delta\end{smallmatrix}\right)\in\mathrm{GL}(2,\mathcal{K})$ with $\gamma,\delta\in\mathfrak{o}$, such that $(M\tau)_j>C_j$ holds for all $1\leq j\leq n$. Blumenthal wrongly assumes $M\in\mathrm{GL}(2,\mathfrak{o})$, so he obtains the existence of exactly one cusp for $\mathbb{H}^n/\,\mathrm{GL}(2,\mathfrak{o})$, as this was conjectured by Hilbert (cf. [Ge88, p. 4]).

The second part of Blumenthal's work deals with Poincaré series (cf. Section 2.3), he shows their convergence and the existence of $n+1$ algebraically independent Poincaré series. He uses the result of the first part, but the proof can easily be amended by treatment of all the finitely many cusps instead of the single cusp ∞. Equivalently he shows the existence of n independent modular functions which are quotients of the $n+1$ algebraically independent Hilbert modular forms.

The third part (cf. [Bl04a]) proves the theorems of Weierstraß, that

I) all rational functions of the fundamental domain can be algebraically expressed by n independent functions,

II) they can be rational expressed by $n+1$ appropriate functions.

This result is independent of the mistake at the beginning. We will refer to this fact in Section 6.2. An alternative proof of the Theorem of Weierstraß can be found in [Th54, Hauptsatz II, p. 457], some further explanations and a good overview in [Re56, p. 277, 278].

Architecture of the thesis

Much progress has been made since Blumenthal's work. We focus on concrete calculations of rings of Hilbert modular forms, where a number of rings already have been calculated. But only in the case of $\mathbb{Q}(\sqrt{5})$ the full ring of Hilbert modular forms, in this case there is only the trivial multiplier system, has been calculated. For example Hammond's modular embedding delivers the subring of symmetric Hilbert modular forms with trivial multiplier system of even weight in case $\mathbb{Q}(\sqrt{8})$ and probably less in case of larger determinants (cf. [Ha66a] and [Re74]). We will apply the method of Borcherds products and obtain the complete ring for symmetric multiplier systems respectively the complete ring for the extended Hilbert modular group.

This is done in several steps. In the first chapter we will introduce automorphic forms, Hilbert modular forms and some modular groups with their appropriate operation. We will give quite general definitions and some equivalent notions in order to enable the reader to classify the further results on Hilbert modular forms on $\mathbb{H}^2$ and show the interrelation of the different notions.

Following the definitions we introduce three important examples of Hilbert modular forms in the second chapter, as there are Eisenstein series, Theta series embedded via Hammond's modular embedding and the Poincaré series, the latter more important for theoretical investigations than for concrete calculations. Additionally we introduce elliptic modular forms, especially with characters and for congruence subgroups and define vector valued modular forms, all of which we will need later.

The third chapter presents Borcherds products in the case of Hilbert modular forms following Bruinier and Bundschuh's paper [BB03]. We further investigate $\mathcal{K}$ and its ring $\mathfrak{o}$ of integers, Weyl vectors and Weyl chambers and Hirzebruch-Zagier divisors, such that the parameters of the Borcherds-lift can be calculated explicitely.

We include a chapter about general properties of Hilbert modular forms and, in particular, about Borcherds products. We apply Gundlach's method of determining all multiplier systems, we investigate symmetric and skew-symmetric Hilbert modular forms with respect to two reflections and present two methods to obtain new Hilbert modular forms by differentiation. Both are not needed in our cases, but could be beneficial in other context and differentiation poses a way to obtain Hilbert modular forms of inhomogeneous weight.

Chapter number five deals with the calculation of Bocherds products, especially we give several sources for the elements of the plus space of the elements needed for the Bocherds lift and determine the weight and multiplier systems and Fourier expansions of the calculated products.

In the sixth chapter we compose the various results of the preceding chapters to determine the rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. The ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ is generated by four modular forms and we succeed in expressing the known results with help of Borcherds products. In this case all Hilbert modular forms have trivial multiplier system. In case $\mathbb{Q}(\sqrt{13})$ the ring of extended Hilbert modular forms is also generated by four modular forms, the subring of Hilbert modular forms with trivial multiplier system is generated by seven modular forms. In case $\mathbb{Q}(\sqrt{17})$ the ring of extended Hilbert modular forms is generated by five modular forms, the subring for trivial multiplier systems needs eleven generators. All these rings have transcendence degree three.

The last chapter poses new questions possibly connected to this work and presents some approaches to their solution.

Main results

The main results of this work are the calculation of some rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. We write M^p for the ring of extended Hilbert modular forms for

$\mathbb{Q}(\sqrt{p})$ and $M^p(1)$ for the ring of extended Hilbert modular forms with trivial multiplier system. In case $\mathbb{Q}(\sqrt{5})$ we reformulate the already known ring $M^5 = M^5(1)$ of Hilbert modular forms using Borcherds products into

Theorem 6.3.1. *M^5 is generated by the Eisenstein series E_2^H and E_6^H and the Borcherds products Ψ_1 and Ψ_5 (cf. table 6.2) and all relations in between the given generators are induced by the relation R_{30}:*

$$\begin{aligned}\Psi_5^2 - &\left(\frac{67}{25}E_6^H - \frac{42}{25}\left(E_2^H\right)^3\right)\left(\frac{67}{43200}\left(\left(E_2^H\right)^3 - E_6^H\right)\right)^4 \\ &= \Psi_1^2\Big(3125\,\Psi_1^4 + \frac{1}{1728}\Psi_1^2\left(335\left(E_2^H\right)^2 E_6^H - 227\left(E_2^H\right)^5\right) \\ &+ \frac{4486}{89579520000}\left(43\left(E_2^H\right)^{10} - 153\left(E_2^H\right)^7 E_6^H + 177\left(E_2^H\right)^4\left(E_6^H\right)^2 - 67E_2^H\left(E_6^H\right)^3\right)\Big)\end{aligned}$$

In other words if we write $X_2 = E_2^H$, $X_5 = \Psi_1$, $X_6 = e_6$ and $X_{15} = \Psi_5$ we get

$$M^5 = \mathbb{C}[X_2, X_5, X_6, X_{15}]/\left\langle R_{30}\right\rangle.$$

By a comparison of Fourier expansions we can easily show that the Theta series s_5 and s_{15} introduced by Müller (cf. [Mü85]) are Borcherds products. Note that the index of the Borcherds products does not indicate their weight.

In the case of $\mathbb{Q}(\sqrt{13})$ there are non-trivial multiplier systems and we calculate the ring M^{13} using the Borcherds products Ψ_1, Ψ_4 and Ψ_{13} and the Eisenstein series E_2^H of weights 1, 3, 7 and 2.

Theorem 6.4.1. *M^{13} is generated by Ψ_1, $\frac{\Psi_4}{2\Psi_1}$, E_2^H and Ψ_{13} (cf. table 6.3) and the relations in between the given generators are induced by*

$$\begin{aligned}R_{14}:\ \Psi_{13}^2 - &\left(\frac{\Psi_4}{2\Psi_1}\right)^4\left(\left(E_2^H\right)^3 - 2^6 3^3\left(\frac{\Psi_4}{2\Psi_1}\right)^3\right) = -108\Psi_1^{12}\Psi_2 - \frac{27}{16}\Psi_1^{10}\left(E_2^H\right)^2 \\ &+ \frac{495}{8}\Psi_1^8\Psi_2^2E_2^H - \frac{1459}{16}\Psi_1^6\Psi_2^4 + \frac{41}{8}\Psi_1^6\Psi_2E_2^H - 512\Psi_1^6\left(\frac{\Psi_4}{2\Psi_1}\right)^4 + \frac{1}{16}\Psi_1^4\left(E_2^H\right)^5 \\ &- \frac{97}{4}\Psi_1^4\Psi_2^3\left(E_2^H\right)^2 - \frac{1}{8}\Psi_1^2\Psi_2^2\left(E_2^H\right)^4 - 144\Psi_1^2\left(\frac{\Psi_4}{2\Psi_1}\right)^5 E_2^H + \frac{189}{8}\Psi_1^2\Psi_2^5E_2^H.\end{aligned}$$

In other words if we write $X_1 = \Psi_1$, $X_2 = \frac{\Psi_4}{2\Psi_1}$, $Y_2 = E_2^H$ and $X_7 = \Psi_{17}$ we get

$$M^{13} = \mathbb{C}[X_1, X_2, Y_2, X_7]/\left\langle R_{14}\right\rangle.$$

As a Corollary we get the subring

Corollary 6.4.2. *We write* $X_4 = E_2^H$, $X_6 = \Psi_1^3$, $X_8 = \Psi_1^2 \frac{\Psi_4}{2\Psi_1}$, $X_{10} = \Psi_1 \left(\frac{\Psi_4}{2\Psi_1}\right)^2$, $X_{12} = \left(\frac{\Psi_4}{2\Psi_1}\right)^3$, $X_{16} = \Psi_1\Psi_{13}$ *and* $X_{18} = \frac{\Psi_4}{2\Psi_1}\Psi_{13}$ *and define the relations*

$$
\begin{aligned}
R_{18}: &\quad X_{10}X_8 = X_{12}X_6, \qquad\qquad R_{20}: \quad X_{10}^2 = X_{12}X_8,\\
R_{24}: &\quad X_{16}X_8 = X_6X_{18},\\
R_{36}: &\quad X_{18}^2 = X_{12}^2X_4^3 - 1728X_{12}^3 - 108X_3X_6^4 + \tfrac{1}{16}X_8^2X_4^5 + \tfrac{41}{8}X_{12}X_6^2X_4^3 - \tfrac{1459}{16}X_{12}^2X_6^2\\
&\qquad + \tfrac{495}{8}X_{10}^2X_6^2X_4 - \tfrac{97}{4}X_8X_4^2X_{10}^2 - \tfrac{27}{16}X_{10}X_6^3X_4^2 - \tfrac{1}{8}X_{10}^2X_4^4 + \tfrac{189}{8}X_4X_{12}^2X_8.
\end{aligned}
$$

Then

$$M^{13}(1) = \mathbb{C}[X_4, X_6, X_8, X_{10}, X_{12}, X_{16}, X_{18}] \,/\, (R_{18}, R_{20}, R_{24}, R_{36}).$$

In the case of Hilbert modular forms for $\mathbb{Q}(\sqrt{17})$, we can describe the ring M^{17} with the the Hilbert modular forms η_2 of weight $\frac{3}{2}$ defined by Theta series (cf. [He81]), the Borcherds products Ψ_1, Ψ_2, Ψ_{17} of weight $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{9}{2}$ and with the Eisenstein series E_2^H of weight 2. We get

Theorem 6.5.1. M^{17} *is generated by* $X_{\frac{1}{2}} = \Psi_1$, $X_{\frac{3}{2}} = -\Psi_2$, $Y_{\frac{3}{2}} = \eta_2$, $X_2 = E_2^H$ *and* $X_{\frac{9}{2}} = \Psi_{17}$. *Together with the two relations of weight 3 and 9,*

$$R_3 : \eta_2^2 - 64\Psi_2^2 = 16\Psi_1^2E_2^H$$

and

$$
\begin{aligned}
R_9 : \Psi_{17}^2 - \Psi_2^2\left(E_2^H\right)^3 + 216\Psi_2^5\eta_2 = &-256\Psi_1^{18}\\
&- 176\Psi_1^{12}\Psi_2\eta_2 - \frac{2671}{4096}\Psi_1^6\eta_2^4 + \frac{103}{8}\Psi_1^4\left(E_2^H\right)^2\Psi_2\eta_2\\
&- \frac{87}{16}\Psi_1^{10}\left(E_2^H\right)^2 - \frac{99}{128}\Psi_1^2E_2^H\Psi_2\eta_2^3 + \frac{1387}{128}\Psi_1^8E_2^H\eta_2^2,
\end{aligned}
$$

we have $M^{17} = \mathbb{C}[X_{\frac{1}{2}}, X_{\frac{3}{2}}, Y_{\frac{3}{2}}, X_2, X_{\frac{9}{2}}]/(R_3, R_9)$.

As a corollary we get the subring $M^{17}(1)$ of Hilbert modular forms with trivial multiplier systems:

Corollary 6.5.2. *We write*

$$
\begin{aligned}
&X_2 = E_2^H, \quad X_6 = -\Psi_2^3\,\eta_2/8, \quad X_9 = \Psi_2^2\Psi_{17}\,\eta_2/8, \quad X_5 = -\Psi_1\Psi_2^3,\\
&X_8 = \Psi_1\Psi_2^2\Psi_{17}, \quad X_4 = -\Psi_1^2\Psi_2\,\eta_2/8, \quad X_7 = \Psi_1^2\Psi_{17}\,\eta_2/8, \quad X_3 = -\Psi_1^3\Psi_2,\\
&Y_6 = \Psi_1^3\Psi_{17}, \quad Y_5 = \Psi_1^7\,\eta_2/8, \quad Y_4 = \Psi_1^8,
\end{aligned}
$$

and define the relations

$$
\begin{array}{rlrl}
R_9: & X_4X_5 = X_3X_6, & R_{10}: & Y_4X_6 = X_3^2X_4, \\
R_{11}: & Y_5X_6 = X_3X_4^2, & R_{12}: & X_4X_8 = X_5X_7, \\
R'_{12}: & X_6Y_6 = X_5X_7, & R_{13}: & X_6X_7 = X_9X_4, \\
R_{14}: & X_5X_9 = X_6X_8, & & \\
R_{18}: & \multicolumn{3}{l}{X_9^2 = X_3^2(X_3 + X_2^3) - 256X_4Y_4^2X_6 - 1408X_3^2X_4^3 - \frac{2671}{4}X_2X_4^4} \\
& \multicolumn{3}{l}{-2671X_4^2X_5^2 + \frac{2671}{4}X_2X_3X_4^2X_5 - 103X_2^2X_4^2X_6 - \frac{87}{16}X_2^2X_4Y_4X_6} \\
& \multicolumn{3}{l}{+\frac{99}{128}X_2X_4X_6^2 + \frac{99}{512}X_2^2X_4^2X_6 + \frac{1387}{2}X_2X_4^4.}
\end{array}
$$

Then

$$M^{17}(1) = \mathbb{C}[X_2, X_3, X_4, Y_4, X_5, Y_5, X_6, Y_6, X_7, X_8, X_9]/(R_{18}, R_{14}, R_{13}, R_{12}, R_9, R'_{12}, R_{11}, R_{10}).$$

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1 Definitions of Hilbert Modular Forms

Starting with the general case, we will restrict our research to Hilbert Modular forms for real quadratic number fields. We give several equivalent definitions of Hilbert modular forms. We will mainly use the first one, but need the orthogonal and the vector valued one for the original formulation of Borcherds' work. The definition of vector valued modular forms is not given in this chapter, but in section 2.6, after an equivalent subspace of nearly elliptic modular forms for a congruence subgroup is introduced. The contents of this chapter are taken from the books of Freitag [Fr90] and Leutbecher [Le96] and from Bruinier [Br98].

1.1 Automorphic Forms

This section is based on the first chapter of Freitag's book [Fr90], but introduces automorphic forms with multiplier system. Given a subgroup Γ of $\mathrm{SL}(2,\mathbb{R})^n$ we define its operation on $\mathbb{H}^n$ and its cusps and the notion of automorphic forms with respect to Γ.

Definition 1.1.1 (Operation of subgroups of $\mathrm{SL}(2,\mathbb{R})^n$ on $\mathbb{H}^n$). We denote the upper half plane $\{z \in \mathbb{C};\ \mathrm{Im}\,(z) > 0\}$ by $\mathbb{H}$. Let $n \in \mathbb{N}$. Given a subgroup Γ of $\mathrm{SL}(2,\mathbb{R})^n$ we define its operation on $\mathbb{H}^n$ by $\mathrm{SL}(2,\mathbb{R})^n \times \mathbb{H}^n \longrightarrow \mathbb{H}^n$,

$$(M,\tau) \longmapsto M\tau := \left(\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, \ldots, \frac{a_n\tau_n + b_n}{c_n\tau_n + d_n}\right),$$

where

$$M = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\right) \quad \text{and} \quad \tau = (\tau_1, \ldots, \tau_n)\,.$$

This operation can be continuously extended to an operation on $(\mathbb{H} \cup \mathbb{R} \cup \{\infty\})^n$.

Definition 1.1.2 (Extension of $\mathrm{SL}(2,\mathbb{R})^n$). The group S_n of permutations of $\{1,\ldots,n\}$ acts naturally on $\mathrm{SL}(2,\mathbb{R})^n$ and on $\mathbb{H}^n$ by permutation of the n components. We define the extended group $\widehat{\mathrm{SL}(2,\mathbb{R})}^n$ as semidirect product $\mathrm{SL}(2,\mathbb{R})^n \rtimes S_n$ with

$$((M_1,\ldots,M_n)\,,\pi_1) \cdot ((N_1,\ldots,N_n)\,,\pi_2) = \big((M_1,\ldots,M_n)\left(N_{\pi_1(1)},\ldots,N_{\pi_1(n)}\right), \pi_1\pi_2\big)$$

for all $(M_1,\ldots,M_n), (N_1,\ldots,N_n) \in \mathrm{SL}(2,\mathbb{R})^n$, $\pi_1, \pi_2 \in S_n$.

Remark 1.1.3. *We can embed* $\widehat{\mathrm{SL}(2,\mathbb{R})}^n$ *in the symplectic group*

$$\mathrm{Sp}(n,\mathbb{R}) := \left\{M \in \mathbb{R}^{2n\times 2n};\ M^{\mathrm{tr}} \left(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & E_n \\ -E_n & 0 \end{smallmatrix}\right)\right\}$$

by

$$\begin{cases} \widehat{\mathrm{SL}(2,\mathbb{R})}^n & \longrightarrow & \mathrm{Sp}(n,\mathbb{R}) \\ ((M_1,\dots,M_n),\pi) & \longmapsto & (M_1\times\dots\times M_n)\cdot \left(\begin{smallmatrix} P_\pi & 0 \\ 0 & P_\pi \end{smallmatrix}\right) \end{cases},$$

where P_π *is the permutation matrix corresponding to* π *and*

$$(M_1\times\dots\times M_n) := \begin{pmatrix} \mathrm{Diag}(a_1,\dots,a_n) & \mathrm{Diag}(b_1,\dots,b_n) \\ \mathrm{Diag}(c_1,\dots,c_n) & \mathrm{Diag}(d_1,\dots,d_n) \end{pmatrix}$$

for $M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{Sp}(1,\mathbb{R})$, $1 \le j \le n$*. It is easy to see that this embedding is well defined.*

Simple calculations show the following

Remark 1.1.4. *The group* $\widehat{\mathrm{SL}(2,\mathbb{R})}^n$ *operates on* $\mathbb{H}^n$ *and the operation is given by*

$$\widehat{\mathrm{SL}(2,\mathbb{R})}^n \times \mathbb{H}^n \to \mathbb{H}^n, \qquad ((M,\pi)\tau) \mapsto M\pi(\tau).$$

Definition 1.1.5 (Cusp). For $\lambda, \varepsilon \in \mathbb{R}^n$ and $\tau \in \mathbb{H}^n$ we define

$$\tau + \lambda := (\tau_1 + \lambda_1, \dots, \tau_n + \lambda_n)$$

and

$$\varepsilon\tau + \lambda := (\varepsilon_1\cdot\tau_1 + \lambda_1, \dots, \varepsilon_n\cdot\tau_n + \lambda_n).$$

For a discrete subgroup $\hat{\Gamma} < \widehat{\mathrm{SL}(2,\mathbb{R})}^n$ we define the **group of translations** by

$$\mathbf{t}_{\hat{\Gamma}} := \left\{\lambda \in \mathbb{R}^n;\ \text{there is } M \in \hat{\Gamma} : M\tau = \tau + \lambda \text{ for all } \tau \in \mathbb{H}^n\right\}$$

and the **group of multipliers** by

$$\mathbf{\Lambda}_{\hat{\Gamma}} := \left\{\varepsilon \in \mathbb{R}^n;\ \varepsilon \gg 0,\ \text{There are } M \in \hat{\Gamma}, \lambda \in \mathbb{R}^n : M\tau = \varepsilon\tau + \lambda \text{ for all } \tau \in \mathbb{H}^n\right\},$$

where $\varepsilon \gg 0$ means $\varepsilon_1 > 0, \dots, \varepsilon_n > 0$. We say that $\hat{\Gamma}$ has **cusp infinity**, iff $\mathbf{t}_{\hat{\Gamma}}$ is isomorphic to $\mathbb{Z}^n$ and $\mathbf{\Lambda}_{\hat{\Gamma}}$ is isomorphic to $\mathbb{Z}^{n-1}$. We will write $\hat{\Gamma}$ has cusp ∞.

We say that $\hat{\Gamma}$ has **cusp** κ for some $\kappa \in (\mathbb{R}\cup\{\infty\})^n$, iff there is an $M \in \widehat{\mathrm{SL}(2,\mathbb{R})}$ with $M\kappa = (\infty,\dots,\infty)$ such that $M\hat{\Gamma}M^{-1}$ has cusp infinity.

Remark 1.1.6. *For every κ in $(\mathbb{R}\cup\{\infty\})^n$ there exists an $M \in \widehat{\mathrm{SL}(2,\mathbb{R})}^n$ with $M\kappa = (\infty,\ldots,\infty)$. The definition of cusp κ is independent of the choice of M, i.e. if $M\kappa = N\kappa = (\infty,\ldots,\infty)$ for two elements M, N of $\widehat{\mathrm{SL}(2,\mathbb{R})}^n$, then either both $M\hat{\Gamma}M^{-1}$ and $N\hat{\Gamma}N^{-1}$ have cusp infinity or neither has.*

Definition 1.1.7. We define

$$(\mathbb{H}^n)^* := \mathbb{H}^n \cup \text{ set of cusps of } \hat{\Gamma}.$$

From now on let $\hat{\Gamma} = \langle \Gamma, S\rangle$ with a subgroup S of S_n and a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})^n$ such that S operates on Γ, $(\mathbb{H}^n)^*/\hat{\Gamma}$ is compact and each of the projections $p_j : \Gamma \to \mathrm{SL}(2,\mathbb{R}), M \mapsto \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$, $1 \le j \le n$, is injective.

Remark 1.1.8. *The compactness of $(\mathbb{H}^n)^*/\Gamma$ and $(\mathbb{H}^n)^*/\hat{\Gamma}$ are equivalent.*

Proof. Since $\Gamma < \hat{\Gamma}$, the compactness of $(\mathbb{H}^n)^*/\Gamma$ implies the compactness of $(\mathbb{H}^n)^*/\hat{\Gamma}$.

Write $S = \{\pi_1,\ldots,\pi_m\}$ and let $(\mathbb{H}^n)^*/\hat{\Gamma} = (\mathbb{H}^n)^*/\Gamma/S$ be compact. Consider an open covering $\cup_{i\in I} U_i$ of $(\mathbb{H}^n)^*/\Gamma$. Then also $\cup_{i\in I^m}\left(\cap_{j=1}^m \pi_j U_{i_j}\right)$ is an open covering of $(\mathbb{H}^n)^*/\Gamma$. It induces the open covering $\cup_{i\in I^m} S\left(\cap_{j=1}^m \pi_j U_{i_j}\right)/S$ on $(\mathbb{H}^n)^*/\Gamma/S$, which has a finite subcovering corresponding to some finite set $J \subset I^n$. Then $\cup_{i\in J, 1\le j\le m} U_{i_j}$ is a finite subcovering for $\cup_{i\in I} U_i$, since for every $x \in \mathbb{H}^m/\Gamma$ there is $i \in J$ such that $Sx \in S\left(\cap_{j=1}^m \pi_j U_{i_j}\right)/S$, i.e. there is $\pi_j \in S$ with $x \in \pi_j^{-1} \cap_{j=1}^n \pi_j U_{i_j}$, so $x \in U_{i_j}$. ☐

Remark 1.1.9. *If $(\mathbb{H}^n)^*/\Gamma$ is compact, then there are only finitely many cusps. If ε is a multiplier, then $\mathrm{N}(\varepsilon) = 1$. For a proof we refer to [Fr90, Remark I.2.3], where also further properties of $(\mathbb{H}^n)^*$ and of $(\mathbb{H}^n)^*/\Gamma$ can be found. From the next section on we will restrict to the Hilbert modular group $\Gamma = \mathrm{SL}(2,\mathbb{Q}(\sqrt{p}))$ of some real quadratic field $\mathbb{Q}(\sqrt{p})$ of prime discriminant $p \equiv 1 \pmod 4$ and S either the trivial group or S_2. Then $(\mathbb{H}^n)^*/\Gamma$ is compact and the projections p_j are injective.*

Definition 1.1.10 (Trace and dual lattice). Given $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define the **trace**

$$\mathrm{S}(ax) = a_1x_1 + \cdots + a_nx_n$$

and for a lattice $\mathbf{t} \subset \mathbb{R}^n$ we define the **dual lattice** $\mathbf{t}^\#$ by

$$\mathbf{t}^\# = \{a \in \mathbb{R}^n;\ \mathrm{S}(ax) \in \mathbb{Z} \text{ for all } x \in \mathbf{t}\}.$$

Lemma 1.1.11 (Fourier expansion). *Let $V \subset \mathbb{R}^n_{>0}$ be an open, connected set. Define the tube-domain $D := \{\tau \in \mathbb{H}^n;\ \mathrm{Im}(\tau) \in V\}$ corresponding to V. Let*

$$f : D \to \mathbb{C}$$

be a holomorphic function on D satisfying

$$f(\tau + a) = f(\tau), \quad \textit{for all } a \in \mathfrak{t} \textit{ and all } \tau \in \mathbb{H}^n$$

for some lattice $\mathfrak{t} \subset \mathbb{R}^n$. Then f has an unique Fourier expansion

$$f(\tau) = \sum_{g \in \mathfrak{t}^{\#}} a_g e^{2\pi i \,\mathrm{S}(g\tau)}$$

and the series converges absolutely and uniformly on compact subsets of D.

Definition 1.1.12 (Norm). Given $c, d \in \mathbb{R}^n$, $r \in \mathbb{Q}^n$ and $\tau \in \mathbb{H}^n$ we define the r^{th} power of the norm of $c\tau + d$ by

$$\mathrm{N}(c\tau + d)^r := (c_1\tau_1 + d_1)^{r_1} \cdot \dots \cdot (c_n\tau_n + d_n)^{r_n}$$

where the r_j^{th} power is defined using the main branch of the logarithm $\mathbb{C}^* \to \mathbb{R} + i(-\pi, \pi]$.

Definition 1.1.13 (Slash operator). Given a holomorphic map $f : \mathbb{H}^n \to \mathbb{C}$, $r = (r_1, \dots, r_n) \in \mathbb{C}^n$, a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and a map $\mu : \hat{\Gamma} \to \mathbb{C}^*$ we define

$$f|_r^\mu M : \begin{cases} \mathbb{H}^n & \longrightarrow \quad \mathbb{C} \\ \quad \tau & \longmapsto \quad \mu(M)^{-1} \cdot \mathrm{N}(c\tau + d)^{-r} \cdot f(M\tau). \end{cases}$$

For a permutation $\pi \in S_n$ we define

$$f|_r^\mu \pi : \begin{cases} \mathbb{H}^n & \longrightarrow \quad \mathbb{C}, \\ \quad \tau & \longmapsto \quad \mu(\pi)^{-1} \cdot f(\pi\tau). \end{cases}$$

For every element $(M, \pi) \in \widehat{\mathrm{SL}(2, \mathbb{R})}^n$ with $M \in \mathrm{SL}(2, \mathbb{R})^n$ and $\pi \in S_n$ we define $f|_r^\mu(M, \pi) = f|_r^\mu M|_r^\mu \pi$. For sake of consistency we additionally require $\mu(M) = f|_k^1(M)/f|_k^\mu(M)$ for all $M \in \widehat{\Gamma}$. We will write $|_k$ for $|_k^1$, where 1 is the constant map $\widehat{\mathrm{SL}(2, \mathbb{R})}^n \to \{1\}$.

Remark 1.1.14. *Note that $\mathrm{N}(c\tau + d)^{-r} = 1/(\mathrm{N}(c\tau + d)^r)$ holds for every $r \in \mathbb{C}^n$ independent of the chosen branch of the complex logarithm.*

Remark 1.1.15. *We are interested in functions f on $\mathbb{H}^n$ satisfying $f|_r^\mu M = f$ for all $M \in \hat{\Gamma}$ and some fixed r and μ, so need the condition*

$$f|_r^\mu M|_r^\mu N = f|_r^\mu(MN) \qquad \textit{for all } M, N \in \hat{\Gamma}.$$

Hence we are interested in $\pi r = r$ for all $\pi \in S_n$.

Remark 1.1.16. *If we embed $\widehat{\mathrm{SL}(2, \mathbb{R})}^n$ in $\mathrm{Sp}(n, \mathbb{R})$ as described in Remark 1.1.3, we obtain*

$$f|_k^\mu M(\tau) = \mu(M) \prod_{i=1}^n \left(\sum_{j=1}^n N(c_{ij}, d_{ij}, \tau, k_i) \right) f(M\tau)$$

for all $M = \left(\begin{smallmatrix} (a_{ij})_{ij} & (b_{ij})_{ij} \\ (c_{ij})_{ij} & (d_{ij})_{ij} \end{smallmatrix}\right)$ $(1 \leq i, j \leq n)$, $\tau \in \mathbb{H}^n$, *where*

$$N(c_{ij}, d_{ij}, \tau, k_i) = \begin{cases} (c_{ij}\tau_j + d_{ij})^{-k_i}, & \text{if } (c_{ij}, d_{ij}) \neq (0,0), \\ 0, & \text{if } c_{ij} = d_{ij} = 0. \end{cases}$$

Note that in each of the factors, all but one summand vanishes. This form of $f|_k^\mu M$ *motivates the restriction posed upon* μ *in Definition 1.1.13 of the slash operator.*

Definition 1.1.17 (Regularity at a cusp). Let $V = \mathbb{R}^n_{>0}$ and $D = \mathbb{H}^n$. If $f : D \to \mathbb{C}$ is a function satisfying the requirements in Lemma 1.1.11 and $\hat{\Gamma}$ has cusp infinity, then f is called **regular at cusp** ∞, if

$$a_g \neq 0 \Longrightarrow g_j \geq 0 \quad (\text{for all } 1 \leq j \leq n).$$

We say that f **vanishes at cusp** ∞ if

$$a_g \neq 0 \Longrightarrow g_j > 0 \quad (\text{for all } 1 \leq j \leq n).$$

Let κ be a cusp of $\hat{\Gamma}$ and let N in $\mathrm{SL}(2,\mathbb{R})^n$ be a matrix with $N^{-1}\kappa = (\infty, \ldots, \infty)$. If there is $r \in \mathbb{Q}^n$ and a map $\mu : \Gamma \to \mathbb{C}^*$ such that f satisfies

$$f|_r^\mu M = f \quad \text{for all } M \in \hat{\Gamma},$$

then we say that f is **regular at cusp** κ (resp. **vanishes at cusp** κ) if $f|_r N$ has cusp ∞ with respect to the group $N^{-1}\Gamma N$ and is regular at ∞ (resp. vanishes at ∞).

Remark 1.1.18. *Note that* μ *is not needed for the definition of regularity. A constant does not change the regularity at a cusp and there is no unique way to extend* μ *to a map* $\widehat{\mathrm{SL}(2,\mathbb{R})}^n \to \mathbb{C}^*$.

Definition 1.1.19 (Automorphic form). Let $n \in \mathbb{N}$, $\hat{\Gamma}$ as in Definition 1.1.7 and let $\mu : \Gamma \to \mathbb{C}$ be a map of finite order, i.e. let $\left\{\mu^k;\ k \in \mathbb{N}\right\}$ be a finite set. An **automorphic form** of weight $r = (r_1, \ldots, r_n) \in \mathbb{Q}^n$ with respect to $\hat{\Gamma}$ with multiplier system μ is a holomorphic function

$$f : \mathbb{H}^n \to \mathbb{C}$$

with the properties

a) $f|_r^\mu M = f$ for all $M \in \hat{\Gamma}$,

b) f is regular at the cusps.

If f vanishes at all cusps, we call f a **cusp form**. If f is an automorphic form of weight r with multiplier system μ, we will sometimes write $f|M$ for $f|_r^\mu M$.

Remark 1.1.20. *The definition of an automorphic form is based on the one in Freitag's book, cf. [Fr90], but includes multiplier systems and the extended group $\hat{\Gamma}$, since both occur naturally in the theory of Borcherds. Freitag mentions the problem of formulating a general theory of multiplier systems. In the case of the Hilbert modular group and of subgroups of finite index, this was done by Gundlach, cf. [Gu88]. We restrict to multiplier systems of finite order, since this will do for us and we can easily deduce the important properties of an automorphic form f with multiplier system of order n from the properties of the automorphic form f^n with trivial multiplier system.*

Proposition 1.1.21. *Each automorphic form f of weight $0 = (0, \dots, 0)$ is constant.*

Proof. (cf. [Fr90, Proposition I.4.7])

Let us first assume that $\mu \equiv 1$ is the trivial multiplier system. f induces a holomorphic map on $\mathbb{H}^n/\hat{\Gamma}$ which can be continuously extended to a map $(\mathbb{H}^n)^*/\hat{\Gamma}$ which we also denote by f. Its absolute value $|f|$ attains its maximum in $(\mathbb{H}^n)^*/\hat{\Gamma}$ because this set is compact. If the maximum is attained in $(\mathbb{H}^n)/\hat{\Gamma}$, then f is constant by the maximum principle. Else we consider the finite product $\prod(f(\tau) - f(\kappa_j))$ where κ_j are representatives of the cusps modulo $\hat{\Gamma}$. This function is a cusp form and the induced function on $(\mathbb{H}^n)^*/\hat{\Gamma}$ attains its maximum in $\mathbb{H}^n/\hat{\Gamma}$, so it is constant. If $\mu \not\equiv 1$, then there is $k \in \mathbb{N}$ such that $\mu^k \equiv 1$ holds. Hence f^k is an automorphic form of weight 0 with trivial multiplier system μ^k and thus constant. Therefore the continuous function f is constant too. ☐

From Freitag [Fr90, after Proposition 4.7] we take

Lemma 1.1.22 (Action of multipliers). *Let f be an automorphic form of weight r with respect to $\hat{\Gamma}$ with trivial multiplier system, let ∞ be a cusp of $\hat{\Gamma}$ and let $\varepsilon \in \mathbf{\Lambda}_{\hat{\Gamma}}$. Then $\mathbf{t} = \varepsilon\mathbf{t}$, $\mathbf{t}^\# = \varepsilon\mathbf{t}^\#$ and the Fourier expansion*

$$f(\tau) = \sum_{g \in \mathbf{t}^\#} a_g e^{2\pi i \operatorname{S} g\tau}$$

satisfies

$$|a_{g\varepsilon}| = |a_g| \operatorname{N}(\varepsilon)^r \qquad \text{for all } g \in \mathbf{t}.$$

Proof. For all $\varepsilon \in \mathbf{\Lambda}_{\hat{\Gamma}}$ there are $b \in \mathbb{R}^n$ and $M \in \Gamma$ with operation $M(\tau) = \varepsilon\tau + b$ for all $\tau \in \mathbb{H}^n$. Let $a \in \mathbf{t}$ and $K \in \Gamma$ with operation $K(\tau) = \tau + a$ for all $\tau \in \mathbb{H}^n$. Then $MKM^{-1} \in \Gamma$ satisfies

$$MKM^{-1}(\tau) = MK(\varepsilon^{-1}\tau - \varepsilon^{-1}b) = M(\varepsilon^{-1}\tau - \varepsilon^{-1}b + a) = \tau + \varepsilon a \qquad \text{for all } \tau \in \mathbb{H}^n$$

showing $\varepsilon\mathbf{t} \subset \mathbf{t}$ and vice versa $M^{-1}KM(\tau) = \tau + \varepsilon^{-1}a$ for all $\tau \in \mathbb{H}^n$ shows $\varepsilon\mathbf{t} \supset \mathbf{t}$, so we have $\varepsilon\mathbf{t} = \mathbf{t}$. Since the condition on $a \in \mathbf{t}^\#$ is $\operatorname{S}(ax) \in \mathbb{Z}$ for all $x \in \mathbf{t}$ and ε operates on $\mathbf{t}$, the multiplier ε also operates on the dual lattice $\mathbf{t}^\#$.

We use $\varepsilon^{-1}\mathbf{t} = \mathbf{t}$ to calculate

$$\begin{aligned} \mathrm{N}(\varepsilon)^{-r} f(\tau) = f(\varepsilon\tau + b) &= \sum_{g\in\mathbf{t}^{\#}} a_g e^{2\pi i\,\mathrm{S}\,(g\varepsilon\tau+b)} \\ &= \sum_{g\in\mathbf{t}^{\#}} e^{2\pi i\,\mathrm{S}\,(gb)} a_{g\varepsilon^{-1}} e^{2\pi i\,\mathrm{S}\,(g\tau)} \qquad \text{for all } \tau\in\mathbb{H}^n \end{aligned}$$

and by comparison of Fourier coefficients we get

$$\mathrm{N}(\varepsilon)^{-r} a_g = a_{g\varepsilon^{-1}} e^{2\pi i\,\mathrm{S}(gb)} \qquad \text{for all } g\in\mathbf{t},$$

hence the absolute values (remember $\varepsilon \in \Lambda_{\hat{\Gamma}}$ implies $N(\varepsilon) > 0$) satisfy

$$|a_{g\varepsilon}| = |a_g|\,\mathrm{N}(\varepsilon)^r \qquad \text{for all } g\in\mathbf{t}.$$

□

We get the simple

Corollary 1.1.23 (Remark I.4.8 in [Fr90]). *If f is an automorphic form, but not a cusp form, then*

$$r_1 = \cdots = r_n$$

Proof. Let f be an automorphic form of weight r. Choose j such that r_j is minimal. Since Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})^n$, the group of translations $\mathbf{t}_\Gamma$ is a discrete subgroup of $\mathbb{R}^n$. Then $\mathbf{\Lambda}_\Gamma$ is a discrete subgroup of $\mathbb{R}^n$, since it operates naturally on the discrete group $\mathbf{t}_\Gamma$. Moreover $\mathbf{\Lambda}_\Gamma$ is isomorphic to $\mathbb{Z}^{n-1}$ and for all multipliers ε in $\mathbf{\Lambda}_\Gamma$ we have $\mathrm{N}(\varepsilon) = 1$, so there is a multiplier ε in $\mathbf{\Lambda}_\Gamma$ with $\varepsilon_j > 1$ and $\varepsilon_k < 1$ for all $k \neq j$ (similar as in [Fr90, Proof of Corollary after Proposition I.4.9]). Hence we have

$$1 = N(\varepsilon)^r \cdot \varepsilon^{(r_j,\ldots,r_j)} = \varepsilon_1^{r_1-r_j} \cdot \cdots \cdot 1 \cdot \cdots \cdot \varepsilon_n^{r_n-r_j}.$$

Since all $\varepsilon_k < 1$ ($k \neq j$) and $r_k - r_j$ is nonnegative for all $k \neq j$, this equation only holds if $r_k = r_j$ for all $k \neq j$. □

Another Corollary from Lemma 1.1.22 is

Lemma 1.1.24 (Götzky-Koecher principle). *In case $n \geq 2$ the regularity condition in the Definition of automorphic forms can be omitted.*

Proof. Let $n \geq 2$ and $f : \mathbb{H}^n \to \mathbb{C}$ be an automorphic form of weight r with multiplier system μ with respect to $\hat{\Gamma}$. Assume that μ is the trivial multiplier system (compare Freitag's book, [Fr90, Corollary after Proposition I.4.9]). As in the proof of Corollary 1.1.23, we choose a multiplier ε with $\varepsilon_1 > 1$ and $\varepsilon_j < 1$ for all $j \geq 2$. Let $g \in \mathbf{t}$ with $g_1 < 0$. Since $\varepsilon_j < 1$ for all $j \geq 2$, the set

$$\left\{ \left| \sum_{j=2}^{n} g_j \varepsilon_j^m \right| = |S(g\varepsilon^m) - g_1\varepsilon_1^m|\,;\ m\in\mathbb{N} \right\}$$

is bounded by some $M \in \mathbb{R}$ and from the absolute convergence of the Fourier expansion of f and Lemma 1.1.22 we get the convergence of

$$\begin{aligned}\sum_{m=1}^{\infty} |a_g| \, \mathrm{N}(\varepsilon^m)^r e^{2\pi |g_1| \varepsilon_1^m} &= \sum_{m=1}^{\infty} |a_{g\varepsilon^m}| e^{2\pi i (g_1 \varepsilon_1^m) i} \\ &\leq \sum_{m=1}^{\infty} |a_{g\varepsilon^m}| e^{2\pi i S(g\varepsilon^m \tau) + 2\pi M} \\ &\leq e^{2\pi M} \sum_{m=1}^{\infty} |a_{g\varepsilon^m}| e^{2\pi i S(g\varepsilon^m \tau)} \\ &\leq e^{2\pi M} \sum_{g \in \mathbf{t}^{\#}} \left| a_g e^{2\pi i S(g\tau)} \right| ,\end{aligned}$$

where $\tau := (i, \ldots, i) \in \mathbb{H}^n$. The left side converges, hence we get $|a_g| = 0$. Since for every automorphic form f there is $k \in \mathbb{N}$ such that f^k has trivial multiplier system, together with f^k surely f is regular at the cusps. $\square$

1.2 Standard Definition of Hilbert Modular Forms

We identify $\mathrm{SL}(2, \mathcal{K})$ with a subgroup of $\mathrm{SL}(2, \mathbb{R})^n$ and define Hilbert modular forms as certain automorphic forms. In this case, notations can be simplified. We restrict our investigations on Hilbert modular forms $\mathbb{H}^2 \to \mathbb{C}$ for the modular group. This definition will be used throughout this work.

Definition 1.2.1 ($\mathcal{K}$, $\mathfrak{o}$, operation of $\mathrm{SL}(2, \mathcal{K})$ on $\mathbb{H}^n$). Let $\mathcal{K}$ be a totally real number field of degree $n := [\mathcal{K} : \mathbb{Q}] := \dim_{\mathbb{Q}}(\mathcal{K})$. Then there are exactly n different embeddings of $\mathcal{K}$ into $\mathbb{R}$, or, if we assume $\mathcal{K} \subset \mathbb{R}$, there are n different automorphisms $\mathcal{K} \to \mathcal{K}$. We denote them by $\mathcal{K} \to \mathbb{R}, a \mapsto a^{(j)}$ where j ranges from 1 to n and $a = a^{(1)}$ holds for all $a \in \mathcal{K}$. We denote the ring of integers of $\mathcal{K}$, i.e. the set of all $x \in \mathcal{K}$, such that there is a monic polynomial $p \in \mathbb{Z}[X]$ with $p(x) = 0$, by $\mathfrak{o}$. We define the operation of $\mathrm{SL}(2, \mathcal{K})$ on $\mathbb{H}^n$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{a^{(n)}\tau_n + b^{(n)}}{c^{(n)}\tau_n + d^{(n)}} \right).$$

Remark 1.2.2. *The operation on $\mathbb{H}^n$ of the group $\mathrm{SL}(2, \mathcal{K})$ and of its image with respect to*

$$\mathrm{SL}(2, \mathcal{K}) \longrightarrow \mathrm{SL}(2, \mathbb{R})^n, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right),$$

are the same. Two groups are commensurable, if their intersection has finite index in each of the two groups. Freitag [Fr90] defines Hilbert modular forms as automorphic forms with respect to groups commensurable to the image of $\mathrm{SL}(2,\mathfrak{o}) \subset \mathrm{SL}(2,\mathcal{K})$ *in* $\mathrm{SL}(2,\mathbb{R})^n$*. We will only consider* $\mathrm{SL}(2,\mathfrak{o})$ *and can thus simplify notations.*

Remark 1.2.3. *The operation of* $\mathrm{SL}(2,\mathcal{K})$ *shows a common principle of Hilbert modular forms. The images* $\lambda^{(j)}$ *of an element* $\lambda \in \mathcal{K}$ *with respect to the field automorphisms of* $\mathcal{K}$ *and the* j^{th}*-component* τ_j *of a point* $\tau \in \mathbb{H}^n$ *belong together. Thus we give the following definitions:*

Definition 1.2.4. An element λ of $\mathcal{K}$ is called **totally positive** , if $\lambda^{(j)} > 0$ holds for all $1 \leq j \leq n$. Then we write $\lambda \gg 0$.

Definition 1.2.5 (Norm and trace). For $\lambda \in \mathcal{K}$ we define

- the **norm** $\mathrm{N}(\lambda) = \lambda^{(1)} \cdot \dots \cdot \lambda^{(n)}$ and
- the **trace** $\mathrm{S}(\lambda) = \lambda^{(1)} + \dots + \lambda^{(n)}$.

We define

- the **trace** $\mathrm{S}(\lambda\tau) = \lambda^{(1)}\tau_1 + \dots + \lambda^{(n)}\tau_n$ for all $\lambda \in \mathcal{K}$ and $\tau \in \mathbb{H}^n$,
- the **norm** $\mathrm{N}(c\tau + d) = (c^{(1)}\tau_1 + d^{(1)}) \cdot \dots \cdot (c^{(n)}\tau_n + d^{(n)})$ for all $c, d \in \mathcal{K}$ and $\tau \in \mathbb{H}^n$,
- $\mathrm{N}(c\tau + d)^r := (c^{(1)}\tau_1 + d^{(1)})^{r_1} \cdot \dots \cdot (c^{(n)}\tau_n + d^{(n)})^{r_n}$ for all $c, d \in \mathcal{K}$, $\tau \in \mathbb{H}^n$ and $r = (r_1, \dots, r_n) \in \mathbb{Q}^n$, where $z^{r_j} := e^{r_j \ln z}$ is defined using the main branch $\ln : \mathbb{C}^* \to \mathbb{R} + i(-\pi, \pi]$ of the complex logarithm,
- the translation $\tau \mapsto \tau + \lambda$ with $\lambda \in \mathcal{K}$ as the map
$$\mathbb{H}^n \longrightarrow \mathbb{H}^n,\ \tau \longmapsto \tau + \lambda := \left(\tau_1 + \lambda^{(1)}, \dots, \tau_n + \lambda^{(n)}\right)$$
and
- the multiplication $\tau \mapsto \lambda \cdot \tau$ with $\lambda \in \mathcal{K}$, $\lambda \gg 0$ as the map
$$\mathbb{H}^n \longrightarrow \mathbb{H}^n,\ \tau \longmapsto \lambda \cdot \tau := \left(\lambda^{(1)} \cdot \tau_1, \dots, \lambda^{(n)} \cdot \tau_n\right).$$

Definition 1.2.6 ((extended) Hilbert modular form). Let $n \in \mathbb{N}$ and let $\mu : \mathrm{SL}(2,\mathfrak{o}) \to \mathbb{C}$ be a map of finite order. A **Hilbert (Blumenthal) modular form** for $\mathcal{K}$ of weight $r = (r_1, \dots, r_n) \in \mathbb{Q}^n$ with **multiplier system** μ is a holomorphic function

$$f : \mathbb{H}^n \to \mathbb{C}$$

with the properties

a) $f|_r^\mu M = f$ for all $M \in \mathrm{SL}(2,\mathfrak{o})$,

b) f is regular at the cusps of $\mathrm{SL}(2, \mathfrak{o})$.

If f vanishes at all cusps, we call f a **cusp form** . If f has homogeneous weight $r = (k, \dots, k) \in \mathbb{Q}^n$ we will also say that f has weight $k \in \mathbb{Q}$. If f satisfies $f|_r^\mu M = f$ for all $M \in S_n$ we call it **extended Hilbert modular form** for $\mathcal{K}$ of weight $r = (r_1, \dots, r_n)$.

Remark 1.2.7. *Since* $(\mathbb{H}^n)^* / \mathrm{SL}(2, \mathfrak{o})$ *is compact, every Hilbert modular form is an automorphic form. If* $\mathcal{K} \neq \mathbb{Q}$*, then the Götzky-Koecher principle grants that condition b) can be omitted.*

We want to restrict the notion of multiplier systems to the relevant cases, i. e.

Definition 1.2.8 (Multiplier system). Let $\Gamma = \mathrm{SL}(2, \mathfrak{o})$ and $\hat{\Gamma} = \langle \Gamma, S_n \rangle$ or $\hat{\Gamma} = \Gamma$. A map $\mu : \hat{\Gamma} \to \mathbb{C}^*$ is called **multiplier system** , if it is of finite order and there is $k \in \mathbb{Q}$ such that

$$\mu(M_{(1)} M_{(2)}) \,\mathrm{N}(c\tau + d)^k = \mu\left(M_{(1)}\right) \mathrm{N}\left(c_{(1)} M_{(2)} \tau + d_{(1)}\right)^k \mu\left(M_{(2)}\right) \mathrm{N}\left(c_{(2)} \tau + d_{(2)}\right)^k$$

holds for all

$$\tau \in \mathbb{H}^n, M_{(1)} = \begin{pmatrix} a_{(1)} & b_{(1)} \\ c_{(1)} & d_{(1)} \end{pmatrix} \in \Gamma, M_{(2)} = \begin{pmatrix} a_{(2)} & b_{(2)} \\ c_{(2)} & d_{(2)} \end{pmatrix} \in \Gamma \text{ and } M_{(1)} M_{(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and in case $\pi_1, \pi_2 \in \hat{\Gamma} \cap S_n$ additionally

$$\mu(\pi_1)\mu(M)\mu(\pi_2) = \mu(\pi_1 M \pi_2) \qquad \text{holds for all } M \in \Gamma.$$

Lemma 1.2.9. *If* $f \neq 0$ *is an (extended) Hilbert modular form of weight* k *with multiplier system* μ *(in the sense of Definition 1.2.6) , then* μ *is a multiplier system in the sense of Definition 1.2.8.*

Proof. The first equation follows directly from $f|_k^\mu M_1 M_2 = f|_k^\mu M_1|_k^\mu M_2$. Let $\pi_1, \pi_2 \in \hat{\Gamma} \cap S_n$ and $M \in \Gamma$. We calculate for all $\tau \in \mathbb{H}^n$:

$$\begin{aligned} f|_k^\mu \pi_1|_k^\mu M|_k^\mu \pi_2(\tau) &= \mu(\pi_2)^{-1} \cdot f|_k^\mu \pi_1|_k^\mu M(\pi_2 \tau) \\ &= \mu(\pi_2)^{-1} \mu(M)^{-1} \,\mathrm{N}(c(\pi_2\tau) + d)^k \cdot f|_k^\mu \pi_1(M \pi_2 \tau) \\ &= \mu(\pi_2)^{-1} \mu(M)^{-1} \,\mathrm{N}((\pi_2^{-1} c)\tau + (\pi_2^{-1} d))^k \mu(\pi_1)^{-1} \cdot f(\pi_1 M \pi_2 \tau) \end{aligned}$$

and

$$\begin{aligned} f|_k^\mu(\pi_1\pi_2)|_k^\mu(\underbrace{\pi_2^{-1} M \pi_2}_{\in\Gamma})(\tau) &= \mu(\pi_2^{-1} M \pi_2)^{-1} \,\mathrm{N}((\pi_2^{-1} c)\tau + (\pi_2^{-1} d))^k \cdot f|_k^\mu(\pi_1\pi_2)(\pi_2^{-1} M \pi_2 \tau) \\ &= \mu(\pi_2^{-1} M \pi_2)^{-1} \,\mathrm{N}((\pi_2^{-1} c)\tau + (\pi_2^{-1} d))^k \mu(\pi_1\pi_2)^{-1} \cdot f(\pi_1 M \pi_2 \tau). \end{aligned}$$

If we insert $\mu \equiv 1$, we get

$$f|_k(\pi_1\pi_2)|_k(\pi_2^{-1} M \pi_2)(\tau) = \mathrm{N}((\pi_2^{-1} c)\tau + (\pi_2^{-1} d))^k f(\pi_1 M \pi_2 \tau),$$

so

$$\mu(\pi_1 M\pi_2) = \frac{f|_k(\pi_1 M\pi_2)}{f|_k^\mu(\pi_1 M\pi_2)} := \frac{f|_k(\pi_1\pi_2)|_k(\pi_2^{-1}M\pi_2)}{f|_k^\mu(\pi_1\pi_2)|_k^\mu(\pi_2^{-1}M\pi_2)} = \mu(\pi_2^{-1}M\pi_2)\mu(\pi_1\pi_2).$$

Since

$$f|_k^\mu \pi_1|_k^\mu M|_k^\mu \pi_2 = f|_k^\mu(\pi_1 M\pi_2) = f|_k^\mu(\pi_1\pi_2(\pi_2^{-1}M\pi_2)) := f|_k^\mu(\pi_1\pi_2)|_k^\mu(\pi_2^{-1}M\pi_2),$$

we get

$$\mu(\pi_2)^{-1}\mu(M)^{-1}\mu(\pi_1)^{-1} = \mu(\pi_2^{-1}M\pi_2)^{-1}\mu(\pi_1\pi_2)^{-1} = \mu(\pi_1 M\pi_2)^{-1}.$$

□

Remark 1.2.10. *Gundlach [Gu88] showed that the restriction on the order of μ is obsolete, compare Remark 4.1.7.*

Lemma 1.2.11 (Integral weight). *If $\mu : \hat{\Gamma} \to \mathbb{C}^*$ is a multiplier system of integral weight k, then μ is an abelian character, i. e. $\mu(MN) = \mu(M)\mu(N)$ holds for all $M, N \in \hat{\Gamma}$.*

Proof. For $\mu|_\Gamma$ one calculates

$$\mathrm{N}(c\tau + d) = \mathrm{N}\left(c_{(1)}M_{(2)}\tau + d_{(1)}\right)\mathrm{N}\left(c_{(2)}\tau + d_{(2)}\right)$$

or compares Remark 4.1.4 and Remark 4.1.7. Together with Lemma 1.2.9 this proves the assertion. □

We will see in Proposition 2.3.3, that for every multiplier system there exists a nontrivial Hilbert modular form of some weight with this multiplier system.

Clearly extended Hilbert modular forms are Hilbert modular forms. We investigate the relation between the corresponding multiplier systems:

Lemma 1.2.12 (multiplier systems of (extended) Hilbert modular forms). *Let μ be a multiplier system of a Hilbert modular form $f \not\equiv 0$. It can be extended to a multiplier system $\hat{\mu} : \langle \Gamma, S_n\rangle \to \mathbb{C}^*$ if and only if μ satisfies $\mu(\pi^{-1}M\pi) = \mu(M)$ for all $\pi \in S_n$ and $M \in \mathrm{SL}(2, \mathfrak{o})$. The extension can be realized by continuation of $\hat{\mu}|_\Gamma = \mu$ and $\hat{\mu}|_{S_n} = 1$. On the other hand, if $\mu : \hat{\Gamma} \to \mathbb{C}^*$ is a multiplier system of an extended Hilbert modular form, then for every $m \in \mathbb{N}$ with $\pi^m = 1$, the value $\mu(\pi)$ is an m-th root of unity. $\mu|_\Gamma$ satisfies $\mu|_\Gamma(\pi^{-1}M\pi) = \mu|_\Gamma(M)$ for all $M \in \mathrm{SL}(2, \mathfrak{o})$ and $\pi \in S_n$.*

Proof. This follows almost directly from Lemma 1.2.9 since we do not demand that there is an extended Hilbert modular form for this multiplier system. Note that

$$\mu(\pi^{-1}E\pi) = \mu(E) = \mu(\pi^{-1})\mu(E)\mu(\pi),$$

so $\mu(\pi^{-1}) = \mu(\pi)^{-1}$ for all $\pi \in S_n \cap \hat{\Gamma}$. Since by assumption

$$\mu(\pi^m E\pi^n) = \mu(\pi^m)\mu(E)\mu(\pi^n) = \mu(\pi^m)\mu(\pi^n),$$

we have $\mu(\pi^n) = \mu(\pi)^n$. □

1.2.1 Restriction to Quadratic Number Fields

In the rest of this paper, we will restrict our investigations to Hilbert modular forms of homogeneous weights $k \in \mathbb{Q}$ and for real quadratic number fields $\mathcal{K} = \mathcal{K}_p := \mathbb{Q}(\sqrt{p})$ for prime numbers p which are congruent to 1 modulo 4. Then $n = 2$ and we do not need the condition b) in the definition of Hilbert modular forms by the Götzky-Koecher principle. For calculation of at least some non-homogeneous weights see section 4.4.

Definition 1.2.13 ($\hat{\Gamma}$, Γ, $\hat{\Gamma}_\infty$, Γ_∞, $\mathcal{K}_p$, $\mathfrak{o}$). For a prime $p \equiv 1 \pmod 4$ we write $\Gamma := \mathrm{SL}(2, \mathfrak{o})$ where $\mathfrak{o}$ is the ring of integers of $\mathcal{K} = \mathcal{K}_p = \mathbb{Q}(\sqrt{p})$. It is given by

$$\mathfrak{o} = \mathbb{Z} + \frac{1+\sqrt{p}}{2}\mathbb{Z}.$$

We denote the group of the elements of Γ fixing $\infty = (i\infty, i\infty)$ by Γ_∞. We write

$$\overline{\lambda} := \lambda^{(2)} = \lambda_1 - \lambda_2\sqrt{p} \qquad \text{for } \lambda = \lambda_1 + \lambda_2\sqrt{p} \in \mathcal{K}, \lambda_1, \lambda_2 \in \mathbb{Q}$$

for the nontrivial field automorphism of $\mathcal{K}_p$ and extend Γ and Γ_∞ to the groups $\hat{\Gamma} = \langle \Gamma, \pi \rangle$ and $\hat{\Gamma}_\infty = \langle \Gamma_\infty, \pi \rangle$, where $\pi : \mathbb{H}^2 \to \mathbb{H}^2$ is the map exchanging the components, $\pi(\tau_1, \tau_2) = (\tau_2, \tau_1)$. We define the **fundamental unit** ε_0 by $\varepsilon_0 = \min\{x \in \mathfrak{o}^*;\ x > 1\}$ and have $\mathfrak{o}^* = \pm\varepsilon_0^{\mathbb{Z}}$. For example we have $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$ in case $p = 5$, $\varepsilon_0 = \frac{3+\sqrt{13}}{2}$ in case $p = 13$ and $\varepsilon_0 = 4 + \sqrt{17}$ in case $p = 17$ (compare Leutbecher [Le96, p. 97, 98]).

For $\lambda = \lambda_1 + \lambda_2\sqrt{p} \in \mathcal{K}_p$ with $\lambda_1, \lambda_2 \in \mathbb{Q}$ we then have

- the **norm** $\mathrm{N}(\lambda) = \lambda\overline{\lambda} = \lambda_1^2 - p\lambda_2^2$,
- the **trace** $\mathrm{S}(\lambda) = \lambda + \overline{\lambda} = 2\lambda_1$.

Definition 1.2.14 (Symmetric multiplier system). We say that a multiplier system $\mu : \Gamma \to \mathbb{C}^*$ is **symmetric** , if it holds

$$\mu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

In the case of real quadratic number fields we can rewrite Lemma 1.2.12 into

Remark 1.2.15. *A multiplier system $\mu : \Gamma \to \mathbb{C}^*$ can be extended to a multiplier system $\hat{\mu} : \hat{\Gamma} \to \mathbb{C}^*$ if and only if μ is symmetric.*

Definition 1.2.16. We define the following sets:

- $M_k^p(\mu)$: vector space of extended Hilbert modular forms for $\mathcal{K}_p = \mathbb{Q}(\sqrt{p})$ of weight k with multiplier system μ. The trivial multiplier system is the constant map to 1 and will be denoted by 1.

- $M^p := \sum_{k,\mu} M_k^p(\mu)$ where the summation ranges over all $k \in \mathbb{Q}$ and all multiplier systems μ.

  We will see in Corollary 4.2.6 that all Borcherds products have symmetric multiplier systems.

- $M^p(1)$: graded ring of all Hilbert modular forms for $\mathcal{K}_p$ with trivial multiplier system.

  Note that $M^p(1)$ is not the ring of extended Hilbert modular forms for $\mathcal{K}_p$ with trivial multiplier systems.

From Lemma 1.1.11 and Gundlach (cf. [Gu88] and Remark 4.1.7) we get

Remark 1.2.17 (Fourier expansion). *For each (extended) Hilbert modular form $f : \mathbb{H}^2 \to \mathbb{C}$ with multiplier system μ there is a lattice $\mathfrak{t} \subset \mathbb{R}^2$ such that*

i) $f(\tau + a) = f(\tau)$ *for all* $a \in \mathfrak{t}$,

ii) $\mathfrak{t}$ *is maximal in* $\left\{(\lambda, \overline{\lambda});\ \lambda \in \mathfrak{o}\right\}$ *under the restriction i) and*

iii) f *has the Fourier expansion* $f(\tau) = \sum_{g \in \mathfrak{t}^\#} a_g e^{2\pi i \,\mathrm{S}(g\tau)}$ *with* $a_g \in \mathbb{C}$ *for all* $g \in \mathfrak{t}^\#$ *and* $a_g \neq 0$ *only if* $g \geq 0$ *and* $\overline{g} \geq 0$. *The Fourier expansion converges absolutely and uniformly on compact subsets of* $\mathbb{H}^2$.

The number of cusp classes can be easily deduced in the case of the Hilbert modular group for quadratic number fields:

Lemma 1.2.18 (Corollary I.3.5$_2$ in [Fr90]). *The Hilbert modular group Γ has only finitely many cusp classes. Their number equals the class number of $\mathcal{K}$.*

Remark 1.2.19. *In the case of $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$ there is only one cusp class of Γ, for these fields have class number 1.*

1.2.2 The groups Γ and Γ_∞

Remark 1.2.20. *The operation of $\Gamma = \mathrm{SL}(2, \mathfrak{o})$ on $\mathbb{H}^2$ is given by*

$$\gamma\tau = \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{\overline{a}\tau_2 + \overline{b}}{\overline{c}\tau_2 + \overline{d}}\right) ,$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$.

The following lemma is a special case of a theorem of Vaserstein [Va72], a corrected proof can be found in [Li81] and [Le78, section 2]:

Lemma 1.2.21. Γ *is generated by the set*

$$\left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} ;\ \lambda \in \mathfrak{o} \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} ;\ \mu \in \mathfrak{o} \right\}.$$

Corollary and Definition 1.2.22. Γ *is generated by the matrices*

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \textit{ and } \quad T_w := \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \ (w = \frac{1}{2} + \frac{1}{2}\sqrt{p}).$$

More generally we will write

$$T_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

for all $\lambda \in \mathcal{K}$.

Proof. We have $\mathfrak{o} = \langle 1, w \rangle$, so the set $\{T, T_w\}$ generates the upper triangular matrices given in Lemma 1.2.21. In addition we get

$$J^3 \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix} J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

for every $\mu \in \mathfrak{o}$, so the lower triangular matrices in Lemma 1.2.21 are generated by J and the upper triangular matrices. □

Lemma 1.2.23. Γ_∞ *is generated by the matrices*

$$T, T_w, -E = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = J^2 \textit{ and } D_{\varepsilon_0} = \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix} = J^3 T_{\varepsilon_0^{-1}} J T_{\varepsilon_0} J T_{\varepsilon_0^{-1}}$$

and consist of all matrices of the type $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ *in* Γ.

Proof. The matrices $-E$, T, T_w and D_{ε_0} are of the given type. The group Γ operates on the first component of $\mathbb{H}^2$ like a group of Moebius transformations. So we already know that every element of Γ_∞ necessarily is of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. One easily checks that $-E$, T, T_w and D_{ε_0} fix ∞. Consider a matrix $M = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma$. Then $\det M = ad = 1$ implies $a = d^{-1} \in \mathfrak{o}^* = \pm\varepsilon_0^{\mathbb{Z}}$. So there is $k \in \mathbb{Z}$ such that $M' = -ED_{\varepsilon_0}^k M$ or $M' = D_{\varepsilon_0}^k M$ is of the form $M' = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ with $\lambda \in \mathfrak{o} =< 1, w >$. This proves both assertions. □

Remark 1.2.24. *Since the exchange of variables fixes* $\infty = (\infty, \infty)$, *we get a generating system of* $\hat{\Gamma}$ *resp. of* $\hat{\Gamma}_\infty$ *by extending a generating system of* Γ *resp. of* Γ_∞ *by the exchange of variables* $\bar{\cdot}$.

1.3 Orthogonal Hilbert Modular Forms

We define orthogonal Hilbert modular forms and will see, that for integral weight, they are essentially Hilbert modular forms, while they vanish for nonintegral weight.

Definition 1.3.1 ($Sym^2(\mathcal{K})$,q,b_j). We define the quadratic vector space $(\mathrm{Sym}^2(\mathcal{K}), q)$ over $\mathbb{Q}$ by

$$\mathrm{Sym}^2(\mathcal{K}) := \left\{ \begin{pmatrix} h_0 & h_1 \\ \overline{h_1} & h_2 \end{pmatrix} | h_0, h_2 \in \mathbb{Q}, h_1 \in \mathcal{K}_p \right\}$$

and

$$q \begin{pmatrix} h_0 & h_1 \\ \overline{h_1} & h_2 \end{pmatrix} = -\det \begin{pmatrix} h_0 & h_1 \\ \overline{h_1} & h_2 \end{pmatrix} = \mathrm{N}(h_1) - h_0 h_2 \qquad \text{(for all } \begin{pmatrix} h_0 & h_1 \\ \overline{h_1} & h_2 \end{pmatrix} \in \mathrm{Sym}^2(\mathcal{K})).$$

We equip it with the basis $\{b_1, b_2, b_3, b_4\}$ given by

$$b_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, b_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } b_4 := \begin{pmatrix} 0 & \frac{1+\sqrt{p}}{2} \\ \frac{1-\sqrt{p}}{2} & 0 \end{pmatrix}$$

and extend it to the quadratic space $(\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}, q)$, where

$$\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}b_1 + \mathbb{C}b_2 + \mathbb{C}b_3 + \mathbb{C}b_4$$

and

$$q(H) = -\det(H) \qquad \text{(for all } H \in 2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}).$$

We define the bilinear form $(\cdot, \cdot)$ corresponding to q by

$$(x, y) = q(x + y) - q(x) - q(y) \qquad \text{(for all } x, y \in \mathrm{Sym}^2(\mathcal{K}) \times_{\mathbb{Q}} \mathbb{C}, \text{ i.e. } (x, x) = 2q(x))$$

Lemma 1.3.2. *The vector space* $\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}$ *equipped with the bilinear form* $(\cdot, \cdot)$ *is an orthogonal space of signature* $(2, 2)$ *with Gram matrix*

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & \frac{1-p}{2} \end{pmatrix}$$

with respect to the basis $(b_1, \dots, b_4)$. $\mathcal{L} := \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_4$ *is an even lattice. So* $(\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}, (\cdot, \cdot))$ *is often referred to as* $O(2, 2)$.

Proof. We calculate

$$q(b_1) = q(b_2) = 0, \qquad q(b_3) = -\det\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 1$$

$$q(b_4) = -\det\begin{pmatrix}0 & \frac{1+\sqrt{p}}{2}\\ \frac{1-\sqrt{p}}{2} & 0\end{pmatrix} = \frac{1-p}{4}, \qquad (b_1, b_2) = -\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -1$$

$$(b_3, b_4) = -\det\begin{pmatrix}0 & \frac{3+\sqrt{p}}{2}\\ \frac{3-\sqrt{p}}{2} & 0\end{pmatrix} - q(b_3) - q(b_4) = \frac{9-p}{4} - 1 - \frac{1-p}{4} = 1.$$

Additionally we get the equation

$$(b_j, b_k) = q(b_j + b_k) - q(b_j) - q(b_k) = -\det(b_j + b_k) + \det(b_k) = 0$$

for all $j \in \{1, 2\}$ and $k \in \{3, 4\}$. It is obvious from the Gram matrix, that $\mathcal{L}$ is an even lattice. □

We restrict $\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}$ to the subspace

$$\left\{H \in \mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C};\ q(H) = 0\right\}$$

and consider the space

$$\tilde{\mathcal{H}} = \left\{H \in \mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C};\ q(H) = 0, (H, \widetilde{H}) > 0\right\},$$

where $\widetilde{H}$ is the matrix derived from H by component wise complex conjugation (We use $\widetilde{H}$ instead of $\overline{H}$ to avoid confusion with the field automorphism $\overline{\lambda_1 + \lambda_2\sqrt{p}} = \lambda_1 - \lambda_2\sqrt{p}$ of $\mathcal{K}$). Every element $H = \left(\begin{smallmatrix}h_0 & h_1\\ \overline{h_1} & h_2\end{smallmatrix}\right)$ of $\widetilde{\mathcal{H}}$ has $h_2 \neq 0$, as otherwise $0 = q(H) = h_1\overline{h_1}$ implies $h_1 = \overline{h_1} = 0$ and then $(H, \widetilde{H}) = (0, \tilde{0}) = 0$. In addition, for every $\delta \in \mathbb{C}^*$ and $H \in \mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}$ we have $q(\delta H) = \delta^2 q(H) = 0$ if and only if $q(H) = 0$ and we have $(\delta H, \widetilde{\delta H}) = |\delta|^2 (H, \tilde{H}) > 0$ if and only if $(H, \widetilde{H}) > 0$. So

$$\tilde{\mathcal{H}} = \left\{H = \delta\begin{pmatrix}h_0 & h_1\\ \overline{h_1} & 1\end{pmatrix};\ q\Big(\frac{1}{\delta}H\Big) = h_1\overline{h_1} - h_0 = 0, (H, \widetilde{H}) > 0, \delta \in \mathbb{C}^*, h_0 \in \mathbb{C}, \lambda \in \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{C}\right\}$$

$$= \left\{H = \delta\begin{pmatrix}h_1\overline{h_1} & h_1\\ \overline{h_1} & 1\end{pmatrix};\ (H, \widetilde{H}) > 0, \delta \in \mathbb{C}^*, h_1 \in \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{C}\right\}$$

We write $\tau_1 := h_1$ and $\tau_2 := \overline{h_1}$ and get

$$\begin{aligned}
\left(\left(\begin{smallmatrix}\tau_1\tau_2 & \tau_1\\ \tau_2 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}\widetilde{\tau_1\tau_2} & \widetilde{\tau_1}\\ \widetilde{\tau_2} & 1\end{smallmatrix}\right)\right) &= -\det\left(\left(\begin{smallmatrix}\tau_1\tau_2 & \tau_1\\ \tau_2 & 1\end{smallmatrix}\right) + \left(\begin{smallmatrix}\widetilde{\tau_1\tau_2} & \widetilde{\tau_1}\\ \widetilde{\tau_2} & 1\end{smallmatrix}\right)\right) + \det\left(\begin{smallmatrix}\tau_1\tau_2 & \tau_1\\ \tau_2 & 1\end{smallmatrix}\right) + \det\left(\begin{smallmatrix}\widetilde{\tau_1\tau_2} & \widetilde{\tau_1}\\ \widetilde{\tau_2} & 1\end{smallmatrix}\right)\\
&= -\det\left(\begin{smallmatrix}\tau_1\tau_2 + \widetilde{\tau_1\tau_2} & \tau_1 + \widetilde{\tau_1}\\ \tau_2 + \widetilde{\tau_2} & 2\end{smallmatrix}\right)\\
&= -4\,\mathrm{Re}\,(\tau_1\tau_2) + 4\,\mathrm{Re}\,(\tau_1)\,\mathrm{Re}\,(\tau_2)\\
&= 4\,\mathrm{Im}\,(\tau_1)\,\mathrm{Im}\,(\tau_2)
\end{aligned}$$

We choose one of the two connected components of $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}^+ = \{H = \delta \left(\begin{smallmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{smallmatrix} \right);\ \delta \in \mathbb{C}^*, \mathrm{Im}\,(\tau_1) > 0, \mathrm{Im}\,(\tau_2) > 0\}\,.$$

We give the definition of a divisor. Later on we will investigate Hirzebruch-Zagier divisors (cf. Definition 3.1.1), which are the divisors (set of zeros, sometimes counted with multiplicity) of the Borcherds products. For example in [Fr01, p. 4] we find:

Definition 1.3.3. For a subspace

$$W \subset \mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}$$

the orthogonal group $O(W)$ is embedded into $O(\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}})$ in a natural way and for every subgroup Γ of $O(\Gamma)$ we can define the projection

$$\Gamma' := \Gamma \cap O(W).$$

Moreover we choose $\mathcal{H}' := \{P \in \mathcal{H}, P \subset W\}$ and get the natural map

$$\left\{ \begin{array}{ccc} \mathcal{H}'/\Gamma' & \longrightarrow & \mathcal{H}/\Gamma \\ \Gamma'\mathbb{C}^* \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} & \longmapsto & \Gamma\mathbb{C}^* \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \end{array} \right. ,$$

which can be extended to the cusps of $\Gamma' \setminus \mathcal{H}'$.

Definition 1.3.4 (Heegner divisor). Choose $W = a^\perp$ for some $a \in \mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}$ with $q(a) < 0$ in the definition 1.3.3. Then the natural image of $\Gamma' \setminus \mathcal{H}'$ in $\Gamma \setminus \mathcal{H}$ is called a **Heegner divisor** .

An equivalent and more abstract definition can be found in [Bo99, p. 6], where the group of Heegner divisors is introduced.

1.3.1 The Operation of $\mathrm{SL}(2, \mathfrak{o})$, $G(\mathcal{K})$ and $\hat{G}(\mathcal{K})$.

The group

$$G(\mathcal{K}) := \{M \in \mathrm{GL}(2, \mathcal{K}) |\, \det M > 0, \mathrm{N}(\det M) = 1\}$$

operates on $\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathbb{C}$ by

$$(M, H) \longmapsto MHM',$$

where M' is the matrix derived from M by transposing and component wise conjugation in $\mathcal{K}$. We extend $G(\mathcal{K})$ to the group $\hat{G}(\mathcal{K})$ by

$$\hat{G}(\mathcal{K}) = G(\mathcal{K}) \cup G(\mathcal{K})\sigma$$

where

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix} \sigma.$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathfrak{o})$ and we define the operation of σ on $\mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} (\mathcal{K})$ by

$$\sigma \begin{pmatrix} h_0 & h_1 \\ \overline{h_1} & h_2 \end{pmatrix} = \begin{pmatrix} h_2 & -h_1 \\ -\overline{h_1} & h_0 \end{pmatrix}.$$

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $G(\mathcal{K})$ and a normalized element $\left(\begin{smallmatrix} \tau_1\tau_2 & \tau_2 \\ \tau_1 & 1 \end{smallmatrix}\right)$ in $\tilde{\mathcal{H}}$ we calculate

$$M \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} M' = (c\tau_1 + d)(\overline{c}\tau_2 + \overline{d}) \begin{pmatrix} \frac{a\tau_1+b}{c\tau_1+d}\frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}} & \frac{a\tau_1+b}{c\tau_1+d} \\ \frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}} & 1 \end{pmatrix}.$$

The operation of σ on $\tilde{\mathcal{H}}$ is given by

$$\sigma \left(\delta \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \right) = \delta \begin{pmatrix} 1 & -\tau_1 \\ -\tau_2 & \tau_1\tau_2 \end{pmatrix} = \tau_1\tau_2\delta \begin{pmatrix} \frac{1}{\tau_1\tau_2} & \frac{-1}{\tau_2} \\ \frac{-1}{\tau_1} & 1 \end{pmatrix}$$

which is similar to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\delta \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tau_1\tau_2\delta \begin{pmatrix} \frac{1}{\tau_1\tau_2} & \frac{-1}{\tau_1} \\ \frac{-1}{\tau_2} & 1 \end{pmatrix}$$

but interchanges τ_1 and τ_2, so we get

Remark 1.3.5. *The exchange of half planes is given by σJ:*

$$(\sigma J)\delta \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} = \delta \begin{pmatrix} \tau_2\tau_1 & \tau_2 \\ \tau_1 & 1 \end{pmatrix}.$$

Definition 1.3.6 ((Extended) orthogonal Hilbert modular form). Let $k \in \mathbb{Q}$ and $\mu : \mathrm{SL}(2, \mathfrak{o}) \cup \sigma\,\mathrm{SL}(2, \mathfrak{o}) \to \mathbb{C}^*$ a map. A holomorphic function $F : \tilde{\mathcal{H}}^+ \to \mathbb{C}$ satisfying

i) $F(tH) = t^{-k}F(H)$ for all $t \in \mathbb{C}^*$ and $H \in \tilde{\mathcal{H}}^+$,

ii) $F(MHM') = \mu(M)F(H)$ for all $M \in \mathrm{SL}(2, \mathfrak{o}) \subset G(\mathcal{K})$ and $H \in \tilde{\mathcal{H}}^+$,

is called **orthogonal Hilbert modular form** of weight k with multiplier system μ for $\mathcal{K}_p$. If F it holds $F(M\langle H \rangle) = \mu(H)F(H)$ for all $M \in \mathrm{SL}(2, \mathfrak{o}) \cup \sigma\,\mathrm{SL}(2, \mathfrak{o})$ and all $H \in \tilde{\mathcal{H}}^+$, then F is called **extended orthogonal Hilbert modular form**.

Remark 1.3.7. *We could pose a third restriction, the regularity in the cusps, but in the 2-dimensional case the Götzky-Koecher principle automatically gives the necessary growth conditions.*

Remark 1.3.8 (Integral weight). *As the first condition works only for holomorphic $F \neq 0$ if k is integral, no matter which branch of the $-k^{th}$ power we apply, all nontrivial (extended) orthogonal Hilbert modular forms have integral weight.*

Lemma 1.3.9. *For integral weights, there is a natural bijection between (extended) Hilbert modular forms and (extended) orthogonal Hilbert modular forms respecting weight and multiplier.*

Proof. Given an orthogonal Hilbert modular form F of weight $k \in \mathbb{Z}$ with multiplier system μ define

$$f : \begin{cases} \mathbb{H}^2 & \longrightarrow & \mathbb{C} \\ (\tau_1, \tau_2) & \longmapsto & F \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \end{cases}$$

Then we have for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathfrak{o})$ and $\tau \in \mathbb{H}^2$:

$$\begin{aligned} f(M\tau) &= F \begin{pmatrix} \frac{a\tau_1+b}{c\tau_1+d}\frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}} & \frac{a\tau_1+b}{c\tau_1+d} \\ \frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}} & 1 \end{pmatrix} \\ &= F\left(\frac{1}{(c\tau_1+d)(\overline{c}\tau_2+\overline{d})} M \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} M' \right) \\ &= \left(\frac{1}{(c\tau_1+d)(\overline{c}\tau_2+\overline{d})} \right)^{-k} F\left(M \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} M' \right) \\ &= \mathrm{N}(c\tau+d)^k \mu(M) F \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \\ &= \mathrm{N}(c\tau+d)^k \mu(M) f(\tau) \end{aligned}$$

and the holomorphic function f is a Hilbert modular form of weight k with multiplier systems μ. Given a Hilbert modular form f of weight $k \in \mathbb{Z}$ with multiplier system μ we define

$$F : \begin{cases} \tilde{\mathcal{H}}^+ & \longrightarrow & \mathbb{C} \\ \delta \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} & \longmapsto & \delta^{-k} f(\tau_1, \tau_2). \end{cases}$$

This holomorphic function satisfies $F(tz) = t^{-k}F(z)$ for every $t \in \mathbb{C}^*$ and $z \in \mathrm{Sym}^2(\mathcal{K}) \otimes_{\mathbb{Q}} \mathcal{K}$ and has the transformation property

$$\begin{aligned}
F\left(M \cdot \delta \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \cdot M'\right) &= F\left(\delta\, \mathrm{N}(c\tau + d) \begin{pmatrix} \frac{a\tau_1+b}{c\tau_1+d}\frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}} & \frac{a\tau_1+b}{c\tau_1+d} \\ \frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}} & 1 \end{pmatrix}\right) \\
&= \delta^{-k}\, \mathrm{N}(c\tau + d)^{-k} f\left(M\tau\right) \\
&= \delta^{-k}\, \mathrm{N}(c\tau + d)^{-k} \mu(M)\, \mathrm{N}(c\tau + d)^{k} f(\tau) \\
&= \mu(M)\delta^{-k} F \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \\
&= \mu(M) F\left(\delta \begin{pmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix}\right)
\end{aligned}$$

for all $M \in SL(2, \mathfrak{o})$ and $\delta \left(\begin{smallmatrix} \tau_1\tau_2 & \tau_1 \\ \tau_2 & 1 \end{smallmatrix}\right) \in \tilde{\mathcal{H}}$. So F is an orthogonal Hilbert modular form of weight k with multiplier system μ. The case of extended (orthogonal) Hilbert modular forms follows directly from the non-extended case, since $F(\sigma J \langle H \rangle) = \mu(\sigma J) F(H)$ corresponds to $f(\overline{\tau}) = \mu(\overline{\cdot}) f(\tau)$. $\square$

1.3.2 The Dual Lattice

Clearly $\mathrm{Sym}^2 \mathcal{K}$ is isomorphic to $\mathbb{Q}^2 \times \mathcal{K}$ by the isomorphism

$$\left(\begin{smallmatrix} a & \lambda \\ \lambda & b \end{smallmatrix}\right) \longmapsto (a, b, \lambda).$$

Therefore we can identify $(a, b, \lambda) = \left(\begin{smallmatrix} a & \lambda \\ \lambda & b \end{smallmatrix}\right)$ for all elements $\left(\begin{smallmatrix} a & \lambda \\ \lambda & b \end{smallmatrix}\right)$ of $\mathrm{Sym}^2 \mathcal{K}$ and obtain the quadratic form

$$q(a, b, \lambda) = \mathrm{N}(\lambda) - ab$$

on $\mathbb{Q}^2 \times \mathcal{K}$. In this isomorphism, b_1 corresponds to $(1, 0, 0)$, b_2 to $(0, 1, 0)$ and the basis elements b_3 and b_4 correspond to the basis elements $(0, 0, 1)$ and $\left(0, 0, \frac{1+\sqrt{p}}{2}\right)$ of $\mathfrak{o}$.

Lemma 1.3.10 (Dual lattice). *We write* $\mu \cdot \left(\begin{smallmatrix} 0 & \lambda \\ \lambda & 0 \end{smallmatrix}\right) := \left(\begin{smallmatrix} 0 & \mu\lambda \\ \mu\lambda & 0 \end{smallmatrix}\right)$. *With this, the dual lattice* $\mathcal{L}^{\#}$ *is given by* $\mathbb{Z}b_1 + \mathbb{Z}b_2 + \mathbb{Z}\frac{1}{\sqrt{p}} \cdot b_3 + \mathbb{Z}\frac{1}{\sqrt{p}} \cdot b_4$, *respectively by* $\mathbb{Z}^2 \times \frac{1}{\sqrt{p}}\,\mathfrak{o} = \mathbb{Z}^2 \times \mathfrak{d}^{-1}$, *where the discriminant* $\mathfrak{d}$ *is the ideal* $(\sqrt{p})$ *in* $\mathfrak{o}$.

Proof. We have

$$\frac{1}{\sqrt{p}} = \frac{-1 + 2\frac{1+\sqrt{p}}{2}}{p} = \frac{-1}{p} \cdot 1 + \frac{2}{p} \cdot \frac{1+\sqrt{p}}{2} \quad \text{and}$$

$$\frac{1}{\sqrt{p}}\frac{1+\sqrt{p}}{2} = \frac{\frac{p-1}{2} + \frac{1+\sqrt{p}}{2}}{p} = \frac{p-1}{2p} \cdot 1 + \frac{1}{p} \cdot \frac{1+\sqrt{p}}{2}, \tag{1.1}$$

so

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{p} & \frac{p-1}{2p} \\ 0 & 0 & \frac{2}{p} & \frac{1}{p} \end{pmatrix}$$

changes the coordinates with respect to the basis $(b_1, b_2, \frac{1}{\sqrt{p}}b_3, \frac{1}{\sqrt{p}}b_4)$ into the coordinates with respect to the basis (b_1, b_2, b_3, b_4). We consider the lattice $\mathcal{L}^{\#} = \mathbb{Z}b_1 + \mathbb{Z}b_2 + \mathbb{Z}\frac{1}{\sqrt{p}}b_3 + \mathbb{Z}\frac{1}{\sqrt{p}}b_4 = U\mathcal{L}$. It is the dual lattice of $\mathcal{L}$ if and only if $U^{\mathrm{tr}}G$ is an element of $\mathrm{GL}(4, \mathbb{Z})$, for the product of an element m of the dual lattice $\mathfrak{L}'$ with an element l of $\mathfrak{L}$, each in the corresponding basis, is given by $(Um)^t\, Gl = m^t\,(U^tG)\, l$. We calculate

$$U^tG = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL(4, \mathbb{Z}),$$

so $\mathcal{L}^{\#}$ is the dual lattice. □

Definition 1.3.11. Define $\mathfrak{e} := \left(1 + \sqrt{p}\right) / \left(2\sqrt{p}\right) + \mathcal{L}$.

Lemma 1.3.12. $\mathcal{L}^{\#}/\mathcal{L} = (\mathbb{Z}/p\mathbb{Z})\mathfrak{e}$.

Proof. We write $e := \frac{1}{\sqrt{p}}\frac{1+\sqrt{p}}{2}$. Clearly e is not an element of $\mathfrak{o}$, but $pe = \sqrt{p}\frac{1+\sqrt{p}}{2}$ is an element of $\mathfrak{o}$. Since p is prime, there is no $0 < m < p$ such that me is contained in $\mathfrak{o}$. We have

$$\frac{1}{\sqrt{p}} = -1 + 2\left(\frac{1}{2\sqrt{p}} + \frac{1}{2}\right) = -1 + 2\frac{1+\sqrt{p}}{2\sqrt{p}},$$

so $\frac{1}{\sqrt{p}} - 2e$ is an element of $\mathfrak{o}$ and we have $\mathfrak{d}^{-1} = \frac{1+\sqrt{p}}{2\sqrt{p}}\mathbb{Z} + \mathfrak{o}$. Therefore

$$\begin{aligned} \mathfrak{L}'/\mathfrak{L} &= \left\{ n\frac{1+\sqrt{p}}{2p} + b_1\mathbb{Z} + b_2\mathbb{Z} + \mathfrak{o};\ n \in \{0, 1, \ldots, p-1\} \right\} \\ &= \left\{ n\frac{1+\sqrt{p}}{2p} + \mathfrak{L};\ n \in \{0, 1, \ldots, p-1\} \right\} \\ &= (\mathbb{Z}/p\mathbb{Z})\mathfrak{e}. \end{aligned}$$

□

1.3.3 The quadratic form q on the Dual Lattice and on $\mathcal{L}^{\#}/\mathcal{L}$

On the dual lattice, the Gram matrix of q is given by

$$U^t G U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{p} & -\frac{1}{p} \\ 0 & 0 & -\frac{1}{p} & \frac{p-1}{2p} \end{pmatrix}$$

and $q(x) = \frac{1}{2} x^t U^t G U x$ holds for all $x \in \mathcal{L}^{\#}$, if $x = (x_1, x_2, x_3, x_4)$ is used for $x_1 b_1 + x_2 b_2 + x_3 \frac{1}{\sqrt{p}} b_3 + x_4 \frac{1}{\sqrt{p}} b_4$. Then we get

$$q(x_1, x_2, x_3, x_4) = -x_1 x_2 - \frac{x_3^2 + x_3 x_4}{p} + \underbrace{\frac{p-1}{4}}_{\in \mathbb{Z}} \frac{x_4^2}{p} \in \frac{1}{p}\mathbb{Z}.$$

We know that $q|_{\mathcal{L}}$ takes only integral values and since $q(b_1 + b_2) = -\det E = 1$ and $\mathcal{L}^{\#}/\mathcal{L} = (\mathbb{Z}/p\mathbb{Z})\mathfrak{e}$ we can easily define q on$\mathcal{L}^{\#}/\mathcal{L}$ modulo $\mathbb{Z}$ by

$$q(m\mathfrak{e}) = m^2 q(\mathfrak{e}) + \mathbb{Z} = m^2 q\left(\frac{1+\sqrt{p}}{2\sqrt{p}}\right) + Z = m^2 q(0,0,0,1) + \mathbb{Z} = m^2 \, \frac{p-1}{p} + \mathbb{Z}.$$

Lemma 1.3.13 (Quadratic forms on $\mathbb{F}_p = (\mathbb{Z}/p\mathbb{Z})$). *For every prime number $p \neq 2$ there are exactly 2 types of quadratic forms $q \not\equiv 0 : \mathbb{F}_p \to \mathbb{F}_p$ in the following sense: For two quadratic forms q_1, q_2 of the same type there exists $c \in \mathbb{F}_p$ such that $q_1(cx) = q_2(x)$ for all $x \in \mathbb{F}_p$. One type maps $\mathbb{F}_p$ surjectively onto the subspace of squares, the other maps surjectively onto the union of the complement and $\{0\}$.*

Proof. i) *Two quadratic forms are equivalent, if the intersection of their images does not only contain 0.:*

For every quadratic form q we have $q(n) = n^2 q(1)$ and especially $q(0) = 0^2 q(1) = 0$. Let q_1 and q_2 be quadratic forms $\mathbb{F}_p = (\mathbb{Z}/p\mathbb{Z}) \to \mathbb{F}_p$. If there are x_1 ,x_2 in $\mathbb{F}_p \setminus \{0\}$ such that $q_1(x_1) = q_2(x_2)$, then $x_1 \neq 0 \neq x_2$ holds and since $\mathbb{F}_p$ is a field, we have

$$q_1(x) = q_1(x x_1^{-1} x_1) = (x x_1^{-1})^2 q_1(x_1) = (x x_1^{-1})^2 q_2(x_2) = q_2(x x_1^{-1} x_2)$$

and q_1 and q_2 belong to the same type.

ii) *One equivalence class of quadratic forms contains the quadratic form $n \mapsto n^2$ and contains all quadratic forms whose image is the set of squares:*

The map $n \mapsto n^2$ is a quadratic form $\mathbb{F}_p \to \mathbb{F}_p$, whose image is the set of squares (in $\mathbb{F}_p$). Part i) says that if q is a quadratic form and there is $x \in \mathbb{F}_p$ such that $q(x) \neq 0$ is a square, then q is equivalent to n^2.

iii) $\mathbb{F}_p$ *contains exactly* $\frac{p-1}{2}$ *squares:*

For $x \in \mathbb{Z}$ we have $(p-x)^2 \equiv p^2 - 2px + x^2 \equiv x^2 \pmod p$ and $(p+x)^2 \equiv p^2 + 2px + x^2 = x^2$. The second equation guarantees that all squares in $\mathbb{F}_p$ but 0 are given by $1^2, \dots, (p-1)^2$. The first equation shows, that it suffices to consider $1^2, \dots, ((p-1)/2)^2$. So there are at most $(p-1)/2$ squares unequal to 0.

Let $x > y$ be in $\{1, 2, \dots, \frac{p-1}{2}\}$. Then

$$x^2 - y^2 = \underbrace{(x-y)}_{\in\{1,\dots,\frac{p-3}{2}\}} \cdot \underbrace{(x+y)}_{\in\{3,\dots,p-2\}} \not\equiv 0 \pmod p \,.$$

Hence there are exactly $\frac{p-1}{2}$ non-zero squares in $\mathbb{F}_p$.

iv) *Every two not identically vanishing quadratic forms not equivalent to* n^2 *are equivalent:*

We denote by Q the set of non-zero square numbers in $\mathbb{F}_p$. So for every quadratic form q its image is given by $Q \cdot q(1) \cup \{0\}$. Since Q has exactly $\frac{p-1}{2}$ elements, there are exactly 2 nontrivial Q orbits in $\mathbb{F}_2$ one of which contains the squares.

$\square$

Definition 1.3.14. The **Legendre symbol** is given by

$$\left(\frac{d}{p}\right) = \begin{cases} 0, & \text{if } d \equiv 0 \pmod p, \\ 1, & \text{if } d \not\equiv 0 \pmod p \text{ and } d \text{ is a square modulo } p, \\ -1, & \text{else.} \end{cases}$$

In order to calculate the Legendre symbol, we can either calculate all squares $0^2, 1^2, \dots (p-1)^2$ (this suffices, for all squares are one of those modulo p), or we use the Euler criterion (cf. [Le96, chapter 5.1]):

Theorem 1.3.15 (Euler criterion). *For every prime number* $p \neq 2$ *and every integer* m *we have*

$$\left(\frac{m}{p}\right) \equiv m^{\frac{1}{2}(p-1)} \pmod p$$

So $\left(\frac{-1}{p}\right) \equiv (-1)^{(p-1)/2} \equiv 1 \pmod p$, since $p \equiv 1 \pmod 4$. We check

$$q(0,0,1,0) = q(1/\sqrt{p}) = -1/p + \mathbb{Z}$$

and obtain

Remark 1.3.16. *q represents the squares, i.e. the image of q contains the squares in $\mathbb{Z}/p\mathbb{Z}$. Hence there is $\alpha \in (\mathbb{Z}/p\mathbb{Z})$ such that*

$$q(n\mathfrak{e}) = \alpha n^2/p$$

and $\alpha = (p-1)/4 + p\mathbb{Z}$ is a square, i.e.

This is of course equivalent to

Remark 1.3.17. *There is* $v \in \mathcal{L}^{\#}/\mathcal{L}$ *with*

$$q(nv) = n^2/p.$$

Proof. By Remark 1.3.16 α is a square modulo p, so there is β with $\alpha = \beta^2$. Then β^{-1} is an element of the field $\mathbb{F}_p$ and $q(\beta^{-1}\mathfrak{e}) = \alpha\beta^{-2}/p = 1/p$. $\square$

2 Some Modular Forms

We give some examples of Hilbert modular forms and of other modular forms important in our case. These are Theta series, which are Siegel modular forms and can be restricted to Hilbert modular forms using the modular embedding of Hammond, elliptic modular forms with character, since the restriction of Hilbert modular forms to the diagonal yields elliptic modular forms with characters and elliptic modular forms for congruence subgroups, which are isomorphic to vector valued modular forms and arise in Borcherds' theory.

2.1 Hilbert Eisenstein Series

We define Hilbert Eisenstein series and state that the ring of Hilbert modular forms for even weight and trivial multiplier system is the direct sum of the space of Eisenstein series and the space of cusp forms. The proofs can be found in [Fr90, p. 60 - 66]. Additionally we give Hecke's way of calculating the Fourier coefficients of the Eisenstein series as they are explained in [Si69].

Definition 2.1.1. Given $k \in \mathbb{N}$ we define

$$E_{2k}^H : \begin{cases} \mathbb{H}^2 & \longmapsto & \mathbb{C} \\ \tau & \longrightarrow & \sum_{M \in \Gamma_\infty \backslash \Gamma} 1|_{2k} M = \sum_{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \mathrm{N}(c\tau + d)^{-2k}. \end{cases}$$

The function E_{2k}^H is called **Eisenstein series** of weight $2k$ with respect to the cusp ∞.

Proposition 2.1.2. *The Eisenstein series E_{2k}^H converges absolutely for $k \geq 1$ and represents an extended Hilbert modular form of weight $2k$ with trivial multiplier system, which has the value 1 at the cusp ∞. It vanishes in all the other cusps.*

Proof. Freitag proves most of this, but only shows that E_{2k}^H is a Hilbert modular form not an extended Hilbert modular form. Since it is $\hat{\Gamma}/\hat{\Gamma}_\infty = \Gamma/\Gamma_\infty$ (compare Remark 1.2.24), it follows $E_{2k}^H(\overline{\tau}) = E_{2k}^H(\tau)$ for all $\tau \in \mathbb{H}^2$ immediately. $\square$

The importance of Eisenstein series is given by the following proposition, which can be found in [Fr90].

Proposition 2.1.3. *For every Hilbert modular form f of even weight $2k \geq 2$ with trivial multiplier system, there is an unique element E in the space spanned by all Eisenstein series of weight $2k$, such that $f - E$ is a cusp form.*

In case of a single cusp of Γ, this is the trivial observation that for every Hilbert modular form f of even weight $2k$, $f - \alpha_f(0)E^H_{2k}$ is a cusp form.

Siegel [Si69] described a way of calculating the Fourier coefficients of Hilbert Eisenstein series. He considers a more general definition for Hilbert Eisenstein series (he calls Hecke Eisenstein series) than we did so far and uses a notation which has to be explained before using it. He denotes the degree of the number field $\mathcal{K}$ by g, its discriminant by d and in case that there are units in $\mathfrak{o}$ with negative norm he considers an even natural number $k > 0$. Then in our case $g = 2$ and $d = p$. For every ideal $\mathfrak{u}$ and the fundamental ideal $\mathfrak{d} = (\sqrt{p}) = \sqrt{p} \cdot \mathfrak{o}$ he defines $\mathfrak{u}^* = (\mathfrak{u}\mathfrak{d})^{-1}$. Siegel defines the Hecke Eisenstein series

$$F_k(\mathfrak{u}, z) = \mathrm{N}(\mathfrak{u}^k) \sum_{\mathfrak{u}|(\lambda,\mu)}{}' \mathrm{N}(\lambda z + \mu)^{-k}, \quad z \in \mathbb{H}^2 \qquad \text{for all} z \in \mathbb{H}^2 \text{ and all ideals } \mathfrak{u} \text{ in } \mathcal{K},$$

where the summation ranges over a set of representatives $(\lambda, \mu) \neq (0, 0)$ of $\mathfrak{u} \times \mathfrak{u}/\ \mathfrak{o}^*$, where $\mathfrak{o}^*$ operates on $\mathfrak{u} \times \mathfrak{u}$ by componentwise multiplication. The series has the Fourier expansion:

$$\begin{aligned} F_k(\mathfrak{u}, z) &= \zeta(\mathfrak{u}, k) + \left(\frac{(2\pi i)^k}{(k-1)!}\right)^2 d^{\frac{1}{2}-k} \sum_{\mathfrak{d}^{-1}|\nu\gg 0} \sigma_{k-1}(\mathfrak{u}, \nu) e^{2\pi i S(\nu z)}, \text{ where} \\ \zeta(\mathfrak{u}, k) &= \mathrm{N}(\mathfrak{u}^k) \sum_{\mathfrak{u}|(\mu)} \mathrm{N}(\mu^{-k}) \text{ and} \\ \sigma_{k-1}(\mathfrak{u}, \nu) &= \sum_{\mathfrak{d}^{-1}|(\alpha)\mathfrak{u}|\nu} \mathrm{sign}(\mathrm{N}(\alpha^k))\, \mathrm{N}((\alpha)\mathfrak{u}\mathfrak{d})^{k-1}. \end{aligned}$$

There he summarizes over all principal ideals (μ), (α) under the given restrictions and ν ranges over all totally positive numbers in $\mathfrak{d}^{-1}$. In our case $\mathrm{N}(\varepsilon_0) = -1$ and thus k is even. So σ_{k-1} can be rewritten into

$$\sigma_{k-1}(\mathfrak{u}, \nu) = \sum_{\substack{t\in\mathfrak{u}\mathfrak{d} \\ t|(\nu)\mathfrak{d}}} \mathrm{N}(t^{k-1})$$

Now we can substitute k by $2k$, write τ for z and define $\mathfrak{u} = \mathfrak{o}$ and $d = p$:

$$\begin{aligned} \zeta(2k)E^H_{2k}(\tau) = F_{2k}(\mathfrak{o}, \tau) &= \zeta(2k) + \left(\frac{(2\pi i)^{2k}}{(2k-1)!}\right)^2 \sqrt{p}^{1-4k} \sum_{\substack{\nu\in\mathfrak{d}^{-1} \\ \nu\gg 0}} \sigma_{2k-1}(\nu) e^{2\pi i S(\nu\tau)}, \text{ where} \\ \zeta(2k) &= \sum_{\text{ideals } (\mu)} \mathrm{N}(\mu^{-2k}) = \sum_{\text{ideals } (\mu)} \mathrm{N}(\mu)^{-2k} \text{ and} \\ \sigma_{2k-1}(\nu) &= \sum_{\substack{(t)|(\sqrt{p}\nu) \\ t\in\sqrt{p}\,\mathfrak{o}}} \mathrm{N}((t)^{2k-1}) = \sum_{\substack{(t)|(\sqrt{p}\nu) \\ t\in\sqrt{p}\,\mathfrak{o}}} \mathrm{N}(t)^{2k-1} \end{aligned}$$

Note that we have written $\zeta(2k)E^H_{2k}(\tau) = \zeta(2k) + \sum_\nu a_\nu e^{2\pi i S(\nu\tau)}$ with some known a_ν. Siegel advises to restrict the Fourier expansion of E^H_{2k} to the diagonal $\mathrm{Diag} = \{\tau \in \mathbb{H}^2;\ \tau_1 = \tau_2\}$, which yields an elliptic modular form as we will see in section 4.2. Its weight and character are known, so we can give a finite dimensional space of elliptic modular forms in which it is contained. As its Fourier expansion has constant coefficient $\zeta(2k)$ and the other coefficients can be calculated explicitely, we can easily calculate $\zeta(2k)$ by linear algebra.

Remark 2.1.4. *Some of the (truncated) Fourier expansions of Eisenstein series can be found in the tables A.8, A.10 and A.12 in the appendix.*

2.2 Theta Series and Modular Embedding

Siegel modular forms can be restricted to Hilbert modular forms by the modular embedding of Hammond. It is described in Hammond's two papers [Ha66a] and [Ha66b], of which the second one is just a short summary of the first one, so both papers share the same title: "The Modular Groups of Hilbert and Siegel". Note that Hammond uses the term "modular imbedding", while we will use the term "modular embedding" instead. We will use Hammond's embedding for theta products and give a first result for the ring of Hilbert modular forms.

Definition 2.2.1 (Modular embedding, $\mathfrak{S}_n$ and $\mathrm{Sp}(n,\mathbb{R})$). We denote the Siegel half space by $\mathfrak{S}_n := \{X + iY;\ X, Y \in \mathbb{R}^{n\times n}, X + iY \text{ symmetric}, Y > 0\}$ and the symplectic group by $\mathrm{Sp}(n,\mathbb{R}) := \left\{M \in \mathbb{R}^{2n\times 2n};\ M^{\mathrm{tr}} \left(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & E_n \\ -E_n & 0 \end{smallmatrix}\right)\right\}$. We define the diagonal embedding (φ_0, Ψ_0) by

$$\varphi_0 : \begin{cases} \mathbb{H}^n & \longrightarrow & \mathfrak{S}_n \\ x & \longmapsto & \mathrm{Diag}(x) \end{cases} \qquad \text{and}$$

$$\Psi_0 : \begin{cases} \mathrm{Sp}(1,\mathbb{R})^n & \longrightarrow & \mathrm{Sp}(n,\mathbb{R}) \\ \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)\right) & \longmapsto & \left(\begin{smallmatrix} \mathrm{Diag}(a_1,\ldots,a_n) & \mathrm{Diag}(b_1,\ldots,b_n) \\ \mathrm{Diag}(c_1,\ldots,c_n) & \mathrm{Diag}(d_1,\ldots,d_n) \end{smallmatrix}\right) \end{cases}.$$

A **modular embedding** of $\mathcal{K}$ is a pair (φ, Ψ) consisting of a holomorphic injection φ from $\mathbb{H}^n$ into the Siegel half space $\mathfrak{S}_n$ and a monomorphism Ψ from $\mathrm{Sp}(1,\mathbb{R})^n$ to $\mathrm{Sp}(n,\mathbb{R})$, such that

(i) there is $N \in \mathrm{Sp}(n,\mathbb{R})$ such that $\varphi(\tau) = N\varphi_0(\tau)$ and $\Psi(m) = N\Psi_0(m)N^{-1}$ for all $\tau \in \mathbb{H}^n$ and $m \in \mathrm{Sp}(1,\mathbb{R})^n$,

(ii) $\Psi(\mathrm{Sp}(1,\mathfrak{o})) \subset \mathrm{Sp}(n,\mathbb{Z})$,

(iii) if f is a Siegel modular form of weight k, then the composition $f \circ \varphi$ is a Hilbert modular form of weight k for $\mathcal{K}$.

Proposition 2.2.2 (Proposition 2.2 in [Ha66a]). *The restriction (iii) of Definition 2.2.1 can be replaced by* *(iii') the matrix N from (i) holds $c = 0_n$.*

Definition 2.2.3. We call two modular embeddings (φ_1, Ψ_1) and (φ_2, Ψ_2) equivalent, if there is an element $M \in \mathrm{Sp}(n, \mathbb{Z})$ such that $\varphi_2 = M\varphi_1$ and $\Psi_2 = M\Psi_1 M^{-1}$ hold.

In the case of Hilbert modular forms for quadratic number fi elds we obtain the following

Theorem 2.2.4 (Theorem 3.4 in [Ha66a]). *Let $\mathcal{K}$ be the real quadratic number field of discriminant D. The orthogonal modular embeddings for K correspond in an one-to-one manner to ordered pairs (u, v) of integers such that:*

1) $D = u^2 + v^2$

2) v is even.

This can be reformulated into

Theorem 2.2.5 (Theorem 3.6 in [Ha66a]). *Let $\mathcal{K}$ be a totally real quadratic number field of discriminant D and let t be the number of prime divisors of D. There are modular embeddings for $\mathcal{K}$ if and only if D contains no prime divisor of the type $4m+3$ (where $m \in \mathbb{N}_0$). In this case, the number of modular embeddings for $\mathcal{K}$ is given by 2^{t-1}.*

Remark 2.2.6. *In case $p \in \{5, 13, 17\}$ there is exactly one equivalence class of modular embeddings by Theorem 2.2.5 (then $p = D \equiv 1 \pmod 4$). We have $5 = 1^2 + 2^2$, $13 = 3^2 + 2^2$ and $17 = 1^2 + 4^2$.*

Müller [Mü83] gives an explicit formulation of the modular embedding for totally real quadratic number fi elds:

Example 2.2.7. Let $\mathcal{K} = \mathbb{Q}(\sqrt{D})$ where $D = u^2 + v^2$, u, v in $\mathbb{Z}$ and v even and $\omega := \frac{1}{2}(u + \sqrt{D})$. Then a modular embedding is given by the pair (ψ, Ψ), where

$$\psi(\zeta) = \begin{pmatrix} \mathrm{S}\left(\frac{\omega}{\sqrt{D}}\zeta\right) & \mathrm{S}\left(\frac{v}{2\sqrt{D}}\zeta\right) \\ \mathrm{S}\left(\frac{v}{2\sqrt{D}}\zeta\right) & \mathrm{S}\left(\overline{\left(\frac{\omega}{\sqrt{D}}\right)}\zeta\right) \end{pmatrix}$$

and

$$\Psi(M) = \begin{pmatrix} \psi(a) & \psi(b) \\ \psi(c) & \psi(d) \end{pmatrix}.$$

In this $\mathrm{S}\,(\alpha\zeta) = \alpha\zeta_1 + \overline{\alpha}\zeta_2$ for $\alpha \in \mathfrak{o}$ and $\zeta \in \mathbb{H}^2$. Details about Fourier coeffi cients of Hilbert modular forms, which can be obtained this way, can be found in [Mü83].

The following defi nition of θ_m and θ can be found in Müller [Mü85], Hammond [Ha66a] and Hermann [He81], of which the last forgets the $\frac{m'}{2}$ in the defi nition.

Definition 2.2.8 (θ_m, θ, Θ_m and Θ). Given m' and m'' in $\{0,1\}^2$ with $m_1'm_1''+m_2'm_2'' \in 2\mathbb{Z}$ write $m=(m',m'')$ and define

$$\theta_m(\tau)=\sum_{g\in\mathbb{Z}^2}\exp\pi i\left(\left(g+\frac{m'}{2}\right)^t\tau\left(g+\frac{m'}{2}\right)+g^tm''+(m')^tm''/2\right),\quad \tau\in\mathfrak{S}_2$$

There are exactly 10 such theta series and we denote their product by θ. Additionally we define $\Theta_m=\theta_m\circ\psi$ and $\Theta=\theta\circ\psi$.

Hammond [Ha66a, p. 507] writes that his modular embedding produces symmetric modular forms, so we have

Lemma 2.2.9. *If f is a polynomial in the Θ_m, then $f(\overline{\tau})=f(\tau)$ holds for all $\tau\in\mathbb{H}^2$. Hence all modular embeddings of Siegel modular forms, which are polynomials in theta products, are extended Hilbert modular forms.*

Theorem 2.2.10 (Theorem 4.1 in [Ha66a]). *If $\mathcal{K}$ is a totally real quadratic number field, which's discriminant is the sum of two squares, then there are three algebraically independent (extended) Hilbert modular forms for $\mathcal{K}$ of weight 4, 6 and 10, namely E_4^H, E_6^H and Θ^2.*

In case $\mathcal{K}=\mathbb{Q}(\sqrt{17})$ Hermann (cf. [He81]) constructs another Hilbert modular form coming from theta products:

Lemma 2.2.11 ([He81]). *In case $\mathcal{K}=\mathbb{Q}(\sqrt{17})$ there is an extended Hilbert modular form we denote by η_2 of weight $\frac{3}{2}$ with multiplier system μ_{17} ($\mu_{17}(J)=-i$, $\mu_{17}(T)=i$ and $\mu_{17}(T_w)=e^{5\pi i/4}$):*

$$\begin{aligned}\eta_2 :=&\Theta_{1100}\Theta_{0011}\Theta_{0000}+\Theta_{1100}\Theta_{0010}\Theta_{0001}\\&+\Theta_{1001}\Theta_{0110}\Theta_{0000}-\Theta_{1001}\Theta_{0100}\Theta_{0010}\\&+\Theta_{1000}\Theta_{0100}\Theta_{0011}-\Theta_{1000}\Theta_{0110}\Theta_{0001}.\end{aligned}$$

Remark 2.2.12. *In order to calculate a finite number of summands of the Fourier expansion of Θ_m, we have to use*

$$\theta_m(\tau)\approx\sum_{g\in\{-N,\dots,N\}^2}\exp\pi i\left(\left(g+\frac{m'}{2}\right)^t\tau\left(g+\frac{m'}{2}\right)+g^tm''+(m')^tm''/2\right)$$

and choose N large enough. We calculate

$$\begin{aligned}e^{\pi i\left(\left(g+\frac{m'}{2}\right)^t\tau\left(g+\frac{m'}{2}\right)+g^tm''+(m')^tm''/2\right)}&=c_ge^{\pi i\left(g^{\mathrm{tr}}\tau g+g^{\mathrm{tr}}\tau\frac{m'}{2}+\left(\frac{m'}{2}\right)^{\mathrm{tr}}\tau g\right)}\\&=c_ge^{\pi i\left(g^{\mathrm{tr}}\tau g+g^{\mathrm{tr}}\left(\tau+\tau^{\mathrm{tr}}\right)\frac{m'}{2}\right)}\end{aligned}$$

with c_g depending only on g, m' and m'' and since τ is symmetric, this can be reformulated into

$$\begin{aligned} c_g e^{\pi i \left(g^{\mathrm{tr}} \tau g + g^{\mathrm{tr}} \left(\tau + \tau^{\mathrm{tr}} \right) \frac{m'}{2} \right)} &= c_g e^{\pi i \left(g^{\mathrm{tr}} \tau g + g^{\mathrm{tr}} \tau m' \right)} \\ &= c_g e^{\pi i g^{\mathrm{tr}} \tau (g + m')}. \end{aligned}$$

We write $\tau = A^{\mathrm{tr}} A$ and get $g^{\mathrm{tr}} \tau (g + m') = (Ag)^{\mathrm{tr}} (Ag + Am')$. In the Fourier expansion, the coefficient of $g^m := e^{\pi i m (\tau_1 + \tau_2)}$ is a polynomial in $h := e^{\pi i m (\tau_1 - \tau_2)/\sqrt{p}}$ and h^{-1}, so we focus on the powers of g (compare section 5.3) and get

$$c_g e^{\pi i g^{\mathrm{tr}} \tau (g + m')} = c_g d(\tau_1 - \tau_2) e^{\pi i (\tau_1 + \tau_2) g^{\mathrm{tr}} M (g + m')}$$

with a real symmetric matrix M of rank 2. So, for large $\|g\|_\infty$, it is $\|g^{\mathrm{tr}} M (g + m')\|_\infty \geq \|g^{\mathrm{tr}} M g\|_\infty - \|g^{\mathrm{tr}} M m'\|_\infty =: (g, g)_M - (g, m')_M$ and the equivalency of norms can be used to find N appropriate for a given number of Fourier coefficients (depending on the concrete shape of M).

2.3 Hilbert Poincaré Series

If we take a bounded holomorphic function $\mathbb{H}^2 \to \mathbb{C}$ and summarize over the shifted quotients of this function over some factor of automorphy, we obtain a Hilbert modular form, which we call Hilbert Poincaré series. For each multiplier system there is a Hilbert Poincaré series which does not vanish identically.

We modify Freitag's defi nition [Fr90, Prop I 5.3] in order to get Hilbert modular forms with nontrivial multiplier systems:

Proposition and Definition 2.3.1. *Let $\varphi : \mathbb{H}^2 \to \mathbb{C}$ be a bounded holomorphic function, $k \in \mathbb{Q}$, $k \geq 4$, $w \in \mathbb{H}^2$ and $\mu : \Gamma \to \mathbb{S}^1 = \{z \in \mathbb{C};\ |z| = 1\}$ a multiplier system. The series*

$$F(z) = F^{(k)}_{\varphi,\mu}(\tau) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M \in \Gamma} \frac{\varphi(M\tau)}{\mathrm{N}(M\tau - \overline{w})^k\, \mathrm{N}(c\tau + d)^k \mu(M)} = \sum_{M \in \Gamma} \frac{\varphi|^\mu_k M(\tau)}{\mathrm{N}(M\tau - \overline{w})^k}$$

converges absolutely and uniformly on compact subsets of $\mathbb{H}^2$. It therefore represents a holomorphic function on $\mathbb{H}^2$. This function satisfies the transformation law

$$F|^\mu_k M = F \qquad \text{for all } M \in \mathrm{SL}(2, \mathfrak{o}).$$

Proof. In case $\mu \equiv 1$ and $k \in \mathbb{Z}$, $k \geq 4$, the proposition is a corollary of [Fr90, Proposition I 5.3], where Γ is any discrete subgroup of $\mathrm{SL}(2, \mathbb{R})^n$. In our case, $\Gamma = \mathrm{SL}(2, \mathfrak{o})$ is isomorphic to such a subgroup with the same operation on $\mathbb{H}^2$.

We have

$$\left|\frac{\varphi(M)\tau}{\mathrm{N}(M\tau-\overline{w})^k\,\mathrm{N}(c\tau+d)^k\mu(M)}\right|$$
$$\leq \max\left(\left|\frac{\varphi(M)\tau}{\mathrm{N}(M\tau-\overline{w})^{\lfloor k\rfloor}\,\mathrm{N}(c\tau+d)^{\lfloor k\rfloor}}\right|,\left|\frac{\varphi(M)\tau}{\mathrm{N}(M\tau-\overline{w})^{\lceil k\rceil}\,\mathrm{N}(c\tau+d)^{\lceil k\rceil}}\right|\right),$$

where $\lfloor k\rfloor = \max\{n\in\mathbb{Z};\ n\leq k\}$ and $\lceil k\rceil = \min\{n\in\mathbb{Z};\ n\geq k\}$, for all $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ and $\tau\in\mathbb{H}^2$. Thus absolute and uniform convergence follow immediately from the case $\mu\equiv 1$. We have $F=\sum_M f|M$ for $f(z)=\varphi(z)/\,\mathrm{N}(z-\overline{w})$.

Remember the multiplicative structure of multiplier systems introduced in Defi nition 1.2.8. It says that $f|M|N = f|(MN)$ holds. Thus $F|N=\sum_M f|M|N=\sum_M f|(MN) = F$ proves the proposition. □

Definition 2.3.2. All series of the type $\sum_{M\in\Gamma} f|M$ are called **Poincaré series** .

The following proposition can be found as a remark in [Gu88]. We will give a proof.

Proposition 2.3.3. *The Poincaré series defined in Proposition 2.3.1 define cusp forms of weight k. Given w, μ and k as in the proposition, there is always some $r\in\mathbb{Q}$ and bounded $\varphi:\mathbb{H}^2\to\mathbb{C}$, such that $F^{(r)}_{\varphi,\mu}$ does not vanish identically and defines a Hilbert modular form with multiplier system μ.*

Proof. The weight is clear from the defi nition. Freitag proves that for even k and $\mu\equiv 1$ the Poincaré series $F^{(k)}_{\varphi,1}$ is a cusp form, by showing that each summand of $F^{(k)}_{\varphi,1}$ converges with $\tau\to\infty$ to 0 and summation and limit can be interchanged. We can redo this for rational $k\geq 4$, almost as we did for the proof of the convergence of the Poincaré series, by comparison with $F^{(2\lfloor k/2\rfloor)}_{\varphi,1}$ and $F^{(2\lceil k/2\rceil)}_{\varphi,1}$.

In case $\mu\not\equiv 1$ we will see in Remark 4.1.9, that all Hilbert modular forms with multiplier system μ are cusp forms.

Let k, μ and φ be as in proposition 2.3.1. We want to fi nd $r\in\mathbb{Q}, r\geq 4$ and $\tilde\varphi$ such that $F^{(r)}_{\tilde\varphi,\mu}$ does not vanish identically. We enumerate Γ together with the summands and write $F^{(r)}_{\tilde\varphi,\mu} =: \sum_n a_n\mu(n)$, where $\mu(n)$ equals μ applied to the n-th matrix in Γ. In case $\mu\equiv 1$, Freitag uses the fact that if $\sum_n a_n^m$ converges absolutely and $\sum_n a_n^m = 0$ for all but fi nitely many $m\in\mathbb{N}$, then $a_n=0$ for all $n\in\mathbb{N}$. This is not valid for general μ, but it remains valid, if we require that for some $n_0\in\mathbb{N}$ the term a_{n_0} has larger absolute value than the other a_n:

Lemma 2.3.4. *Given $k\in\mathbb{Q}$, $k>0$, a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n\in\mathbb{C}$, such that $\sum_n a_n^{k+m}$ converges absolutely for all $m\in\mathbb{N}$ (where $a_n^{k+m}:=a_n^k a_n^m$ with a_n^k defined independently of m as some k^{th} power of a_n), and $\mu:\mathbb{N}\to\mathbb{S}^1$, such that $\sum_n \mu(n)a_n^{k+m}=0$ for all $m\in\mathbb{N}$, then there is no $n_0\in\mathbb{N}$ such that $|a_{n_0}|>|a_n|$ for all $n\in\mathbb{N}\setminus\{n_0\}$.*

Proof. We give an indirect proof. Assume that $|a_1| > |a_n|$ for all $n > 1$. Without loss of generality, we have $a_1 = 1$. Since $\sum_{n=2}^{\infty} \mu(n) a_n^{k+m}$ converges absolutely for all $m \in \mathbb{N}$ and $|a_n| < 1$ for all $n \geq 2$, there is $R \in \mathbb{N}$ such that $|\sum_{n=2}^{\infty} \mu(n) a_n^{k+m}| < \frac{1}{2}$ for all $m \geq R$. So we have $\left|\sum_{n=1}^{\infty} \mu(n) a_n^{k+m}\right| \geq 1 - \frac{1}{2} = \frac{1}{2}$ for all $m \geq R$ contradicting $\left|\sum_{n=1}^{\infty} \mu(n) a_n^{k+m}\right| = 0$. $\square$

We choose some $\tau_0 \in \mathbb{H}^2$ and consider

$$a_M^{k,\varphi} = \frac{\varphi(M\tau_0)}{\mathrm{N}(M\tau_0 - \overline{w})^k \, \mathrm{N}(c\tau_0 + d)^k \mu(M)}$$

for all M in Γ. In order to use the lemma, we want to change φ, such that there is an unique M whith $|a_M^{k,\varphi}|$ maximal. We have $|a_M^{k,\varphi}| = |a_{-M}^{k,\varphi}|$, so we choose a set Γ^+ of representatives of $\Gamma / \{\pm E\}$ and get $\sum_{M \in \Gamma} a_M^{k,\varphi} = 2 \sum_{M \in \Gamma^+} a_M^{k,\varphi}$. The group Γ^+ acts properly on $\mathbb{H}^2$. We know that $\sum_{M \in \Gamma^+} a_M^{k,\varphi}$ converges absolutely, so

(1) there is $M_0 \in \Gamma$ such that $|a_{M_0}^{k,\varphi}| \geq |a_M^{k,\varphi}|$ for all $M \in \Gamma^+$,

(2) the set $A := \left\{ M \in \Gamma^+ \setminus \{M_0\} ;\ |a_M^{k,\varphi}| = |a_{M_0}^{k,\varphi}| \right\}$ is fi nite and

(3) $d := \sup \left\{ \frac{|a_M^{k,\varphi}|}{|a_{M_0^{k,\varphi}}|} ;\ M \in \Gamma^+ \setminus A, M \neq M_0 \right\}$ is a positive number smaller than 1.

We defi ne the biholomorphic map

$$\psi : \begin{cases} \mathbb{D}^2 = \{z \in \mathbb{C};\ |z| < 1\}^2 & \longrightarrow & \mathbb{H}^2 \\ (\tau_1, \tau_2) & \longmapsto & \left(\frac{\tau_1 - 1}{i - \tau_1}, \frac{\tau_2 - 1}{i - \tau_2} \right) \end{cases}$$

and the holomorphic map

$$\delta : \begin{cases} \mathbb{H}^2 & \longrightarrow & \mathbb{C} \\ \tau & \longmapsto & \prod_{x \in \mathbb{D}^2, \psi(x) \in A\tau_0} \frac{\|\psi^{-1}(\tau) - x\|_2^2}{\|\psi^{-1}(M_0\tau_0) - x\|_2^2} \end{cases} .$$

The map δ simply is a polynomial on $\mathbb{D}^2$ lifted to a map on $\mathbb{H}^2$, vanishing in all points $M\tau_0$ with $M \in A$ and of value $\delta(M_0\tau_0) = 1$. Here we need the proper action of Γ^+ to guarantee $M\tau_0 \neq M_0\tau_0$ for all $M \in A$. So $|\delta \circ \psi|$ obtains a maximum on $\mathbb{D}^2$ which we denote by δ_{Max}. We defi ne the function

$$\tilde{\varphi} : \begin{cases} \mathbb{H}^2 & \longrightarrow & \mathbb{C} \\ \tau & \longmapsto & \varphi(\tau) \left(1 + \frac{1-d}{d} \frac{\delta(\tau)}{\delta_{\mathrm{Max}}} \right) \end{cases}$$

and get

$$|\tilde{\varphi}(M\tau_0)| = \begin{cases} |\varphi(M_0\tau_0)| \cdot \left(1 + \frac{1-d}{d}\frac{1}{\delta_{\text{Max}}}\right) & \\ \qquad > |\varphi(M_0\tau_0)|, & \text{if } M = M_0 = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right), \\ |\varphi(M\tau_0)|, & \text{if } M \in A, \\ |\varphi(M\tau_0)| \cdot \left|1 + \frac{1-d}{d}\frac{\delta(M\tau_0)}{\delta_{\text{Max}}}\right| & \\ \qquad \le |\varphi(M\tau_0)| \cdot \left(1 + \frac{1-d}{d}\right) & \\ \qquad = \frac{1}{d}\left|\varphi\left(M\tau_0\right)\right|, & \text{if } M \in \Gamma \setminus (A \cup \{M_0\}). \end{cases}$$

The holomorphic function $\tilde{\varphi}$ is bounded, so Proposition 2.3.1 guarantees that for all $m \in \mathbb{N}$ the function $F^{(k+m)}_{\tilde{\varphi},\mu}$ is a Hilbert modular form of weight $k+m$. We get

$$\left|a^{k,\tilde{\varphi}}_M\right| \begin{cases} > \left|a^{k,\varphi}_{M_0}\right|, & \text{if } M = M_0, \\ = \left|a^{k,\varphi}_{M_0}\right|, & \text{if } M \in A, \\ \le \frac{1}{d}\left|a^{k,\varphi}_{M}\right| \le \left|a^{k,\varphi}_{M_0}\right|, & \text{if } M \in \Gamma \setminus (A \cup \{M_0\}). \end{cases}$$

and hence know by Lemma 2.3.4 that there is $m \in \mathbb{N}$ such that $F^{(k+m)}_{\tilde{\varphi},\mu} = \sum_M a^{k+m,\tilde{\varphi}}_M$ does not vanish in τ_0. □

We give a result for Hilbert modular forms in the special case of $\mathbb{H}^2$ and $\Gamma = \mathrm{SL}(2,\mathfrak{o})$. The general case can be found in [Fr90, I.5]

Theorem 2.3.5 (Existence theorems). *I) Let $a, b \in \mathbb{H}^2$ be points which are inequivalent with respect to Γ. There exists a Poincaré series F (hence a cusp form) of suitable weight such that*

$$F(a) = 0, \quad F(b) = 1.$$

II) There exist three Poincaré series

$$F_0, F_1, F_2$$

of a suitable common weight, which are algebraically independent.

2.4 Elliptic Modular Forms with Character

We give a short introduction, fi x notations and give some well known facts about elliptic modular forms with trivial character. We will always consider the normalized form of the modular forms. The ring of elliptic modular forms with character comes as a corollary. Most of this section can be found in [KK98].

Every subgroup of $\mathrm{SL}(2,\mathbb{R})$ operates on $\mathbb{H}$ by the corresponding Moebius transformations and we write $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$.

Definition 2.4.1 (Elliptic modular form). A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called **meromorphic elliptic modular form** of weight $k \in \mathbb{Q}$ with multiplier system $\mu : \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}^*$ if

(i) $f(Mz) = \mu(M)(cz+d)^k f(z) = \mu(M)e^{k\ln(cz+d)} f(z)$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{H}$, where $\ln$ is the main branch of the complex logarithm,

(ii) $f(\infty) := \lim_{\mathrm{Im}(z)\to\infty} f(z)$ exists (in $\mathbb{C} \cup \{\infty\}$) and

(iii) There is $k \in \mathbb{N}$ such that $\mu^k = 1$ (μ has finite order).

If f is a holomorphic function and $f(\infty)$ is a complex number, f is called **(holomorphic) elliptic modular form**. If additionally $f(\infty) = 0$, we call f an elliptic cusp form.

Remark 2.4.2. *Since μ is finite, there is $M \in \mathbb{N}$ such that $f(z+M) = f(z)$ for all $z \in \mathbb{H}$ and the limit in (ii) can be restricted to bounded real part of z. The meromorphic elliptic modular form f is a holomorphic elliptic modular form, if and only if all the Fourier coefficients belonging to negative exponents of $q = e^{2\pi iz}$ vanish.*

We give some important examples of elliptic modular forms:

Definition 2.4.3 (Eisenstein series E_k). For given $k \in 2\mathbb{Z}$, $k \geq 4$, we define

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} (mz+n)^{-k} = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2,\gcd(m,n)=1} (mz+n)^{-k} = \sum_{M\in\mathrm{SL}(2,\mathbb{Z})_\infty\setminus\mathrm{SL}(2,\mathbb{Z})} 1|_k M.$$

This defines an elliptic modular form of weight k with trivial multiplier system $\mu = 1$, we call (normalized) **elliptic Eisenstein series of weight** k. It has the Fourier expansion

$$E_k(z) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{m\geq 1} \sigma_{k-1}(m)\, q^m, \quad z \in \mathbb{H},$$

where $\sigma_k(m) := \sum_{d\in\mathbb{N};d|m} d^k$ and $q := e^{2\pi iz}$. We get $E_k(\infty) = 1$ from the constant term in the Fourier expansion.

Definition 2.4.4 (Discriminant). We define

$$\Delta := \frac{1}{1728}\left(E_4^3 - E_6^2\right).$$

This is an elliptic cusp form of weight 12 with trivial multiplier system without any zeros on $\mathbb{H}$.

Definition 2.4.5 (Dedekind η-function). We define

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} \left(1 - e^{2\pi imz}\right).$$

This defines an elliptic cusp form $\mathbb{H} \to \mathbb{C}$ of weight $\frac{1}{2}$ with multiplier system ν_η induced by

$$\nu_\eta(T) = e^{\pi i/12} \qquad \text{and} \qquad \nu_\eta(J) = e^{\pi i/4} \qquad \text{(compare also [El06]).}$$

It satisfies $\eta^{24} = \Delta$. For prime numbers p we denote by $\eta^{(p)}$ the function $\mathbb{H} \to \mathbb{C}$, $z \mapsto \eta(pz)$.

Definition 2.4.6 (Absolute invariant). We define the meromorphic elliptic modular form

$$j := \frac{E_4^3}{\Delta}.$$

It is a modular form of weight 0 with trivial multiplier system.

Theorem 2.4.7. *The ring of holomorphic elliptic modular forms with trivial multiplier system is generated by the two elliptic modular forms E_4 and E_6, which are algebraically independent.*

Proof. A proof can be found in [Bu97, Propposition 1.3.4] □

Corollary 2.4.8. *Every holomorphic elliptic modular form is a polynomial in E_4, E_6 and η, hence all multiplier systems occurring in Definition 2.4.1 are powers of the multiplier system of η and all weights are half-integral.*

Proof. Let $f \not\equiv 0$ be an elliptic modular form of weight $k \in \mathbb{Q}$ with multiplier system μ. We distinguish two cases:

a) $\mu(T) = 1$. We have $J \cdot T \cdot J \cdot T \cdot J \cdot T = E$ and $f(Tz) = 1 \cdot 1^k f(z) = f(z)$ for all $z \in \mathbb{H}$, so we calculate

$$\begin{aligned}
f|_k^\mu(JT)(z) &= f|_k^\mu J|_k^\mu T(z) = f|_k^\mu J(1+z) \\
&= \mu(J)^{-1}(-(1+z))^k f\left(\frac{-1}{1+z}\right) \\
&= \mu(J)^{-1} \exp\left(-\frac{k\pi i}{2} + k \ln \underbrace{-i(1+z)}_{\in -i\mathbb{H}}\right) f\left(\frac{-1}{1+z}\right)
\end{aligned}$$

and obtain ($JTz = -\frac{1}{1+z}$, $JTJTz = -\frac{1+z}{z}$ and $JTJTJTz = z$)

$$\begin{aligned}
\mu(J)^3 f(z) &= \mu(J)^3 f|_k^\mu(JT)^3(z) \\
&= \exp\Big(-\frac{3k\pi i}{2} + k \ln \underbrace{i^3(1+z)}_{\in -i\mathbb{H}} + k \ln \underbrace{i^3\left(1 - \frac{1}{1+z}\right)}_{\in -i\mathbb{H}} + k \ln \underbrace{i^3\left(1 - \frac{1+z}{z}\right)}_{\in -i\mathbb{H}}\Big) f(z) \\
&= \exp\Big(-\frac{3k\pi i}{2} + k \ln \underbrace{i^3(1+z)}_{\in -i\mathbb{H}} + k \ln -\frac{z}{1+z}\frac{-1}{z}\Big) f(z) \\
&= \exp\Big(-\frac{3k\pi i}{2} + k \ln \underbrace{i^3(1+z)}_{\in -i\mathbb{H}} + k \ln i^3 \cdot \underbrace{i\frac{1}{1+z}}_{\in -i\mathbb{H}}\Big) f(z) \\
&= \exp(-2k\pi i) f(z).
\end{aligned}$$

f does not vanish identically, so $\mu(J)^3 = e^{-2k\pi i}$. Since $f|_k^\mu J(z) = \mu(J)^{-1}\exp(k\ln(-z))f\left(\frac{-1}{z}\right)$ we get

$$\begin{aligned}\mu(J)^2 f(z) &= \mu^2(J) f|_k^\mu (JJ)(z)\\ &= \exp\left(k\ln(-z) + \ln\left(\frac{1}{z}\right)\right) f(z)\\ &= \exp\left(-k\frac{\pi i}{2} + k\ln(i^3 z) - k\frac{\pi i}{2} + \ln\left(\frac{i}{z}\right)\right) f(z)\\ &= \exp\left(-k\pi i\right) f(z)\end{aligned}$$

and conclude $\mu(J)^2 = e^{-k\pi i}$. So

$$\mu(J) = \frac{\mu(J)^3}{\mu(J)^2} = e^{-k\pi i} = \mu(J)^2$$

implies $\mu(J) = 1$ and $k \in 2\mathbb{Z}$ and f is a polynomial in E_4 and E_6 by Theorem 2.4.7.

b) $\mu(T) \neq 1$. Then we have

$$\mu(T)f(\infty) = \lim_{\mathrm{Im}(z)\to\infty} \mu(T)f(z) = \lim_{\mathrm{Im}(z)\to\infty} f(Tz) = \lim_{\mathrm{Im}(z)\to\infty} f(z) = f(\infty)$$

and conclude that f is a cusp form. The index of the first non-vanishing coefficient of its Fourier expansion gives the order m of which f vanishes in ∞. So we can divide f by η^m (which vanishes only in ∞ and there of order 1) and get an elliptic modular form of weight $k - \frac{m}{2}$, which is no cusp form. Therefore its multiplier system $\mu\nu_\eta^{-m}$ is trivial and $f\eta^{-m}$ is a polynomial in E_4 and E_6 by Theorem 2.4.7.

$\square$

2.5 Elliptic Modular Forms for Congruence Subgroups

Borcherds products are lifts of nearly holomorphic modular forms for congruence subgroups. We give some definitions and examples and investigate a certain subspace, which is needed for the Borcherds lift.

2.5.1 Basic Notions

We give a number of definitions (based on Koecher and Krieg, [KK98]) and state a result of Rademacher, who gives a set of generators for $\Gamma_0(p)$ for some p.

Remember that p is a prime number with $p \equiv 1 \pmod 4$ throughout this paper.

Definition 2.5.1 ($\Gamma_0(p)$, χ_p). We define

$$\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z});\ c \equiv 0 \pmod p \right\}$$

and

$$\chi_p : \begin{cases} \quad \Gamma_0(p) & \longrightarrow \quad \{-1,1\} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto \quad \chi_p(d) := \left(\frac{d}{p}\right) \end{cases}$$

where $\left(\frac{d}{p}\right)$ is the Legendre symbol defined in Definition 1.3.14. Since $p \nmid d$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, this is an abelian character even if $\chi_p|_{p\mathbb{Z}} \equiv 0$.

Rademacher investigated the ring of congruence subgroups of the modular group $\mathrm{SL}(2,\mathbb{Z})$ in [Ra29] and especially got the following result:

Theorem 2.5.2 ($\Gamma_0(p)$). *The group* $\Gamma_0(p) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0\ (p)\}$ *is generated by*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ V_2 = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix},\ V_3 = \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} \quad \textit{if } p = 5,$$

$$T,\ V_4 = \begin{pmatrix} -3 & -1 \\ 13 & 4 \end{pmatrix},\ V_5 = \begin{pmatrix} -5 & -1 \\ 26 & 5 \end{pmatrix},\ V_8 = \begin{pmatrix} -8 & -1 \\ 65 & 8 \end{pmatrix},\ V_{10} = \begin{pmatrix} -9 & -1 \\ 91 & 10 \end{pmatrix} \quad \textit{if } p = 13 \textit{ and}$$

$$T,\ V_4 = \begin{pmatrix} -4 & -1 \\ 17 & 4 \end{pmatrix},\ V_7 = \begin{pmatrix} -12 & -1 \\ 85 & 7 \end{pmatrix},\ V_9 = \begin{pmatrix} -15 & -1 \\ 136 & 9 \end{pmatrix},\ V_{13} = \begin{pmatrix} -13 & -1 \\ 170 & 13 \end{pmatrix} \quad \textit{if } p = 17.$$

Definition 2.5.3 (Cusp). Remember, that cusps were already defined in Definition 1.1.5, now we need the special case $n = 1$ and $\Gamma = \Gamma_0(p)$. So a **cusp of** $\Gamma_0(p)$ is an element $\kappa \in \mathbb{R} \cup \{\infty\}$ such that there is $M \in \mathrm{SL}(2,\mathbb{R})$ with $M\infty = \kappa$ and the action of the subgroup $(M^{-1}\Gamma_0(p)M)_\infty$ of $M^{-1}\Gamma_0(p)M$ fixing ∞ is generated by one element $z \mapsto z + b$ with some $0 \neq b \in \mathbb{R}$. We say that two cusps $\kappa_1, \kappa_2 \in \mathbb{R} \cup \{\infty\}$ are **equivalent** (with respect to G), if there is matrix $M \in G$ such that $G\kappa_1 = \kappa_2$.

Lemma 2.5.4. $\Gamma_0(p)$ *has exactly two classes of equivalent cusps, one containing* ∞ *and the other one containing* 0.

Proof. Clearly ∞ is a cusp of $\Gamma_0(p)$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ we have

$$M\infty = \begin{cases} \infty, & \text{if } c = 0, \\ \frac{a}{c}, & \text{else.} \end{cases}$$

For every $\frac{a}{c} \in \mathbb{Q}$ with $a, c \in \mathbb{Z}$, $\gcd(a,c) = 1$ and $p|c$, there is $(d,b) \in \mathbb{Z}^2$ such that $ad + cb = 1$. So $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is contained in $\Gamma_0(p)$.

0 is cusp of $\Gamma_0(p)$, because its fix group in $\Gamma_0(p)$ is the set of all $\left(\begin{smallmatrix} 1 & 0 \\ pr & 1 \end{smallmatrix}\right)$ with $r \in \mathbb{Z}$ and so $J^{-1}(\Gamma_0(p))_0 J = (J^{-1}\Gamma_0(p)J)_\infty$ contains (for example) $\left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$. Consider some rational number $\frac{b}{d}$ with $\gcd(b,d) = 1$ not contained in $\Gamma_0(p)\infty$. Then $p \nmid d$ and $\gcd(pb, d) = \gcd(b,d) = 1$. We get $a, b \in \mathbb{Z}$ such that $ad + pbc = 1$, so the matrix $M = \left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right)$ is contained in $\Gamma_0(p)$ and maps 0 to $\frac{b}{d}$.

So every rational number q is a cusp: If there is $M \in \Gamma_0(p)$ such that $M\infty = q$, then MTM^{-1} is a nontrivial element of $\Gamma_0(p)$ fixing q. If there is $M \in \Gamma_0(p)$ such that $M0 = q$, then $MJT^pJ^{-1}M^{-1}$ is a nontrivial element of $\Gamma_0(p)$ fixing q, so in both cases q is a cusp.

Given some irrational κ, the matrix $\left(\begin{smallmatrix} 1 & \kappa \\ 0 & 1 \end{smallmatrix}\right) J \in \mathrm{SL}(2,\mathbb{R})$ maps ∞ to κ. It suffices to look at this matrix by Remark 1.1.6. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ we calculate

$$J^{-1}\begin{pmatrix} 1 & -\kappa \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} J = \begin{pmatrix} \kappa c + d & -c \\ (-a+\kappa c)\kappa - b + \kappa d & a - \kappa c \end{pmatrix}.$$

and compare with a translation matrix $\pm\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. We have equality only in the two cases $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \pm E$, so κ is no cusp. $\square$

We are interested in certain modular forms for $\Gamma_0(p)$:

Definition 2.5.5 (Modular forms for congruence subgroups). Let μ be an abelian character $\Gamma_0(p) \to \mathbb{C}^*$ and $k \in \mathbb{N}_0$ a non negative integer. A holomorphic map $f : \mathbb{H} \to \mathbb{C}$ with the transformation law

$$f|_k^\mu M = f \qquad \text{for all } M \in \Gamma_0(p),$$

for which $f(\infty) := \lim_{\mathrm{Im}(z)\to\infty} f(z)$ and $f(0) := \lim_{z\to\infty} f|_k J(z) = \lim_{z\to 0} z^k f(z)$ exist in $\mathbb{C} \cup \{\infty\}$ is called **nearly holomorphic modular form** for $\Gamma_0(p)$ of weight k with character μ. If $f(\infty)$ and $f(0)$ are complex numbers, then f is called a **(holomorphic) modular form** for $\Gamma_0(p)$ of weight k with character μ. If $f(\infty) = f(0) = 0$, then f is called cusp form.We define the spaces:

$A_k(p,\mu)$: nearly holomorphic modular forms for $\Gamma_0(p)$ of weight k with character μ

$M_k(p,\mu)$: holomorphic modular forms for $\Gamma_0(p)$ of weight k with character μ

$S_k(p,\mu) = \{f \in M_k(p,\mu);\ f \text{ cusp form}\}$

$A_k^\epsilon(p,\chi_p) = \left\{f(z) = \sum_{n\in\mathbb{Z}} a(n)e^{2\pi i n z} \in A_k(p,\chi_p); a(n) = 0 \text{ for } \chi_p(n) = -\epsilon\right\}$

$M_k^\epsilon(p,\chi_p) = A_k^\epsilon(p,\chi_p) \cap M_k(p,\chi_p)$

$S_k^\epsilon(p,\chi_p) = A_k^\epsilon(p,\chi_p) \cap S_k(p,\chi_p)$

where ϵ is $+1$ or -1. For simplicity of notations, we will omit the 1 and write for example $A_k^+(p,\chi_p)$ instead of $A_k^{+1}(p,\chi_p)$.

Remark 2.5.6. *We have $M_0(p,\chi_p)=\{0\}$ and $M_0(p,1)=\mathbb{C}$ (cf. [KK98, Satz III.7.4]). A table of dimensions of the space $S_k(p,\chi_p)$ of cusp forms can be found in [Mi89, p. 295 et seqq.].*

We are mainly interested in modular forms in $\sum_k A_k(p,\chi_p)$, but the ring $\sum_k A_k(p,1)$ of modular forms with trivial character operates on $\sum_k A_k(p,\chi_p)$ by multiplication.

Definition 2.5.7 (Order in a cusp). A nearly modular form f for a group Γ with cusp κ has order m in κ, if for $M\in \mathrm{SL}(2,\mathbb{R})$ with $M\infty=\kappa$ the translations of $M^{-1}\Gamma M$ are generated by a map $z\mapsto z+\delta$ and there are Fourier coeffi cients $a_k\in\mathbb{C}$ $(k\geq m)$, $a_m\neq 0$, such that

$$f|_kM(z)=\sum_{k\geq m}a_ke^{2\pi ikz/\delta}\qquad \text{for all } z\in\mathbb{H}.$$

Remark 2.5.8. *The order in a cusp is a geometric notion. Consider the cusp ∞: We map $\mathbb{H}/\Gamma$ by $z\mapsto e^{2\pi iz/\delta}$ to the unit disc, mapping ∞ to zero. Then the Fourier expansion $f(z)=\sum_{k\geq m}a_ke^{2\pi ikz/\delta}$ maps to the power series $\tilde{f}(z)=\sum_{k\geq m}a_kz^k$ and the notion of zero order or pole order of f and $\tilde{f}$ are the same. The argument translates one to one to other cusps.*

Remark 2.5.9. *It is easy to see that the order of a modular form f in a cusp is the same as the order in all equivalent cusps.*

2.5.2 Examples

The easiest examples of elliptic modular forms for congruence subgroups are elliptic modular forms for the full group $\mathrm{SL}(2,\mathbb{Z})$. The next type of example is similar: For every elliptic modular form f for $\mathrm{SL}(2,\mathbb{Z})$, $z\mapsto f(pz)$ is an elliptic modular form for $\Gamma_0(p)$. Hecke used this to describe a modular form without roots as quotient of powers of the Dedekind η-function. At last we give Eisenstein series for $\Gamma_0(p)$ for trivial character. Eisenstein series for nontrivial character exist, but we need some preparations and postpone them until the next section.

Lemma 2.5.10. *Let f be a nearly holomorphic elliptic modular form for the group $\mathrm{SL}(2,\mathbb{Z})$ of weight k with trivial character and of order m in the cusp ∞. Define $f^{(p)}:\mathbb{H}\to\mathbb{C}$, $z\mapsto f(pz)$. Then f and $f^{(p)}$ are nearly holomorphic elliptic modular forms for the group $\Gamma_0(p)$ of weight k with trivial character. They have the following orders in the cusps 0 and ∞ (as modular forms for $\Gamma_0(p)$):*

cusp	∞	0
f	m	pm
$f^{(p)}$	pm	m

Remark 2.5.11. *If we consider nontrivial characters* μ *in Lemma 2.5.10, the modular form* f *has the character* $\mu|_{\Gamma_0(p)}$ *and the modular form* $f^{(p)}$ *has character* μ_p *where*

$$\mu_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu \begin{pmatrix} a & pb \\ \frac{c}{p} & d \end{pmatrix} \qquad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p).$$

To calculate the order at the cusps, it is necessary to consider $\{M \in \mathrm{SL}(2,\mathbb{Z});\ \mu(M) = 1\}$ *instead of* $\mathrm{SL}(2,\mathbb{Z})$.

Proof of Remark 2.5.11. The transformation property comes from

$$\begin{aligned} f^{(p)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\langle\tau\rangle\right) &= f\left(p\frac{a\tau+b}{c\tau+d}\right) \\ &= f\left(\frac{ap\tau+pb}{(c/p)(p\tau)+d}\right) \\ &= f\left(\begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}\langle p\tau\rangle\right) \\ &= \nu\left(\begin{pmatrix} a & pb \\ c/p & d \end{pmatrix}\right)(c\tau+d)^k\, f^{(p)}(\tau)\ . \end{aligned}$$

□

Proof of Lemma 2.5.10. All cusps of $\mathrm{SL}(2,\mathbb{Z})$ are equivalent. So f has order m in the cusp ∞ and in the cusp 0.

Consider the cusp ∞. The translation subgroup of $\Gamma_0(p)$ is induced by the translation $z \mapsto z+1$ on $\mathbb{H}$ and we have

$$\begin{aligned} f(z) &= \sum_{k\geq m} a_k e^{2k\pi iz} \quad \text{and} \\ f^{(p)}(z) &= \sum_{k\geq m} a_k e^{2pk\pi iz} = \sum_{k\geq pm, k\in p\mathbb{Z}} a_{k/p} e^{2k\pi iz}, \end{aligned}$$

so the given orders at the cusp infi nity are valid.

The subgroup of $\Gamma_0(p)$ fi xing 0 consists of the elements of the type $\pm\left(\begin{smallmatrix} 1 & 0 \\ pc & 1 \end{smallmatrix}\right)$ with $c \in \mathbb{Z}$. So the action of $J^{-1}\Gamma_0(p)J$ is generated by the matrix $T^p = \left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$, while

$$f|_k J(z) = (-z)^{-k} f(Jz) = \sum_{k\geq m} a_k e^{2k\pi iz} = \sum_{k\geq pm, k\in p\mathbb{Z}} a_{k/p} e^{2k\pi iz/p} \text{and}$$

$$f^{(p)}|_k J(z) = (-z)^{-k} f^{(p)}(Jz) = \left(-\frac{z}{p}\right)^k f\left(\frac{-p}{z}\right) = f\left(\frac{z}{p}\right)$$

$$= \sum_{k\geq m} a_k e^{2k\pi iz/p}.$$

We obtain the order pm for f and the order m for $f^{(p)}$ at the cusp 0 with respect to $\Gamma_0(p)$. $\square$

We get the following Theorem of Hecke (cf. [Og73, Theorem 6, p. 28]) as an example:

Theorem 2.5.12 (Hecke). *The function*

$$H^{(1)} = \eta^p/\eta^{(p)} = 1 + O(q) : \mathbb{H} \to \mathbb{C}, z \mapsto \eta(z)^p/\eta(pz)$$

is a modular form of weight $\frac{p-1}{2}$ *for* $\Gamma_0(p)$ *with character* χ_p*.*

We get a simple

Corollary 2.5.13. *In case* $24|(p^2-1)$*, i.e.* $p \neq 2, 3$*, the function*

$$H^{(q)} = (\eta^{(p)})^p/\eta = q^{\frac{p^2-1}{24}} + \ldots : \mathbb{H} \to \mathbb{C}, z \mapsto \eta(pz)^p/\eta(z)$$

is a modular form of weight $\frac{p-1}{2}$ *for* $\Gamma_0(p)$ *with character* χ_p*.*

Proof. In case $24|(p^2-1)$ we have

$$\frac{\eta(pz)^p}{\eta(z)} = \eta^{p^2-1}(z)\left(\frac{\eta(pz)}{\eta^p(z)}\right)^p = \Delta^{\frac{p^2-1}{24}}(z)\left(H^{(1)}\right)^{-p}.$$

For every prime $p \neq 2, 3$ we write $p = 2k+1$ and since $3 \nmid p$ we get $3|(p+1)(p+2) = (2k+2)(2k+3) \equiv (2k+2)2k = 4k(k+1) \pmod 3$. Additionally either k or $k+1$ are even, so $24|4k(k+1) = (p^2-1)$. $\square$

Another Corollary from Lemma 2.5.10 and Remark 2.5.11 is the following

Corollary 2.5.14. *Let* $p \equiv 1 \pmod 4$ *and set* $k = 24/\gcd(p-1, 24)$*. Then*

$$\tilde{H} = \frac{\eta^k}{(\eta^{(p)})^k} : \mathbb{H} \longrightarrow \mathbb{C}, z \longmapsto \frac{\eta(z)^k}{\eta^k(pz)} = q^{(1-p)/\gcd(p-1,24)} + \ldots$$

is a nearly holomorphic modular form of weight 0 *for* $\Gamma_0(p)$ *with trivial character. The exponent* k *is one of the numbers* 1*,* 2*,* 3 *or* 6*.*

Proof. By Lemma 2.5.10 together with Remark 2.5.11 the function $\tilde{H}$ has character $\nu_\eta^{k-k\cdot p} = \mu_\eta^{-24(p-1)/\gcd(p-1,24)} = 1$ and weight $\frac{1}{2}(k-k) = 0$. Its Fourier expansion starts with $q^{(k-kp)/24} = q^{24(1-p)/(24\gcd(p-1,24))} = q^{(1-p)/\gcd(p-1,24))}$. Since $p \equiv 1 \pmod 4$, we have $4|\gcd(p-1,24)$ and k divides $6 = 24/4$. $\square$

Remark 2.5.15. *There is a more geometric proof of Lemma 2.5.10, the lemma can be directly obtained from the following diagrams:*

f at cusp $\infty : ((\Gamma_0(p))_\infty = \langle T\rangle)$

$$\begin{tikzcd}
& z \arrow[dl, "T"'] \arrow[d, "\pi"] \arrow[dr, "f"] & \\
z+1 \arrow[r, "\pi"] & e^{2\pi iz} \arrow[r, "()^m+O(\dots)"'] & e^{2\pi imz}+O(\dots)
\end{tikzcd}$$

where $\pi : \mathbb{H}/\Gamma_\infty \to \mathrm{D}$ is the universal covering and $\Gamma_\infty = \langle T\rangle$.

We calculate $J\Gamma_0(p) = \Gamma^0(p)J$ (where $\Gamma^0(p) = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z});\ b \equiv 0 \pmod p\}$). So we get for the cusp 0

f at cusp $0 : (J(\Gamma_0(p))_0 = \langle T^p\rangle\, J)$

$$\begin{tikzcd}
zJ \arrow[r] & \tau = -\frac{1}{z} \arrow[dl, "T^p"'] \arrow[d, "\pi"] \arrow[dr, "f"] & \\
\tau+p \arrow[r, "\pi"] & e^{2\pi i\tau/p} \arrow[r, "()^{pm}+O(\dots)"'] & e^{2\pi im\tau}+O(\dots)
\end{tikzcd}$$

with universal covering $\pi : \mathbb{H}/\langle T^p\rangle \to \mathrm{D}$.

$f^{(p)}$ at cusp $\infty : ((\Gamma_0(p))_\infty = \langle T\rangle)$

$$\begin{tikzcd}
& z \arrow[r, "p\cdot"] \arrow[dl, "T"'] \arrow[d, "\pi"] \arrow[drr, "f^{(p)}"'] & pz \arrow[dr, "f"] & \\
z+1 \arrow[r, "\pi"] & e^{2\pi iz} \arrow[rr, "()^{pm}+O(\dots)"'] & & e^{2\pi ipmz}+O(\dots)
\end{tikzcd}$$

and

$f^{(p)}$ at cusp $0 : (J(\Gamma_0(p))_0 = \langle T^p\rangle\, J)$

$$\begin{tikzcd}
z \arrow[r, "J"] & \tau = -\frac{1}{z} \arrow[r, "z\mapsto pz"] \arrow[dl, "T^p"'] \arrow[d, "\pi"] \arrow[drr, "f^{(p)}"'] & \tau/p \arrow[dr, "f"] & \\
\tau+p \arrow[r, "\pi"] & e^{2\pi i\tau/p} \arrow[rr, "()^m+O(\dots)"'] & & e^{2\pi im\tau/p}+O(\dots)\,.
\end{tikzcd}$$

From Koecher/Krieg [KK98] we take

Theorem 2.5.16 (Eisenstein series for $\Gamma_0(p)$). *If $S \in Pos(n,\mathbb{Z})$ is an even matrix and $p := \min\{l \in \mathbb{N};\mid lS^{-1}\text{even}\}$, then $\tau \to \Theta(\tau,S)$ is a modular form of weight $\frac{n}{2}$ for $\Gamma_0(p)$, where*

$$\begin{aligned}
\Theta(\tau,S) &= \sum_{g\in\mathbb{Z}^n} e^{\pi i(g^tSg)\tau} \\
&= \sum_{m=0}^{\infty} \sharp(S,2m)q^m
\end{aligned}$$

is the Θ *Nullwert of* S*. In this* $\sharp(S, 2m)$ *is called representation number, that is the number of* $g \in \mathbb{Z}^n$ *with* $g^t S g = 2m$*.*

We formulate a special case of a lemma of Hecke [Og73, Lemma 6, p. 32]:

Lemma 2.5.17. *Let* χ *be a primitive character modulo* p *(i. e. if* χ *is the product of two characters, one of them is trivial, e. g.* χ_p *is primitive and the trivial character* $1 = \chi_p^2$ *is not primitive) and denote the map* $\mathrm{SL}(2, \mathbb{Z}) \to \mathbb{C}$*,* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(d)$ *by* $\tilde{\chi}$*. If there is*

$$f(\tau) = \sum_{\nu=1}^{\infty} a_\nu q^\nu \in A_k(p, \chi)$$

such that $a_\nu = 0$ *for all* $(\nu, p) = 1$*, then* $f = 0$*.*

Proof (Ogg). Since the Fourier expansion only contains a_ν where $p|\nu$, we have $f(\tau + \frac{1}{p}) = f(\tau)$. For $x, y \in \mathbb{Z}$ we define

$$\gamma := \begin{pmatrix} 1 & x/p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & y/p \\ 0 & 1 \end{pmatrix} = \left(\begin{smallmatrix} 1+x & \frac{x+y+xy}{p} \\ p & 1+y \end{smallmatrix}\right)$$

and obtain

$$f|\gamma = f.$$

It is possible to choose x and y such that $\gamma \in \Gamma_0(p)$ and $\chi(1+y) \neq 1$, so we get $f = 0$: We need x, y in Z such that $p|(x+y+xy)$. Choose any y such that $\chi(y+1) \neq 1$. Then there are $\alpha, \beta \in \mathbb{Z}$ such that $\alpha p + \beta(1+y) = 1$. If we set $x := \beta - 1$, we get

$$-\alpha p = \beta(1+y) - 1 = (1+x)(1+y) - 1 = x + y + xy$$

and the right side is divisible by p. $\square$

Bruinier and Bundschuh [Br98, p. 3] derive the following

Corollary 2.5.18. $A_k(p, \chi_p) = A_k^+(p, \chi_p) \oplus A_k^-(p, \chi_p)$

In order to prove this, we need some definitions and Lemma 2.5.23.

Definition 2.5.19 (slash operator, W_p, U_p, V_p). Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function, $k \in \mathbb{Z}$, $z \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}^+(2, \mathbb{R}) = \{M \in \mathrm{GL}(2, \mathbb{R}) \mid \det M > 0\}$, then we write

$$f \mid_k M(z) = (\det M)^{k/2} (cz + d)^{-k} f(Mz)$$

for the slash operator of weight k. One easily checks $f|_k M|_k N = f|_k(MN)$ for all $k \in \mathbb{Z}$ and $M, N \in \mathrm{GL}^+(2, \mathbb{R})$. We define the matrices

$$W_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \quad \text{and} \quad V_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

and the Hecke operator $|_k\, U_p$ (normalized as in [BB03])

$$f\Big|_k U_p = \sum_{j (\operatorname{mod} p)} f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}.$$

Remark 2.5.20. *Both the so called Fricke involution $|_k\, W_p$ and $|_k\, U_p$ act on $A_k(p, \chi_p)$.*

Proof of Remark 2.5.20. For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ it is

$$\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ pa & pb \end{pmatrix} = \begin{pmatrix} d & -c/p \\ -bp & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix},$$

so for all $f \in A_k(p, \chi_p)$ and $M \in \Gamma_0(p)$ we have

$$f\Big|_k W_p\Big|_k M = f\Big|_k \begin{pmatrix} d & -c/p \\ -bp & a \end{pmatrix} \Big|_k W_p = f\Big|_k W_p,$$

hence $f\, |_k\, W_p \in A_k(p, \chi_p)$ and $|_k\, W_p$ acts on $A_k(p, \chi_p)$.
We want to show $f\, |_k\, U_p\, |_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = f\, |_k\, U_p$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ we have

$$f\Big|_k U_p\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{j (\operatorname{mod} p)} f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{j (\operatorname{mod} p)} f\Big|_k \begin{pmatrix} a+jc & b+jd \\ pc & pd \end{pmatrix}.$$

For $j, l \in \{0, 1, \ldots, p-1\}$ define

$$M_{jl} = \begin{pmatrix} a+jc & \frac{b+jd-(a+jc)l}{p} \\ pc & d-lc \end{pmatrix}.$$

Then $\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M_{jl} \left(\begin{smallmatrix} 1 & l \\ 0 & p \end{smallmatrix}\right)$, so we still need to show $M_{jl} \in \Gamma_0(p)$ for appropriate $l = l(j)$ depending on j and that the corresponding map $j \to l(j)$ is bijective.

Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ we have $p \mid c$ and $p \nmid a$, implying $p \nmid (a+jc)$. Additionally $b+jd$ and $a+jc$ are coprime, so for every $j \in \{0, 1, \ldots, p-1\}$ there is $l = l(j) \in \{0, 1, \ldots, p-1\}$ such that $M_{jl} \in \Gamma_0(p)$. Furtheron we get the injectivity of the map $j \mapsto l(j)$ by investigation of M_{12}: If $l(j_1) = l(j_2)$ for $j_1, j_2 \in \{0, 1, \ldots, p-1\}$, then p divides the difference

$$(b + j_1 d - (a + j_1 c)l) - (b + j_2 d - (a + j_2 c)l) = (j_1 - j_2)(d - cl).$$

Since p divides c but not d, clearly p divides $j_1 - j_2$ and necessarily $j_1 = j_2$. $\square$

We need the notion of Dirichlet characters and a fact about Gauß sums:

Definition 2.5.21 (Dirichlet character mod N). (cf. [Za81, p. 34]) Let $N \in \mathbb{N}$. Every character $\chi : (\mathbb{Z}/(N\mathbb{Z}))^* \to \mathbb{C}^*$, where $(\mathbb{Z}/(N\mathbb{Z}))^* = \{n \pmod N | (n, N) = 1\}$, is called Dirichlet character. For every such character χ we identify χ with the map $f : \mathbb{Z} \to \mathbb{C}$, $f(n) = 0$ if $\gcd(n, N) > 1$ and $f(n) = \chi(n \pmod N)$ if $\gcd(n, N) = 1$.

Lemma and Definition 2.5.22 (Gauß sum). *If p is an odd prime number and χ is a primitive Dirichlet character (i.e. it has no nontrivial divisors, cf. [Za81, p. 37]), then the Gauß sum G_χ is given by*

$$G_\chi = \sum_{l \mod p} \chi(l)\zeta^l,$$

where $\zeta = e^{2\pi i/p}$. It holds

$$G_\chi = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod 4, \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

and

$$\sum_{m=1}^{p-1} \chi_p(m)\zeta^{ma} = \chi_p(a)G_\chi$$

for all $a \in \mathbb{N}$ (cf. [Le96, p. 171 et seqq. and Satz 19.8, p. 298]).

From Bruinier and Bundschuh [BB03, Lemma 3] we take the following lemma, adding part (ii) which comes immediately from their proof.

Lemma 2.5.23. *Let $f = \sum_{n \in \mathbb{Z}} a(n)q^n \in A_k(p, \chi_p)$ and $\epsilon \in \{\pm 1\}$. Write $\varepsilon_p = 1$ if $p \equiv 1 \pmod p$ and $\varepsilon_p = i$ if $p \equiv 3 \pmod 4$. Then*

(i) f belongs to $A_k^\epsilon(p, \chi_p)$ if and only if

$$f \mid U_p = \epsilon\varepsilon_p\sqrt{p} f \mid W_p.$$

(ii) $f = f^+ + f^-$ with $f^+ \in A_k^+(p, \chi_p)$ and $f^- \in A_k^-(p, \chi_p)$ for

$$f^+ := \frac{1}{2}f + \frac{\varepsilon_p}{2\sqrt{p}} f \mid_k U_p \mid_k W_p \quad \text{and} \quad f^- := \frac{1}{2}f - \frac{\varepsilon_p}{2\sqrt{p}} f \mid_k U_p \mid_k W_p.$$

For sake of completeness we give the following proof:

Proof (Bruinier, Bundschuh). The function $h = f \mid_k U_p \mid_k W_p$ is contained in $A_k(p, \chi_p)$ and the condition in the lemma is equivalent to

$$h = \epsilon\widetilde{\varepsilon_p}\sqrt{p} f \qquad \text{(where } \widetilde{\varepsilon_p} \text{ is the complex conjugate of } \varepsilon_p).$$

We have

$$\begin{aligned} h &= \sum_{j (\operatorname{mod} p)} f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \Big|_k \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \\ &= f \mid_k W_p \mid_k V_p + \sum_{j (\operatorname{mod} p)^*} f\Big| \begin{pmatrix} j & -1 \\ p & 0 \end{pmatrix} \Big| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where in $\sum_{j (\operatorname{mod} p)^*}$ we summarize over a set of representatives of $(\mathbb{Z}/p\mathbb{Z}) \setminus p\mathbb{Z}$. For a given $j \in \mathbb{Z}$ that is coprime to p let $b, d \in \mathbb{Z}$ such that $jd - pb = 1$. Then $\left(\begin{smallmatrix} j & b \\ p & d \end{smallmatrix}\right) \in \Gamma_0(p)$ and

$$\begin{pmatrix} j & -1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} j & b \\ p & d \end{pmatrix} \begin{pmatrix} 1 & -d \\ 0 & p \end{pmatrix}.$$

Thus

$$\begin{aligned} h &= f \mid_k W_p \mid_k V_p + \sum_{d (\operatorname{mod} p)^*}^{p-1} \chi_p(d) f\Big|_k \begin{pmatrix} p & -d \\ 0 & p \end{pmatrix} \\ &= f \mid_k W_p \mid_k V_p + \sum_{n \in \mathbb{Z}} a(n) q^n \sum_{d (\operatorname{mod} p)^*} \chi_p(d) e(-nd/p). \end{aligned}$$

If we insert the value of the latter Gauß sum (cf. Lemma and Definition 2.5.22), we obtain

$$h = f \mid_k W_p \mid_k V_p + \widetilde{\varepsilon_p} \sqrt{p} \sum_{n \in \mathbb{Z} \setminus p\mathbb{Z}} \chi_p(n) a(n) q^n.$$

By Lemma 2.5.17 it suffices to compare the Fourier coefficients for q^n, $p \nmid n$ in order to decide whether an element of $A_k(p, \chi_p)$ is contained in $A_k^+(p, \chi_p)$ respectively in $A_k^-(p, \chi_p)$. Hence we get all stated results from the Fourier coefficients of $h - f \mid_k W_p \mid_k V_p$ (remember $V_p(z) = pz$). ☐

Proof of Corollary 2.5.18. $A_k(p, \chi_p)$ is the sum of $A_k^+(p, \chi_p)$ and $A_k^-(p, \chi_p)$ by Lemma 2.5.23. This sum is direct by Lemma 2.5.17. ☐

Hecke introduces Eisenstein series for the Haupttypus $(-k, p, 1)$, cf. [He40, Satz 11]:

Theorem 2.5.24 (Eisenstein series for Haupttypus). *The Eisenstein series*

$$E_2^{(p)}(\tau) = \frac{p-1}{24} + \sum_{n=1}^{\infty} \left(\sum_{d \mid n, \gcd(d,p)=1} d \right) z^n$$

is an elliptic modular form of weight 2 for the group $\Gamma_0(p)$*. For even* $k \geq 4$

$$E_k(\tau) = \rho_k + \sum_{n=1}^{\infty} \sigma_{k-1}(n) z^n,$$

where

$$\rho_k = (-1)^{k/2} \frac{(k-1)!}{(2\pi)^k} \zeta(k)$$

and

$$\sigma_r(n) = \sum_{d|n} d^r,$$

which is the Eisenstein series for the full group $\mathrm{SL}(2, \mathbb{Z})$ *of weight* k *and the Eisenstein series* $E_k^{(p)} = z \mapsto E_k(pz)$*, which are both elliptic modular forms of weight* k *for the group* $\Gamma_0(p)$*.*

A more general definition of Eisenstein series can be found in [Mi89, 7.2.14, p.288]. Some more examples can be found at the homepage of William Stein [Ste04]. For a correct usage of his page, we give the following

Definition 2.5.25 (Newform, oldform). Let f be a cusp form for $\Gamma_0(n)$ with character χ of weight k (in case $n = 1$ we write $\Gamma_0(1) = \mathrm{SL}(2, \mathbb{Z})$). Then for every $N > n$, $n|N \in \mathbb{N}$ the map $\tau \mapsto f(N/n\tau)$ is a cusp form for $\Gamma_0(N)$ with character χ of weight k. We denote such modular forms as **"oldforms"** and the set of oldforms by S^{old}. Its orthogonal complement $\left(S^{\text{old}}\right)^{\perp}$ in the space of cusp forms of fixed character we denote by S^{new}, its elements we call **"newforms"**.

2.5.3 Eisenstein series of Nebentypus

We give a second example of Eisenstein series, the Eisenstein series of Nebentypus, for which we need some preparations. Especially we need L*-series and a method to evaluate some values of* L*-series.*

Definition 2.5.26 (L-series). Let $1 < N \in \mathbb{N}$ and let $\chi \neq \chi_0$ be a Dirichlet character modulo N. Then

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

converges for all $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) > 1$. Compare [Za81, p. 41, 42].

For all s with $\mathrm{Re}\,(s) > 1$ we can rewrite $L(s, \chi)$ into the absolutely convergent Euler product

$$L(s, \chi) = \prod_{p \text{ prim}} \frac{1}{1 - \chi(p) p^{-s}}.$$

If χ is distinct from the principal character χ_0 which is given by

$$\chi_0 : \begin{cases} \mathbb{Z} & \longrightarrow \mathbb{C} \\ n & \longmapsto \begin{cases} 1, & \text{if } (n,N)=1, \\ 0, & \text{else,} \end{cases} \end{cases},$$

then the L-series $L(s,\chi)$ converges for all $s \in \mathbb{C}$ with positive real part and can be holomorphically extended to $\mathbb{C}$. Then also $L(1,\chi) \neq 0$. An important general result on Dirichlet series is the following

Theorem 2.5.27 (Satz 1 in I. §7, [Za81]). *Let $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series converging for at least one point $s \in \mathbb{C}$ and let $f(t) = \sum_{n=1}^{\infty} a_n e^{-nt}$ be the corresponding exponential series (converging for all $t > 0$). If for $t \to 0$ the function $f(t)$ has the asymptotic expansion*

$$f(t) \sim b_0 + b_1 t + b_2 t^2 + \dots \quad (t \to 0),$$

then $\varphi(s)$ can be holomorphically extended to the entire complex plane and

$$\varphi(-n) = (-1)^n n!\, b_n \quad (n \in \mathbb{N}) \tag{2.1}$$

holds. More generally, if for $t \to 0$ the function $f(t)$ has the asymptotic expansion

$$f(t) \sim \frac{b_{-1}}{t} + b_0 + b_1 t + b_2 t^2 + \dots,$$

then $\varphi(s)$ can be meromorphically extended. Then $\varphi(s) - \frac{b_{-1}}{s-1}$ is an entire function and the values $\varphi(0), \varphi(-1), \dots$ are given by the formula (2.1).

An important special case is given by

Theorem 2.5.28 (Satz 2 in I. §7, [Za81]). *Let χ be a Dirichlet character modulo N and let $L(s,\chi)$ be the corresponding L-series. Then $L(s,\chi)$ can be meromorphically extended to the entire complex plane $\mathbb{C}$, more precisely holomorphic up to a single pole with residue $\phi(N)/N = \sum_{\substack{1 \le m \le N \\ (m,N)=1}} \frac{1}{N}$ in $s = 1$ in the case of the principal character $\chi = \chi_0$. Additionally we have*

$$L(-n,\chi) = -\frac{N^n}{n+1} \sum_{m=1}^{N} \chi(m) B_{n+1}\left(\frac{m}{N}\right)$$

for all natural n, where B_n is the n-th Bernoulli polynomial.

Remark 2.5.29. *The Bernoulli polynomials (cf. [He40, p.824]) are defined as usual by*

$$\int_x^{x+1} B_k(t)\, dt = x^k$$

and can directly be calculated. We give a few polynomials (cf. also [Za81, p. 25, 51])

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x + \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
&\vdots
\end{aligned}$$

We calculate the L-series $L(s, \chi_p)$ at $s = -1$ for some primes, cf. also [Za81, I. §7]:

p	5	13	17	29	37
$L(-1, p)$	$-\frac{2}{5}$	-2	-4	-6	-10

Zagier gives a method to calculate $L(1, \chi)$ in [Za81, II. §9].

We give a functional equation for further calculations ([Za81, p. 53]): If χ is a primitive Dirichlet character (i.e. it has no nontrivial divisors, cf. [Za81, p. 37]), then

$$\pi^{-\frac{s}{2}} p^{\frac{s}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi) = \frac{G_\chi}{i^\delta \sqrt{p}} \pi^{-\frac{1-s}{2}} p^{\frac{(1-s)}{2}} \Gamma\left(\frac{1-s+\delta}{2}\right) L(1-s, \tilde{\chi}).$$

In this G_χ is the Gauß sum of Definition 2.5.22 and $\tilde{\chi}$ is the character complex conjugated to χ. We have $\delta = 0$ for $\chi(-1) = 1$ and $\delta = 1$ for $\chi(-1) = -1$. Since in our case χ is a real character, we have $\tilde{\chi} = \chi$. Hecke (cf. [He40, p. 823 et seqq.], there $B_k = g_k$ and $q = p$) uses the Bernoulli polynomials to show

$$G_\chi L(k, \chi) = -\frac{1}{2}(2\pi i)^k \sum_{l=1}^{N-1} \chi(l) B_k\left(\frac{l}{N}\right).$$

Note that this is a finite sum. Hence we can directly calculate the Fourier coefficients of the two Eisenstein series for the Nebentypus $(-k, p, \chi_p)$ given in the following Theorem of Hecke [He40, Satz 12], which we normalize as done in [BB03]:

Theorem 2.5.30 (Eisenstein series for Nebentypus). *Let $k \geq 2$. There are two Eisenstein series G_κ and H_κ in $M_\kappa(p, \chi_p)$ of weight $k \geq 2$:*

$$\begin{aligned}
G_k &= 1 + \frac{2}{L(1-k, \chi_p)} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} \chi_p(d) q^n, \\
H_k &= \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} \chi_p(n/d) q^n.
\end{aligned}$$

G_k corresponds to the cusp ∞, H_k to the cusp 0. From its Fourier expansion we see that H_k vanishes in ∞ of first order.

Remark 2.5.31. *In [BB03] Bruinier states that $E_k^+ = G_k + 2H_k/L(1-k,\chi_p)$ belongs to $M_k^+(p,\chi_p)$ (we have $p \equiv 1 \pmod 4$) and is of the form*

$$E_k^+ = 1 + \sum_{n=1}^{\infty} B(n)q^n = 1 + \frac{2}{L(1-k,\chi_p)} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1}\left(\chi_p(d) + \chi_p(n/d)\right) q^n.$$

So we have for all $n \in \mathbb{N}$:

$$B(n) = \frac{2}{L(1-k,\chi_p)} \sum_{d|n} d^{k-1}\left(\chi_p(d) + \chi_p(n/d)\right).$$

This result can also be found in [He40, (38), p. 823], but one has to use the functional equation of the L-series and Lemma 3 in [BB03] to check that both results coincide.

2.5.4 A basis of the plus space

We investigate the plus space $A_k^\epsilon(p,\chi_p)$ and give a criterion for the existence of certain elements with given principal part of the fourier expansion.

First we need two defi nitions:

Definition 2.5.32 (Principal part). If $f = \sum_{n\in\mathbb{Z}} a_n q^n$ is a modular form in $A_k^\epsilon(p,\chi_p)$, then we call $\sum_{n<0} a_n q^n$ the **principal part** of f (at ∞).

Definition 2.5.33. For all integers n defi ne

$$s(n) = 1 + \sum_{j=0}^{p-1} \frac{e^{2\pi i n j/p}}{p} = 2 - \left(\frac{n}{p}\right)^2 = \begin{cases} 2, & \text{if } n \equiv 0 \pmod p \\ 1, & \text{if } n \not\equiv 0 \pmod p \end{cases}$$

In the special case $p \equiv 1 \pmod 4$ (then $\chi_p(-1) = 1$ and q represents the squares by Theorem 1.3.15 and Remark 1.3.16, so $\epsilon = \delta = 1$ in the notation of Bruinier and Bundschuh), we give the following theorem of Bruinier and Bundschuh:

Theorem 2.5.34 (Theorem 6 in [BB03]). *There exists a nearly holomorphic modular form $f \in A_k^+(p,\chi_p)$ with prescribed principal part $\sum_{n<0} a(n)q^n$ (where $a(n) = 0$ if $\chi_p(n) = -1$), if and only if*

$$\sum_{n<0} s(n)a(n)b(-n) = 0$$

for every cusp form $g = \sum_{m>0} b(m)q^m$ in $S_\kappa^+(p,\chi_p)$, where $\kappa = 2-k$. The constant term $a(0)$ of f is given by the coefficients of the Eisenstein series E_κ^+:

$$a(0) = -\frac{1}{2} \sum_{n<0} s(n)a(n)B(-n).$$

We will need the case $k = 0, \kappa = 2, p \in \{5, 13, 17\}$ for which we give a Lemma of Hecke [He40]:

Lemma 2.5.35. *For prime numbers $p \equiv 1 \pmod 4$, the dimension of $S_2(p, \chi_p)$ is given by $2\left\lfloor \frac{p-5}{24} \right\rfloor$.*

So the obstruction space is empty in the case $k = 0, p \in \{5, 13, 17\}$ and we obtain

Corollary 2.5.36. *For all $p \in \{5, 13, 17\}$, there is a nearly holomorphic modular form $f \in A_0^+(p, \chi_p)$ with prescribed principal part $\sum_{n<0} a(n)q^n$ if and only if $a(n) = 0$ for all $n \in \mathbb{N}$ with $\chi_p(n) = -1$.*

By Remark 2.5.6, this nearly holomorphic modular form then is unique. Hence we can define:

Definition 2.5.37. In case $p \in \{5, 13, 17\}$, for all $m \in \mathbb{N}$ with $\chi_p(m) \neq -1$ we write $f_m^{(p)} = f_m$ for the unique nearly holomorphic modular form in $A_0^+(p, \chi_p)$ with principal part $s(m)^{-1}q^{-m}$. The f_m form a basis of $A_0^+(p, \chi_p)$.

Bruinier gives some more information about this basis:

Proposition 2.5.38 (Proposition 7 in [BB03]). *The space $M_k^+(p, \chi_p)$ has a basis of modular forms with integral rational coefficients.*

and

Proposition 2.5.39 (Proposition 8 in [BB03]). *Let $f = \sum a(n)q^n \in \mathcal{A}_k^+(p, \chi_p)$ and suppose that $a(n) \in \mathbb{Q}$ for $n < 0$. Then all coefficients $a(n)$ are rational and have bounded denominator (i.e. there is a positive integer c such that cf has coefficients in $\mathbb{Z}$).*

Remark 2.5.40. *By Proposition 2.5.39, each element of the basis $\{f_m\}$ has rational Fourier coefficients with bounded denominator.*

2.6 Vector Valued Modular Forms

We give an overview over vector valued modular forms related to Hilbert modular forms for quadratic number fields. This section is based upon [BB03].

Definition 2.6.1 (Weil representation). For sake of simplicity we define $\mathbf{e}(z) := e^{2\pi i z}$. ρ is the unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[\mathcal{L}^\#/\mathcal{L}]$ with

$$\rho(T)\mathfrak{e}_\gamma = \mathbf{e}(q(\gamma))\mathfrak{e}_\gamma,$$
$$\rho(J)\mathfrak{e}_\gamma = \frac{1}{\sqrt{p}} \sum_{\delta \in \mathcal{L}^\#/\mathcal{L}} \mathbf{e}(-(\gamma, \delta))\mathfrak{e}_\gamma,$$

where $\mathbb{C}[\mathcal{L}^\#/\mathcal{L}] = \left\langle \mathcal{L}^\#/\mathcal{L} \right\rangle_{\mathbb{C}}$ is the complex vector space generated by $\mathcal{L}^\#/\mathcal{L}$. We fix the basis $\left\{\mathfrak{e}_\gamma = 1 \cdot \gamma;\ \gamma \in \mathcal{L}^\#/\mathcal{L}\right\}$.

Definition 2.6.2 ($\mathcal{A}_{k,\rho}$: vector valued nearly holomorphic modular forms). A nearly holomorphic modular form for $\mathrm{SL}(2,\mathbb{Z})$ of weight k with representation ρ is a holomorphic map $F:\mathbb{H}\longrightarrow\mathbb{C}[\mathcal{L}^\#/\mathcal{L}]$ satisfying

$$F(\gamma\tau)=\rho(\gamma)F(\tau)\qquad\text{for all }\gamma\in\mathcal{L}^\#/\mathcal{L},\tau\in\mathbb{H}$$

with Fourier expansion

$$F(\tau)=\sum_{\gamma\in\mathcal{L}^\#/\mathcal{L}}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\gg-\infty}}a(\gamma,n)\,\mathbf{e}(n\tau)\,\mathfrak{e}_\gamma.$$

In this $n\gg-\infty$ means that there is $M\in\mathbb{Z}$ such that n runs over all integers greater than M and $\mathcal{L}$ is the even lattice defi ned in Lemma 1.3.2. We write

$$F(\tau)=\sum_{\gamma\in\mathcal{L}^\#/\mathcal{L}}F_\gamma(\tau)\mathfrak{e}_\gamma\qquad\text{for all }\tau\in\mathbb{H}$$

and denote by $\mathcal{A}_{k,\rho}$ the space of nearly holomorphic modular forms for $\mathrm{SL}(2,\mathbb{Z})$ of weight k with representation ρ.

Remark 2.6.3. *Bruinier and Bundschuh define $r=b_+-b_-$, where (b_+,b_-) is the signature of the lattice $\mathcal{L}$. In our case this simplifies to $r=b_+-b_-=2-2=0$.*

Lemma 2.6.4 (Lemma 1 of [BB03]). *The Assignment $F\mapsto f$, where*

$$f=\frac{i^{r/2}}{2}p^{(1-k)/2}F_0\mid_k W_p=\frac{1}{2}\sum_{\gamma\in\mathcal{L}^\#/\mathcal{L}}F_\gamma(p\tau),$$

defines an injective homomorphism $\mathcal{A}_{k,\rho}\mapsto A_k^\epsilon(p,\chi_p)$. Here $\epsilon=\chi_p(\alpha)=1$ (cf. Remark 1.3.16) is given by the quadratic form on $\mathcal{L}^\#/\mathcal{L}$. The function f has the Fourier expansion

$$f=\frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{\substack{\gamma\in\mathcal{L}^\#/\mathcal{L}\\ pq(\gamma)\equiv n\ (p)}}a(\gamma,n)q^n,$$

where

$$F_\gamma=\sum_{n\in\mathbb{Z}}a(\gamma,n)q^{n/p}.$$

Proposition 2.6.5 (Proposition 2 of [BB03]). *Let $f=\sum_n a(n)q^n\in A_k(p,\chi_p)$. Then the function*

$$F=\sum_{\gamma\in\mathcal{L}^\#/\mathcal{L}}\mathfrak{e}_\gamma G_\gamma=i^{r/2}p^{k/2-1/2}\sum_{M\in\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})}\big(\rho(M)^{-1}\mathfrak{e}_0\big)\,f\mid_k W_p\mid_k M$$

belongs to $\mathcal{A}_{k,\rho}$*. The components* F_γ *have the Fourier expansion*

$$\begin{aligned} F_0 &= \sum_{\substack{n\in\mathbb{Z}\\ n\equiv 0\ (p)}} a(n)\mathbf{e}(n\tau/p) + i^{r/2}p^{k/2-1/2} f\mid_k W_p, \\ F_\gamma &= \sum_{\substack{n\in\mathbb{Z}\\ n\equiv pq(\gamma)\ (p)}} a(n)\mathbf{e}(n\tau/p) \quad (\gamma\neq 0). \end{aligned} \tag{2.2}$$

Theorem 2.6.6 (Theorem 5 of [BB03]). *Let* $f=\sum_n a(n)q^n \in A_k^\epsilon(p,\chi_p)$ *and define* F *as in Proposition 2.6.5. Then* $F\in\mathcal{A}_{k,\rho}$ *and the components* F_γ *have the Fourier expansion*

$$\begin{aligned} F_0 &= 2\sum_{\substack{n\in\mathbb{Z}\\ n\equiv 0\ (p)}} a(n)\mathbf{e}(n\tau/p), \\ F_\gamma &= \sum_{\substack{n\in\mathbb{Z}\\ n\equiv pq(\gamma)\ (p)}} a(n)\mathbf{e}(n\tau/p) \quad (\gamma\neq 0). \end{aligned} \tag{2.3}$$

The map $f\mapsto F$ *and the map described in Lemma 2.6.4 are inverse isomorphisms between* $A_k^\epsilon(p,\chi_p)$ *and* $\mathcal{A}_{k,\rho}$*.*

3 Theory of Borcherds Products

Borcherds [Bo98] describes a lift of modular forms where the image is given as an infinite product of simple factors. Bruinier and Bundschuh [BB03] reformulate Borcherds theorem for Hilbert modular forms. We start with the Theorem and then investigate the ingredients.

3.1 The Theorem of Borcherds, Bruinier and Bundschuh

Borcherds products are Hilbert modular forms vanishing on Hirzebruch Zagier divisors, which have an absolutely convergent product expansion on so called Weyl chambers. We give the definitions and formulate the theorem of Borcherds products.

Definition 3.1.1.

$$T(m) := \bigcup_{\substack{(a,b,\lambda)\in\mathfrak{L}' \\ -q(a,b,\lambda)=ab-\mathrm{N}(\lambda)=m/p}} \left\{(\tau_1,\tau_2)\in\mathbb{H}\times\mathbb{H};\quad a\tau_1\tau_2+\lambda\tau_1+\overline{\lambda}\tau_2+b=0\right\},$$

$$S(m) := \bigcup_{\substack{\lambda\in\mathfrak{o}/\sqrt{p} \\ -\mathrm{N}(\lambda)=m/p}} M(\lambda), \text{ where}$$

$$\begin{aligned} M(\lambda) &:= \left\{(\tau_1,\tau_2)\in\mathbb{H}\times\mathbb{H};\quad \lambda\,\mathrm{Im}\,(\tau_1)+\overline{\lambda}\,\mathrm{Im}\,(\tau_2)=0\right\} \\ &= \left\{(\tau_1,\tau_2)\in\mathbb{H}\times\mathbb{H};\quad \mathrm{Im}\,(\tau_2)=\frac{-\lambda}{\overline{\lambda}}\,\mathrm{Im}\,(\tau_1)\right\} \end{aligned}$$

$T(m)$ is called **Hirzebruch-Zagier divisor of discriminant** m , where one assigns the multiplicity 1 to every irreducible component of $T(m)$.

Definition 3.1.2 (Weyl chamber). For $f=\sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p,\chi_p)$ we call $W\subset\mathbb{H}\times\mathbb{H}$ a Weyl chamber attached to f, if W is a connected component of

$$\mathbb{H}\times\mathbb{H}\setminus\bigcup_{\substack{n<0 \\ a(n)\neq 0}} S(-n).$$

Definition 3.1.3 ($(W,\lambda)>0$**).** For $W\subset\mathbb{H}\times\mathbb{H}$, especially if W is a Weyl chamber, and $\lambda\in\mathfrak{o}/\sqrt{p}$ we write $(W,\lambda)=(\lambda,W)>0$, if $\lambda\,\mathrm{Im}\,(\tau_1)+\overline{\lambda}\,\mathrm{Im}\,(\tau_2)>0$ holds for all (τ_1,τ_2) in W.

We give the Theorem of Borcherds, Theorem 13.3 in [Bo98], about Borcherds products in the version of Bruinier and Bundschuh, compare Theorem 9 in [BB03] and Theorem 3.1 in [Br04]:

Theorem 3.1.4 (Borcherds, Bruinier, Bundschuh). *Let $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p,\chi_p)$ and assume that $s(n)a(n) \in \mathbb{Z}$ for all $n < 0$ (where $s(n)$ is defined in Definition 2.5.33). Then there is a meromorphic function Ψ on $\mathbb{H} \times \mathbb{H}$ with the following properties:*

(i) Ψ is a meromorphic modular form for Γ_K with some multiplier system of finite order. The weight of Ψ is equal to the constant coefficient $a(0)$ of f. It can also be computed using Theorem 2.5.34.

(ii) The divisor of Ψ is determined by the principal part of f. It equals

$$\sum_{n<0} s(n)a(n)T(-n).$$

(iii) Let $W \subset \mathbb{H} \times \mathbb{H}$ be a Weyl chamber attached to f and put $N = \min\{n; a(n) \neq 0\}$. The function Ψ has the Borcherds product expansion

$$\Psi(\tau_1,\tau_2) = \mathbf{e}(\rho_W\tau_1 + \overline{\rho_W}\tau_2) \prod_{\substack{\nu\in\mathfrak{o}/\sqrt{p} \\ (\nu,W)>0}} (1-\mathbf{e}(\nu\tau_1+\overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})}.$$

Here ρ_W and $\overline{\rho_W}$ are algebraic numbers in K that can be computed explicitely. The product converges normally for all $\tau \in W$ with $\mathrm{Im}\,(\tau_1)\,\mathrm{Im}\,(\tau_2) > |N|/p$ outside the set of poles.

(iv) There is a positive integer c such that Ψ^c has integral rational Fourier coefficients with greatest common divisor 1.

Definition 3.1.5. If W is a Weyl chamber and n an integer, we denote by $R(W,n)$ the finite set of all $\lambda \in \mathfrak{o}/\sqrt{p}$ with $\lambda > 0, \mathrm{N}(\lambda) = n/p$ and

$$\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) < 0, \quad \varepsilon_0^2\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\varepsilon_0}^2\overline{\lambda}\,\mathrm{Im}\,(\tau_2) > 0$$

for all $\tau \in W$.

From [BB03] we take

Remark 3.1.6. *Additionally we have*

(i) For all $\tau \in W$ the Weyl vector $(\rho_W, \overline{\rho_W})$ is given by

$$\rho_W\,\mathrm{Im}\,(\tau_1) + \overline{\rho_W}\,\mathrm{Im}\,(\tau_2) = \sum_{n<0} s(n)a(n) \sum_{\substack{\lambda\in\mathfrak{o}/\sqrt{p} \\ \lambda>0 \\ \mathrm{N}(\lambda)=n/p}} \min(|\lambda\,\mathrm{Im}\,(\tau_1)\,|, |\overline{\lambda}\,\mathrm{Im}\,(\tau_2)\,|).$$

In this $\overline{\rho_W}$ is the element conjugated to ρ_W. We have

$$\sum_{\substack{\lambda\in\mathfrak{o}/\sqrt{p} \\ \lambda>0 \\ \mathrm{N}(\lambda)=n/p}} \min\left\{|\lambda\,\mathrm{Im}\,(\tau_1)\,|, |\overline{\lambda}\,\mathrm{Im}\,(\tau_2)\,|\right\} = \frac{1}{\mathrm{S}(\varepsilon_0)} \sum_{\lambda\in R(W,n)} \left(\varepsilon_0\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\varepsilon_0}\overline{\lambda}\,\mathrm{Im}\,(\tau_2)\right).$$

(ii) Every modular form for $\Gamma_{\mathcal{K}}$, whose divisor is a linear combination of Hirzebruch-Zagier divisors $T(m)$, is given as a Borcherds product as in Theorem 3.1.4.

3.2 Integers in $\mathcal{K}$

Before we can move on with the theory of Borcherds products, we have to investigate some number theoretical properties of $\mathfrak{o}$. Especially we give, for fixed norm, a finite set of representatives of $\mathfrak{o}$ modulo multiplication with $\pm\varepsilon_0^2$ and use this to investigate the sets $S(m)$, which bound the union of all Weyl chambers.

Lemma 3.2.1 (Fundamental unit). *We write $\varepsilon_0 =: x_0 + y_0\sqrt{p}$ for the fundamental unit of $\mathfrak{o}$ with $x_0, y_0 \in \mathbb{Q}$. Then $x_0 > 0$ and $y_0 > 0$.*

Proof. We have $\mathrm{N}(\varepsilon_0) = \varepsilon_0\overline{\varepsilon_0} = \pm 1$ and $\varepsilon_0 = x_0 + y_0\sqrt{p} > 1$. So we get $\varepsilon_0 > 1 > |\overline{\varepsilon_0}| > 0$ and conclude $y_0 = (\varepsilon_0 - \overline{\varepsilon_0})/(2\sqrt{p}) > 0$ and $x_0 = (\varepsilon_0 + \overline{\varepsilon_0})/2 > 0$ independent of the sign of $\overline{\varepsilon_0}$. □

Lemma 3.2.2 (Numbers of fixed norm). *Let p be a prime number, $\mathcal{K} = \mathbb{Q}(\sqrt{p})$ and $\mathfrak{o}$ be the ring of integers in $\mathcal{K}$. For every m in $\mathbb{Z} \setminus \{0\}$ there is a finite set $\mathcal{J}$ which holds*

$$\mathcal{I} := \left\{\lambda \in \frac{\mathfrak{o}}{\sqrt{p}}; \quad \mathrm{N}(\lambda) = -\frac{m}{p}\right\} = \left\{\pm\lambda\varepsilon_0^{2k}; \quad k \in \mathbb{Z}, \lambda \in \mathcal{J}\right\},$$

more precisely, if we write $\check{\lambda}_1 + \check{\lambda}_2\sqrt{p}/p := \varepsilon_0^{-2}\lambda$ for all $\lambda \in \mathfrak{o}\,/\sqrt{p}$, we obtain that

$$\mathcal{J} := \left\{\lambda = \lambda_1 + \lambda_2\sqrt{p}/p \in \frac{\mathfrak{o}}{\sqrt{p}}; \quad \mathrm{N}(\lambda) = -\frac{m}{p}, \lambda_1 > 0, \lambda_2 > 0, \check{\lambda}_1\check{\lambda}_2 \leq 0\right\}$$

is a set of representatives of $\mathcal{I}/\!\sim$ with respect to the equivalence relation $\sim$ induced by multiplication with ε_0^2 and -1. For $\lambda = \lambda_1 + \lambda_2\sqrt{p}/p$ in $\mathcal{J}$ we have depending on $m = -p\,\mathrm{N}(\lambda)$ and p:

p	$m > 0$		$m < 0$	
p	$\lambda_1 \leq \sqrt{\frac{m\alpha_p}{1-p\alpha_p}}$	$\lambda_2 \leq \sqrt{\frac{m}{1-p\alpha_p}}$	$\lambda_1 \leq \sqrt{\frac{-m}{p(1-p\alpha_p)}}$	$\lambda_2 \leq \sqrt{\frac{-mp\alpha_p}{1-p\alpha_p}}$
5	$\lambda_1 \leq \frac{1}{2}\sqrt{m}$	$\lambda_2 \leq \frac{3}{2}\sqrt{m}$	$\lambda_1 \leq \frac{3\sqrt{5}}{10}\sqrt{-m}$	$\lambda_2 \leq \frac{\sqrt{5}}{2}\sqrt{-m}$
13	$\lambda_1 \leq \frac{3}{2}\sqrt{m}$	$\lambda_2 \leq \frac{11}{2}\sqrt{m}$	$\lambda_1 \leq \frac{11}{2\sqrt{13}}\sqrt{-m}$	$\lambda_2 \leq \frac{3\sqrt{13}}{2}\sqrt{-m}$
17	$\lambda_1 \leq 8\sqrt{m}$	$\lambda_2 \leq 33\sqrt{m}$	$\lambda_1 \leq \frac{33\sqrt{17}}{17}\sqrt{-m}$	$\lambda_2 \leq 8\sqrt{17}\sqrt{-m}$

Therein we write $\alpha_p = (2x_0y_0)^2/(x_0^2 + py_0^2)^2$ with the fundamental unit $\varepsilon_0 = x_0 + y_0\sqrt{p}$.

Proof. Let $\mathcal{I}$ be the set of λ in $\mathfrak{o}/\sqrt{p}$ with $-\operatorname{N}(\lambda) = m/p$ and let $\mathcal{I}$ be non empty. Let λ be an element of $\mathcal{I}$. Then we have

$$-\operatorname{N}(\lambda\varepsilon_0^{-2k}) = -\operatorname{N}(\lambda)\underbrace{\operatorname{N}(\varepsilon_0)}_{\pm 1}{}^{-2k} = -\operatorname{N}(\lambda)$$

and $\lambda\varepsilon_0^{-2k}$ is an element of $\mathfrak{o}/\sqrt{p}$, so $\lambda_k := \lambda\varepsilon_0^{-2k}$ is contained in $\mathcal{I}$. We get from $\operatorname{N}(\lambda) = \operatorname{N}(-\lambda)$ that if λ is contained in $\mathcal{I}$, $-\lambda$ is contained in $\mathcal{I}$ too. So the second form of $\mathcal{I}$ is proved except for the finiteness of $\mathcal{J}$. Let $\lambda = \lambda_1 + \lambda_2\frac{\sqrt{p}}{p}$ with $\lambda_1, \lambda_2 \in \mathbb{Q}$. We investigate the behavior of λ under multiplication with $\varepsilon_0^{\pm 2}$:

$$\begin{aligned}
\varepsilon_0^2\left(\lambda_1 + \lambda_2\frac{\sqrt{p}}{p}\right) &= \left(x_0^2 + py_0^2 + 2\sqrt{p}x_0y_0\right)\left(\lambda_1 + \frac{\sqrt{p}}{p}\lambda_2\right)\\
&= \lambda_1(x_0^2 + py_0^2) + \lambda_2(2x_0y_0) + \sqrt{p}\left(\lambda_1(2x_0y_0) + \lambda_2\frac{x_0^2 + py_0^2}{p}\right)\\
&=: \hat{\lambda}_1 + \hat{\lambda}_2\frac{\sqrt{p}}{p}
\end{aligned} \tag{3.1}$$

with $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathbb{Z}/2$. We have $\varepsilon_0^{-1} = \pm(x_0 - y_0\sqrt{p})$ from $\operatorname{N}(\varepsilon_0) = \pm 1$, so

$$\begin{aligned}
\varepsilon_0^{-2}\left(\lambda_1 + \lambda_2\frac{\sqrt{p}}{p}\right) &= \left(x_0^2 + py_0^2 - 2\sqrt{p}x_0y_0\right)\left(\lambda_1 + \frac{\sqrt{p}}{p}\lambda_2\right)\\
&= \lambda_1(x_0^2 + py_0^2) - \lambda_2(2x_0y_0) + \sqrt{p}\left(-\lambda_1(2x_0y_0) + \lambda_2\frac{x_0^2 + py_0^2}{p}\right)\\
&=: \check{\lambda}_1 + \check{\lambda}_2\frac{\sqrt{p}}{p}
\end{aligned} \tag{3.2}$$

with $\check{\lambda}_1, \check{\lambda}_2 \in \mathbb{Z}/2$.

We distinguish the two cases $m > 0$ and $m < 0$, the case $m = 0$ belongs to the trivial case $\lambda = 0$.

$m > 0$ W.l.o.g. let $\lambda_2 \geq 0$. We show that we can achieve $\lambda_1 > 0$ by multiplication with an appropriate power of ε_0^2:

Look at (3.1). Since $p\lambda_1^2 + m = \lambda_2^2$, we have $0 \leq \sqrt{p}|\lambda_1| \leq \lambda_2$ and therefore $\hat{\lambda}_2$ is positive (remember that $(x_0 - \sqrt{p}y_0)^2 = x_0^2 + py_0^2 - 2x_0y_0\sqrt{p} \geq 0$). It remains to look at $\hat{\lambda}_1$. If $\lambda_1 \geq 0$ we have $\hat{\lambda}_1 > 0$, if it is negative we calculate

$$\begin{aligned}
\hat{\lambda}_1 &= \lambda_1(x_0^2 + py_0^2) + \sqrt{p\lambda_1^2 + m}(2x_0y_0)\\
&\overset{m>0}{>} \lambda_1(x_0^2 + py_0^2) + \sqrt{p\lambda_1^2}(2x_0y_0)\\
&= \lambda_1(x_0^2 + py_0^2) - \lambda_1\sqrt{p}(2x_0y_0)\\
&= \lambda_1\underbrace{\left(x_0 - \sqrt{p}y_0\right)^2}_{<1}
\end{aligned}$$

So in case $\lambda_1 < 0$ we have $\hat{\lambda}_1 > \lambda_1$. Since $\lambda_1, \hat{\lambda}_1 \in \frac{1}{2}\mathbb{Z}$, there is $k \in \mathbb{Z}$, such that $\acute{\lambda} = \varepsilon_0^{2k}\lambda = \acute{\lambda}_1 + \sqrt{p}\acute{\lambda}_2$ has coeffi cients $\acute{\lambda}_1, \acute{\lambda}_2 > 0$.

Have a look at (3.2). Analogously we get $\check{\lambda}_2 > 0$, it remains to investigate $\check{\lambda}_1$. In case $\lambda_1 < 0$ we obtain $\check{\lambda}_1 < 0$, otherwise the point of interest $\check{\lambda}_1 = 0$ is given for

$$\lambda_2^2 = p\lambda_1^2 + m = p\lambda_2^2\left(\frac{2x_0y_0}{x_0^2+py_0^2}\right)^2 + m.$$

This determines $\lambda_2 > 0$ uniquely and for larger values of λ_2 we get $\check{\lambda}_1 > 0$. Thus the given set $\mathcal{J}$ is fi nite (note that for given λ_2 and m the choice of λ_1 is unique up to sign) and is a set of representatives. We get explicitely:

$$\begin{aligned}
\lambda_2 &\le \sqrt{m(1-p\alpha_p)^{-1}}, \\
\lambda_1 &\le \sqrt{m\alpha_p(1-p\alpha_p)^{-1}}, \text{ where} \\
\alpha_p &:= \left(\frac{2x_0y_0}{x_0^2+py_0^2}\right)^2,
\end{aligned}$$

and insert the values $\varepsilon_0 = (1+\sqrt{5})/2$, $(3+\sqrt{13})/2$ and $4+\sqrt{17}$ to obtain:

$$\begin{aligned}
\alpha_5 &= \left(\frac{\frac{2}{4}}{\frac{1}{4}+\frac{5}{4}}\right)^2 = \left(\frac{2}{6}\right)^2 = \frac{1}{9}, \\
\alpha_{13} &= \left(\frac{\frac{6}{4}}{\frac{9}{4}+\frac{13}{4}}\right)^2 = \left(\frac{6}{22}\right)^2 = \frac{9}{121} \text{ and} \\
\alpha_{17} &= \left(\frac{2\cdot4\cdot1}{4^2+17}\right)^2 = \frac{2^6}{3^2 11^2} = \frac{64}{1089}.
\end{aligned}$$

$m < 0$ W.l.o.g. let $\lambda_1 \ge 0$. We show that we can achieve $\lambda_2 > 0$ by multiplication with an appropriate power of ε_0:

From $p\lambda_1^2 + m = \lambda_2^2$ we get $\lambda_1 \ge |\lambda_2|\sqrt{p}^{-1}$. Together with the second binomial formula we get that $\hat{\lambda}_1$ in (3.1) is positive.

In case $\lambda_2 \ge 0$ we have $\hat{\lambda}_2 = p\lambda_1(2x_0y_0) + \lambda_2(x_0^2+py_0^2) > 0$. If $\lambda_2 < 0$ we get

$$\begin{aligned}
\hat{\lambda}_2 &= p\lambda_1(2x_0y_0) + \lambda_2(x_0^2+py_0^2) \\
&= p\sqrt{\frac{\lambda_2^2 - m}{p}}2x_0y_0 + \lambda_2\left(x_0^2+py_0^2\right) \\
&\overset{-m>0}{>} \sqrt{p}|\lambda_2|(2x_0y_0) + \lambda_2\left(x_0^2+py_0^2\right) \\
&= \lambda_2(x_0^2+py_0^2-2x_0y_0\sqrt{p}) = \lambda_2\underbrace{(x_0-\sqrt{p}y_0)^2}_{\substack{<1,\text{ since }\varepsilon_0>1\\ \text{and }|\mathrm{N}(\varepsilon_0)|=1}} > \lambda_2.
\end{aligned} \tag{3.3}$$

Due to the discrete possibilities for λ_2 (in $\mathbb{Z}/2$) there is $k \in \mathbb{Z}$, such that $\grave{\lambda} = \varepsilon_0^{2k}\lambda = \grave{\lambda}_1 + \sqrt{p}\grave{\lambda}_2$ has coefficients $\grave{\lambda}_1, \grave{\lambda}_2 > 0$.

Have a look at (3.2). Analogously we get $\check{\lambda}_1 > 0$ and it remains to investigate $\check{\lambda}_2$. If $\lambda_2 < 0$ then we have $\check{\lambda}_2 < 0$, otherwise the critical point $\check{\lambda}_2 = 0$ is given by

$$0 = -\lambda_1(2x_0y_0) + \lambda_2(x_0^2 + py_0^2)/p = -\sqrt{\frac{\lambda_2^2 - m}{p}}2x_0y_0 + \lambda_2\frac{x_0^2 + py_0^2}{p}.$$

This is equivalent to

$$\frac{\lambda_2^2 - m}{p}(2x_0y_0)^2 = \lambda_2^2\frac{(x_0^2 + py_0^2)^2}{p^2} \text{ and } \lambda_2 > 0.$$

Larger values for λ_2 give positive $\check{\lambda}_2$. We precisely get

$$\lambda_2 \leq \sqrt{\frac{-mp\alpha_p}{1 - p\alpha_p}} \text{ and}$$
$$\lambda_1 \leq \sqrt{\frac{-m}{p(1 - p\alpha_p)}}\,.$$

Only those λ belong to $\mathcal{J}$ and its finiteness is proved. Analogously to (3.3) we obtain for $\lambda_2 > 0$

$$\lambda_2 > \check{\lambda}_2 \underbrace{(x_0 - y_0\sqrt{p})^2}_{<1}.$$

Especially $\mathcal{J}$ is a set of representatives of $\mathcal{I}/\sim$. □

Lemma 3.2.3 (Shape of $S(m)$). *For every prime number p and every $m > 0$ the set $S(m)$ is the intersection of $\mathbb{H}^2$ with an empty or an infinite union of hyperplanes of the real vector space $\mathbb{C}^2$. We have*

$$S(m) = \bigcup_{\lambda \in \mathcal{I}} \left\{(z_1, z_2) \in \mathbb{H} \times \mathbb{H}; \quad \lambda \operatorname{Im}(z_1) + \overline{\lambda}\operatorname{Im}(z_2) = 0\right\},$$

where $\mathcal{I}$ is as in Lemma 3.2.2. Especially $S(m)$ is invariant under the stabilizer Γ_∞ of infinity.

Proof. $S(m)$ has the given shape by Definition 3.1.1. Let $\mathcal{I}$ be the set of λ in $\mathfrak{o}/\sqrt{p}$ with $-\operatorname{N}(\lambda) = m/p$ and let $\mathcal{I}$ be nonempty, e.g. let $\lambda \in \mathcal{I}$ be an element. Clearly the set

$$M(\lambda) = \left\{(\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}; \quad \lambda \operatorname{Im}(\tau_1) + \overline{\lambda}\operatorname{Im}(\tau_2) = 0\right\}$$

is mapped onto itself by real transformations $\mathbb{H}^2 \to \mathbb{H}^2, \tau \mapsto \tau + r$, $r \in \mathbb{R}^2$. Let τ in $M(\lambda)$ and $k \in \mathbb{Z}$. Then $\tau^{(k)} := \varepsilon_0^{2k}\tau = \left(\varepsilon_0^{2k}\tau_1, \overline{\varepsilon_0}^{2k}\tau_2\right)$ holds

$$\varepsilon_0^{-2k}\lambda \operatorname{Im}\left(\tau_1^{(k)}\right) + \overline{\varepsilon_0}^{-2k}\overline{\lambda}\operatorname{Im}\left(\tau_2^{(k)}\right) = 0,$$

so $\tau^{(k)}$ is an element of $M(\varepsilon_0^{-2k}\lambda)$.

Γ_∞ is generated by real transformations and multiplication with ε_0^{2k} ($k \in \mathbb{Z}$), so we have shown the invariance under Γ_∞. We rewrite $M(\lambda)$ into

$$M(\lambda) = \left\{ (z_1, z_2) \in \mathbb{H} \times \mathbb{H}; \quad \mathrm{Im}\,(z_2) = \frac{-\lambda}{\overline{\lambda}}\,\mathrm{Im}\,(z_1) \right\}.$$

Since for all $k \in \mathbb{Z} \setminus \{0\}$ we have $\varepsilon_0^{-2k}/\overline{\varepsilon_0}^{-2k} \neq 1$, the sets $M(\lambda)$ and $M(\varepsilon_0^{-2k}\lambda)$ do not coincide, so $\mathcal{I}$ is either empty or has an infinite number of elements. □

Remark 3.2.4 (Calculation of $S(m)$). *Let $m > 0$. If we use both Lemma 3.2.3 and Lemma 3.2.2, we get a program for the calculation of $S(m)$. We take all positive λ_2 in $\frac{1}{2}\mathbb{Z}$ smaller than $\sqrt{\frac{m}{1-p\alpha_p}}$. Then $\lambda_1 > 0$ is uniquely determined by the formula $p\lambda_1^2 + m = \lambda_2^2$. We only have to check whether $\lambda_1 \in \mathbb{Z}/2$ or not. Then we have calculated $S(m)$ modulo multiplication by ε_0^2.*

Lemma 3.2.5. *If $m > 0$ and $\chi_p(m) = -1$ then $S(m) = \emptyset$.*

Proof. We write $\lambda = \lambda_1/2 + \sqrt{p}\lambda_2/(2p)$ where λ_1 and λ_2 are integers and get the equation

$$-4p\,\mathrm{N}(\lambda) = b^2 - pa^2 = 4m.$$

Especially there is no such λ, if $4m$ and therefore m modulo p is no square. □

3.3 Weyl Vector

We calculate the constants ρ_W and $\overline{\rho_W}$ and simplify the representation of $R(W, n)$. Some results can be found in Table A.6.

Lemma 3.3.1 (Empty $S(m)$ and ρ_W). *If $S(m)$ is empty, there is exactly one Weyl chamber, namely $\mathbb{H}^2$. In this case $\rho_W = 0$.*

Proof. The first statement is trivial. Consider the case $S(m) = \emptyset$, so $W = \mathbb{H}^2$ is a Weyl chamber. By Remark 3.1.6 we have

$$\rho_W\,\mathrm{Im}\,(\tau_1) + \overline{\rho_W}\,\mathrm{Im}\,(\tau_2) = \sum_{n<0} s(n)a(n) \sum_{\substack{\lambda \in \mathfrak{o}/\sqrt{p} \\ \lambda > 0 \\ \mathrm{N}(\lambda) = n/p}} \min(|\lambda\,\mathrm{Im}\,(\tau_1)\,|, |\overline{\lambda}\,\mathrm{Im}\,(\tau_2)\,|).$$

If we fix τ_1 in $\mathbb{H}$ and write $\tau_2 = it$, the limit $t \searrow 0$ shows that $\rho_W = 0$. If we fix z_2 and write $z_1 = it$ we get $\overline{\rho_W} = 0$. □

Lemma 3.3.2. *Weyl chambers are open in $\mathbb{H}^2$.*

Proof. Consider $m \in \mathbb{N}$ ($m > 0$). Since there is a finite set $\mathcal{J}$ as in Lemma 3.2.2 with

$$\mathcal{I} = \left\{ \lambda \in \frac{\mathfrak{o}}{\sqrt{p}} \mid \mathrm{N}(\lambda) = -\frac{m}{p} \right\} = \left\{ \pm\lambda\varepsilon_0^{2k} \mid k \in \mathbb{Z}, \lambda \in \mathcal{J} \right\},$$

the set $\mathcal{I}$ is closed in $\mathbb{R}^*$, so also $\bigcup_{\lambda\in\mathcal{I}} \left\{ \tau \in \mathbb{H}^2 \mid \mathrm{Im}\,(y_1) = \frac{\lambda}{\overline{\lambda}}\,\mathrm{Im}\,(y_2) \right\}$ is closed in $\mathbb{H}^2$.
As a consequence, each component of the complement ist open. Since each Weyl chamber is a finite intersection of such components, it is open alike. $\square$

Lemma 3.3.3 (ρ_W and $\overline{\rho_W}$). *Let $p = 5$, $p = 13$ or $p = 17$ and let $m \in \mathbb{N}$ such that $\chi_p(m) \neq -1$. If W is a Weyl chamber attached to f_m (cf. Definition 2.5.37), we have*

$$\rho_W = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,m)} \lambda\varepsilon_0$$

and

$$\overline{\rho_W} = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,m)} \overline{\lambda}\overline{\varepsilon_0}.$$

Proof. By Remark 3.1.6 we have:

$$\rho_W y_1 + \overline{\rho_W} y_2 = \sum_{n<0} \underbrace{s(n)a(n)}_{\delta_{m,n}} \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,n)} \left(\varepsilon_0 \lambda y_1 + \overline{\varepsilon_0}\overline{\lambda} y_2 \right)$$

for all $\tau \in W, y_1 = \mathrm{Im}\,(\tau_1)\,, y_2 = \mathrm{Im}\,(\tau_2)$. The Weyl chamber is an open set, so for sufficiently small $\delta_1, \delta_2 \geq 0$, $\delta_1 + \delta_2 > 0$ and $\tau \in W$ we have $(\tau_1 + i\delta_1, \tau_2 + i\delta_2) \in W$ and get

$$\rho_W(y_1 + \delta_1) + \overline{\rho_W}(y_2 + \delta_2) = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,m)} \left(\varepsilon_0 \lambda (y_1 + \delta_1) + \overline{\varepsilon_0}\overline{\lambda}(y_2 + \delta_2) \right).$$

We substract the equation for τ and obtain

$$\rho_W(\delta_1) + \overline{\rho_W}(\delta_2) = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,m)} \left(\varepsilon_0 \lambda \delta_1 + \overline{\varepsilon_0}\overline{\lambda}\delta_2 \right).$$

If we insert $\delta_2 = 0$ into this equation and divide by $\delta_1 > 0$, we get

$$\rho_W = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,m)} \varepsilon_0 \lambda,$$

while, if we insert $\delta_1 = 0$ and divide by $\delta_2 > 0$, we get

$$\overline{\rho_W} = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{\lambda \in R(W,m)} \overline{\varepsilon_0}\overline{\lambda}.$$

We especially proved that $\overline{\rho_W}$ is the element of $\mathcal{K}$ conjugated to ρ_W by the field automorphism $\bar{\cdot}$. $\square$

Let $p \in \{5, 13, 17\}$, $m \in \mathbb{N}$ with $\chi_p(m) \neq -1$ and consider $f_m(\tau) = \sum_n a(n) q^n$. We want to calculate $R(W, n)$. Define

$$R(n) := \left\{ \lambda \in \frac{\mathfrak{o}}{\sqrt{p}}, \lambda > 0, \mathrm{N}(\lambda) = \frac{n}{p} \right\}.$$

We write $\lambda = \lambda_1 + \lambda_2/\sqrt{p}$ and get

$$\begin{aligned} R(n) &= \left\{ \lambda_1 + \lambda_2/\sqrt{p} \in \frac{\mathfrak{o}}{\sqrt{p}}, \lambda_1 + \lambda_2\sqrt{p} > 0, p\,\mathrm{N}(\lambda_1 + \lambda_2/\sqrt{p}) = p\lambda_1^2 - \lambda_2^2 = n \right\} \\ &= \left\{ \frac{\mu}{\sqrt{p}};\ \mu \in \mathfrak{o}, \mu > 0, \mathrm{N}(\mu) = -n \right\}. \end{aligned}$$

For all $n < 0$ with $a(n) \neq 0$ we have

$$\begin{aligned} R(W, n) &= \left\{ \lambda \in \frac{\mathfrak{o}}{\sqrt{p}}; \lambda > 0, \mathrm{N}(\lambda) = \frac{n}{p}, \forall \tau \in W : \begin{array}{c} \lambda \operatorname{Im}(\tau_1) + \overline{\lambda} \operatorname{Im}(\tau_2) < 0, \\ \varepsilon_0^2 \lambda \operatorname{Im}(\tau_1) + (\overline{\varepsilon_0})^2 \overline{\lambda} \operatorname{Im}(\tau_2) > 0 \end{array} \right\} \\ &= \left\{ \lambda \in \frac{\mathfrak{o}}{\sqrt{p}}; \lambda > 0, \mathrm{N}(\lambda) = \frac{n}{p}, \forall \tau \in W : \begin{array}{c} \operatorname{Im}(\tau_1) < \frac{-\overline{\lambda}}{\lambda} \operatorname{Im}(\tau_2), \\ \operatorname{Im}(\tau_1) > \frac{-(\overline{\varepsilon_0})^2 \overline{\lambda}}{\varepsilon_0^2 \lambda} \operatorname{Im}(\tau_2) \end{array} \right\}. \end{aligned} \tag{3.4}$$

Lemma 3.3.4 (Choice of Weyl chamber). *Let $p \in \{5, 13, 17\}$ (or any other prime p with $\mathrm{N}(\varepsilon_0) = -1$), let $m = -n$ be a natural number and $\tau \in \mathbb{H}^2$. Then $W(\tau)$ defines the Weyl chamber attached to $\tilde{\tau} := \tau + (i\overline{\delta}, i\delta)$ for sufficiently small $\delta \in \mathcal{K}$, $\delta \geq 0$, in the following sense: If τ is contained in a Weyl chamber, then we define $W(\tau)$ to be this Weyl chamber ($\delta = 0$). Else if $\operatorname{Im}(\tau_1) \neq \operatorname{Im}(\tau_2)$ there is a Weyl chamber, which we denote by $W(\tau)$, and some $\delta_0 > 0$ such that for $\delta = (i\delta_1, i\delta_1)$ we have $\tau + \delta \in W(\tau)$ for all $0 < \delta_1 < \delta_0$, $\delta_1 \in \mathbb{Q}$. In the case that τ is not contained in a Weyl chamber and $\operatorname{Im}(\tau_1) = \operatorname{Im}(\tau_2)$, there is an unique Weyl chamber, which we denote by $W(\tau)$, and some $\delta_0 > 0$ such that $\tau + (-i\delta_2\sqrt{p}, i\delta_2\sqrt{p})$ is contained in $W(\tau)$ for all $0 < \delta_2 < \delta_0$ with $\delta_2 \in \mathbb{Q}$.*

Our standard choice for τ will be $\tau = (-i\overline{\varepsilon_0} + i\varepsilon_0)$ and $\tilde{\tau} := (-i\overline{\varepsilon_0} + i\overline{\delta}, i\varepsilon_0 + i\delta)$.

Proof. If $\tau \in \mathbb{H}$ is not contained in a Weyl chamber, then $\tau \in S(m)$. By Lemma 3.2.3 we know that $S(m)$ is modulo multiplication with ε_0^2 a finite union of hyperplanes $M(\lambda)$. The projection of these hyperplanes on the imaginary parts are straight lines through 0 intersected with $\mathbb{H}^2$. Hence for $\tau \in S(m)$ the point $(\operatorname{Im}(\tau_1), \operatorname{Im}(\tau_2))$ lies on the straight line through 0 with direction $(\operatorname{Im}(\tau_1), \operatorname{Im}(\tau_2))$ and the choice of $W(\tau)$ described in the Lemma is unique and well defined. □

Now we can easily calculate $R(W, n)$.

Lemma 3.3.5 (Calculation of $R(W, n)$). *Let p be an odd prime, let $m = -n$ be a natural number and $\tau \in W$ for some Weyl chamber W. Then $R(W, n)$ can be calculated by*

For every element λ in a set of representatives of $R(-m)$ modulo multiplication with ε_0^2 do

- *Multiply λ with ε_0^2 (and denote the result again by λ) until $\lambda y_1 + \overline{\lambda} y_2 > 0$ for the imaginary part y of τ.*
- *Multiply λ with ε_0^{-2}, until $\lambda y_1 + \overline{\lambda} y_2 < 0$.*

The resulting $M(\lambda)$ is an element of $R(W,n)$ and this procedure gives all of its elements when applied to all λ in $R(-m)/\varepsilon_0^2$.

Proof. We have $\varepsilon_0 > 1$ and $\mathrm{N}(\varepsilon_0) = \pm 1$, so $0 < \overline{\varepsilon_0^2} = \varepsilon_0^{-2} < 1$. Let $\tau \in \mathbb{H}^2$ and $\lambda \in \mathfrak{o}/\sqrt{p}$ with $\lambda > 0$. Write $y_1 = \mathrm{Im}\,(\tau_1)$ and $y_2 = \mathrm{Im}\,(\tau_2)$. Then $\overline{\lambda} = \mathrm{N}(\lambda)/\lambda = -\frac{m}{p}\lambda < 0$ and we get

$$\varepsilon_0^{2k} \underbrace{\lambda y_1}_{>0} + \overline{\varepsilon_0^{2k}} \overline{\lambda} y_2 \overset{k\to\infty}{\longrightarrow} +\infty, \qquad \varepsilon_0^{-2k} \lambda y_1 + \overline{\varepsilon_0^{-2k}} \underbrace{\overline{\lambda} y_2}_{<0} \overset{k\to\infty}{\longrightarrow} -\infty$$

and

$$\underbrace{\varepsilon_0^2}_{>1} \underbrace{\lambda y_1}_{>0} + \underbrace{\overline{\varepsilon_0^2}}_{<1} \underbrace{\overline{\lambda} y_2}_{<0} > \lambda y_1 + \overline{\lambda} y_2.$$

So the algorithm described in the lemma gives some $\tilde{\lambda} = \varepsilon_0^{2k}\lambda$ with $\tilde{\lambda} \in R(W,-n)$, $k \in \mathbb{Z}$ and clearly it suffi ces to apply this algorithm on a set of representatives of $\mathcal{I}$ modulo multiplication with ε_0^2. □

Lemma 3.3.6. *Let p be a prime number with $\chi_p(-1) = 1$. Define $x = (-i\overline{\varepsilon_0}, i\varepsilon_0)$. Then x is contained in the Weyl chamber $W(x)$ attached to f_1 and we have $R(W(x), -1) = \left\{1/\sqrt{p}\right\}$.*

Remark 3.3.7. *The restriction $(W,\lambda) > 0$ in the formula of Borcherds products can be replaced by $(\tau,\lambda) > 0$ for a point $\tau \in W$, where we define $(\tau,\lambda) > 0$ if $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) > 0$. This follows directly from Remark 3.4.1.*

From (3.4) we get

Remark 3.3.8 (Interpretation of $R(W,n)$). *If W is a Weyl chamber attached to f_n, then the boundary of W in $\mathbb{H}^2$ is a subset of*

$$\bigcup_{\lambda \in R(W,n)} \left(M(\lambda) \cup M(\varepsilon_0^2 \lambda) \right).$$

Especially the boundary is the union of two $M(\mu)$.

Proof of Lemma 3.3.6. We have $x \in W(x)$, iff $x \notin S(-1)$. For all $\lambda \in \mathfrak{o}/\sqrt{p}$ with $\mathrm{N}(\lambda) = -1/p$ we have

$$\begin{aligned} x \in M(\lambda) \quad &\iff \quad -\lambda\overline{\varepsilon_0} + \overline{\lambda}\varepsilon_0 = 0 \\ &\iff \quad \overline{\lambda}\varepsilon_0 = \lambda\overline{\varepsilon_0} \qquad\qquad \cdot(-\overline{\lambda}\overline{\varepsilon_0}) \\ &\iff \quad \overline{\lambda}^2 = -\lambda\overline{\lambda}\overline{\varepsilon_0}^2 = -\mathrm{N}(\lambda)\overline{\varepsilon_0}^2 = \overline{\varepsilon_0}^2 p^{-1} \\ &\iff \quad \lambda = \pm\overline{\varepsilon_0}/\sqrt{p}. \end{aligned}$$

In this case

$$-\frac{1}{p} = \mathrm{N}(\lambda) = \frac{\varepsilon_0}{\sqrt{p}} \cdot \overline{\frac{\varepsilon_0}{\sqrt{p}}} = \frac{\varepsilon_0}{\sqrt{p}} \cdot \frac{\overline{\varepsilon_0}}{-\sqrt{p}} = \frac{1}{p}$$

shows that x is not contained in $S(-1)$ and $x \in W(x)$ holds. We have $R(-1) = \left\{ \frac{\varepsilon_0^{2k}}{\sqrt{p}} \middle| k \in \mathbb{Z} \right\}$, because $\mathfrak{o}^* = \pm \varepsilon_0^{\mathbb{Z}}$ consists of the elements in $\mathfrak{o}$ of norm ± 1 and $\mathrm{N}(\varepsilon_0) = -1$. We have $\overline{\varepsilon_0^k / \sqrt{p}} = -\overline{\varepsilon_0}^k / \sqrt{p}$, so

$$R(W, -1) = \left\{ \frac{\varepsilon_0^{2k}}{\sqrt{p}}; \quad k \in \mathbb{Z}, \forall z \in W : \begin{array}{c} \varepsilon_0^{2k} \operatorname{Im}(z_\tau) - (\overline{\varepsilon_0})^{2k} \operatorname{Im}(\tau_2) < 0, \\ \varepsilon_0^{2(k+1)} \operatorname{Im}(\tau_1) - (\overline{\varepsilon_0})^{2(k+1)} \operatorname{Im}(\tau_2) > 0 \end{array} \right\}.$$

The restrictions on $R(W, -1)$ for $z = x$ ($\mathrm{N}(\varepsilon_0) = -1$) are:

$$\varepsilon_0^{2k-1} + (\overline{\varepsilon_0})^{2k-1} < 0 \text{ and } \varepsilon_0^{2k+1} + (\overline{\varepsilon_0})^{2k+1} > 0.$$

After multiplication by $\varepsilon_0^{2k \pm 1} > 0$ and addition of 1 this yields

$$\varepsilon_0^{4k-2} < 1 \text{ and } \varepsilon_0^{4k+2} > 1.$$

This is equivalent to $k = 0$. So

$$R(W, -1) \subset \{1/\sqrt{p}\}.$$

The continuous restrictions for τ in $W(x)$ are

$$\operatorname{Im}(\tau_1) - \operatorname{Im}(\tau_2) < 0,$$
$$\varepsilon_0^2 \operatorname{Im}(\tau_1) - (\overline{\varepsilon_0})^2 \operatorname{Im}(\tau_2) > 0,$$

and every point, for which the first or the second inequality is valid as an equality, is contained in

$$S(1) \supset \left\{ \tau \in \mathbb{H}^2; \quad \frac{\operatorname{Im}(\tau_1)}{\sqrt{p}} - \frac{\operatorname{Im}(\tau_2)}{\sqrt{p}} = 0 \right\} \cup \left\{ \tau \in \mathbb{H}^2; \quad \frac{\varepsilon_0^2}{\sqrt{p}} \operatorname{Im}(\tau_1) - \frac{\overline{\varepsilon_0}^2}{\sqrt{p}} \operatorname{Im}(\tau_2) = 0 \right\},$$

so the inequalities hold for all $\tau \in W(x)$ due to their continuity and the connectivity of $W(x)$. $\square$

From Lemma 3.3.6 we get the following

Corollary 3.3.9. *The first factor of the product expansion of the Borcherds product corresponding to f_1 is given by $\mathbf{e}(\rho_W \tau_1 + \overline{\rho_W} \tau_2)$, where*

$$\rho_W = \frac{1 + \sqrt{5}}{2\sqrt{5}} \text{ if } p = 5,$$
$$\rho_W = \frac{3 + \sqrt{13}}{6\sqrt{13}} \text{ if } p = 13,$$
$$\rho_W = \frac{4 + \sqrt{17}}{8\sqrt{17}} \text{ if } p = 17 \text{ and more general}$$
$$\rho_W = \frac{\varepsilon_0}{\mathrm{S}(\varepsilon_0)\sqrt{p}} \text{ if } p \equiv 1 \pmod 4.$$

Proof. By Lemma 3.3.6 we have $R(W,-1) = \left\{1/\sqrt{p}\right\}$, so we get the stated result by the formula $\rho_W = \frac{1}{S(\varepsilon_0)} \sum_{\lambda \in R(W,n)} \lambda\varepsilon_0$ of Lemma 3.3.3 and the values of ε_0 given in Definition 1.2.13. □

3.4 Weyl Chambers

We investigate properties of Weyl chambers. Especially we find that the concrete choice of Weyl chamber in Theorem 3.1.4 influences the resulting Borcherds product only up to a constant factor.

For this section let $j \in \mathbb{N}$ such that $\chi_p(j) \geq 0$ and let $f_j = \frac{1}{s(j)} q^{-j} + O(1) \in A_0^+(p,\chi_p)$.

Remark 3.4.1. *If W is a Weyl chamber attached to $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p,\chi_p)$ and $\lambda \in \mathfrak{o}/\sqrt{p}$, then for every $a(-p\,\mathrm{N}(\lambda)) \neq 0$ the condition $(\lambda, W) > 0$ is equivalent to the existence of a point $(\tau_1,\tau_2) \in W$ with $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) > 0$.*

Proof. In case $\mathrm{N}(\lambda) = 0$, the condition $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) > 0$ holds for no $\tau \in \mathbb{H}^2$. In case $\mathrm{N}(\lambda) = \lambda\overline{\lambda} > 0$, both λ and $\overline{\lambda}$ share the same sign. Thus for all $\tau \in \mathbb{H}^2$ we have

$$\lambda \underbrace{\mathrm{Im}\,(\tau_1)}_{>0} + \overline{\lambda} \underbrace{\mathrm{Im}\,(\tau_2)}_{>0} > 0 \qquad \text{iff} \qquad \lambda > 0,$$

so especially it does not depend on τ or W. In case $\mathrm{N}(\lambda) < 0$ and $a(-p\,\mathrm{N}(\lambda)) \neq 0$ every element τ in W has $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) \neq 0$. This depends continuously on τ, so the sign of $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2)$ is constant on the connected set W. □

From the Definition of $S(m)$ and of Weyl chambers we get:

Remark 3.4.2 (Symmetry).

- *$S(m)$ is a symmetric subset of $\mathbb{H}^2$, i.e. either both or neither (τ_1,τ_2) and (τ_2,τ_1) are contained in $S(m)$, since every $M(\lambda) \cup M(\overline{\lambda})$ is symmetric and for $\lambda \in \mathfrak{o}/\sqrt{p}$ we have $\overline{\lambda} \in \mathfrak{o}/\sqrt{p}$ and $\mathrm{N}(\lambda) = \mathrm{N}(\overline{\lambda})$.*
- *The Weyl chambers are not necessarily symmetric, but if W is a Weyl chamber, then $\overline{W} := \{(\tau_2,\tau_1)|(\tau_1,\tau_2) \in W\}$ is a Weyl chamber, too.*

Remark 3.4.3. *If W is a Weyl chamber attached to $f_j = q^{-j} + O(1)$, then $D_{\varepsilon_0}W$ is a Weyl chamber attached to f_j.*

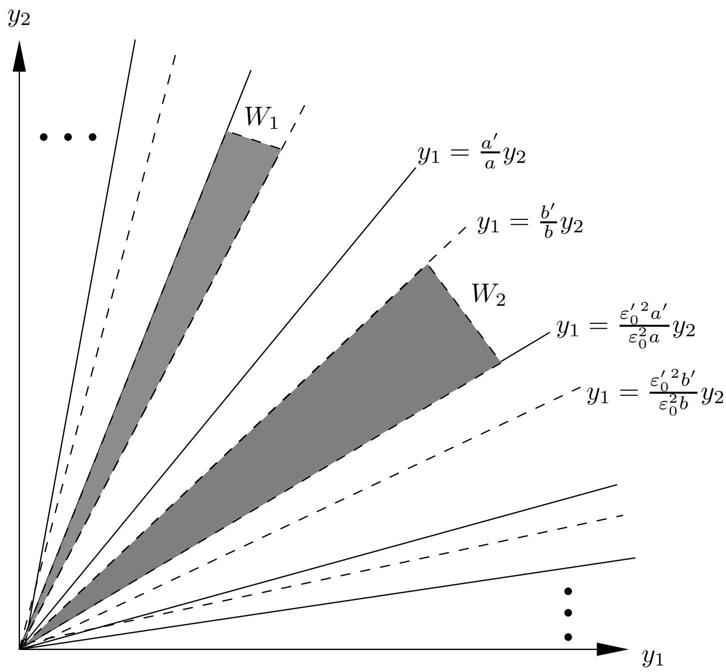

Figure 3.1: Imaginary parts of Weyl chambers

Lemma 3.2.2 together with Lemma 3.2.3 shows that $S(j)$ *is a countable (or empty) union of hyperplanes* $E_\lambda = \{\tau \in \mathbb{H}^2;\ \mathrm{Im}(\tau_1) = \frac{\overline{\lambda}}{\lambda} \mathrm{Im}(\tau_2)\}$ *which is, modulo multiplication with* ε_0^2*, a finite union of hyperplanes. The sketch shows the case* $S(j) = \cup_{m \in \mathbb{Z}} \left\{ E_{\varepsilon_0^{2m} a} \cup E_{\varepsilon_0^{2m} b} \right\}$ *in a projection of* $S(j)$ *and its hyperplanes and the Weyl chambers on the imaginary parts. Each Weyl chamber is the product of its projection on the imaginary part and* $\mathbb{R}^2$*, if we write* $\mathbb{H}^2 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2$*.*

Proof. Let $\lambda \in \mathfrak{o} / \sqrt{p}$ with $-N(\lambda) = j/p$. Then we have

$$\begin{aligned} D_{\varepsilon_0} M(\lambda) &= \left\{ D_{\varepsilon_0} z \in \mathbb{H}^2;\ \lambda \,\mathrm{Im}(z_1) + \overline{\lambda}\,\mathrm{Im}(z_2) = 0 \right\} \\ &\overset{\tau = Dz}{=} \left\{ \tau \in \mathbb{H}^2;\ \lambda \,\mathrm{Im}\left(\varepsilon_0^{-2} \tau_1\right) + \overline{\lambda}\,\mathrm{Im}\left(\overline{\varepsilon_0}^{-2} \tau_2\right) = 0 \right\} \\ &= \left\{ \tau \in \mathbb{H}^2;\ \varepsilon_0^{-2} \lambda \,\mathrm{Im}(\tau_1) + \overline{\varepsilon_0}^{-2} \overline{\lambda}\,\mathrm{Im}(\tau_2) = 0 \right\} \\ &= M(\varepsilon_0^{-2} \lambda)\ . \end{aligned}$$

□

Lemma 3.4.4 (Change of Weyl chamber). *Let* Ψ_1 *and* Ψ_2 *be Borcherds products in the sense of Theorem 3.1.4 for* f *with different Weyl chambers* W_1 *and* W_2 *attached to* f*. Then there is* $c \in \mathbb{C} \setminus \{0\}$ *such that* $\Psi_1 = c\Psi_2$*.*

Indirect proof of Lemma 3.4.4. Ψ_1 and Ψ_2 are Hilbert modular forms of weight k with divisor F. So Ψ_1/Ψ_2 is a Hilbert modular form of weight 0 and divisor 0, so it has a trivial multiplier system and constantly equals some $c \in \mathbb{C} \setminus \{0\}$. □

In a special case we can give a longer but easy and direct proof which uses the product expansion of the Borcherds product:

Direct proof of Lemma 3.4.4 in the case $W_1 = D_{\varepsilon_0}^k W_2$, $k \in \mathbb{Z}$.

Let $W_1 = D_{\varepsilon_0} W_2$. We have

$$\begin{aligned}
\lambda \in R(W_1, j) &\iff \forall \tau \in W_1 : \begin{Bmatrix} \lambda \operatorname{Im}(\tau_1) + \overline{\lambda} \operatorname{Im}(\tau_2) < 0 \text{ and} \\ \varepsilon_0^2 \lambda \operatorname{Im}(\tau_1) + \overline{\varepsilon_0^2 \lambda} \operatorname{Im}(\tau_2) > 0 \end{Bmatrix} \text{ as well as } \lambda \in \frac{\mathfrak{o}}{\sqrt{p}} \\
&\overset{\tau_1 = \varepsilon_0^2 z_1}{\underset{\tau_2 = \overline{\varepsilon_0}^2 z_2}{\iff}} \forall z \in W_2 : \begin{Bmatrix} \varepsilon_0^2 \lambda \operatorname{Im}(z_1) + \overline{\varepsilon_0^2 \lambda} \operatorname{Im}(z_2) < 0 \text{ and} \\ \varepsilon_0^4 \lambda \operatorname{Im}(z_1) + \overline{\varepsilon_0^4 \lambda} \operatorname{Im}(z_2) > 0 \end{Bmatrix} \text{ as well as } \varepsilon_0^2 \lambda \in \frac{\mathfrak{o}}{\sqrt{p}} \\
&\iff \varepsilon_0^2 \lambda \in R(W_2, j) \ ,
\end{aligned}$$

thus

$$\rho_{W_1} = \frac{\varepsilon_0}{\mathrm{S}(\varepsilon_0)} \sum_{\lambda \in R(W_1, j)} \varepsilon_0^{-2} (\varepsilon_0^2 \lambda) \overset{\mu = \varepsilon_0^2 \lambda}{=} \frac{\varepsilon_0}{\mathrm{S}(\varepsilon_0)} \sum_{\mu \in R(W_2, j)} \varepsilon_0^{-2} \mu = \varepsilon_0^{-2} \rho_{W_2}.$$

Additionally for all $\lambda \in \mathfrak{o} / \sqrt{p}$ we have

$$\begin{aligned}
(\lambda, W_1) > 0 &\iff \forall \tau \in W_1 : \ \lambda \operatorname{Im}(\tau_1) + \overline{\lambda} \operatorname{Im}(\tau_2) > 0 \\
&\overset{\tau_1 = \varepsilon_0^2 z_1}{\underset{\tau_2 = \overline{\varepsilon_0}^2 z_2}{\iff}} \forall z \in W_2 : \ \varepsilon_0^2 \lambda \operatorname{Im}(z_1) + \overline{\varepsilon_0^2 \lambda} \operatorname{Im}(z_2) > 0 \\
&\iff (\varepsilon_0^2 \lambda, W_2) > 0.
\end{aligned}$$

We insert this into the product expansion in Theorem 3.1.4 on page 74 and get from the convergence of the products $\Psi_1(\tau)$, where $\tau \in W_1$, and $\Psi_2(z)$, where $z \in W_2$, the equation

$$\begin{aligned}
\Psi_1(\tau) &= \mathbf{e}(\rho_{W_1} \tau_1 + \overline{\rho_{W_1}} \tau_2) \prod_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ (\nu, W_1) > 0}} (1 - \mathbf{e}(\nu \tau_1 + \overline{\nu} \tau_2))^{s(p\nu\overline{\nu}) a(p\nu\overline{\nu})} \\
&\overset{\tau = Dz}{=} \mathbf{e}(\rho_{W_1} \varepsilon_0^2 z_1 + \overline{\rho_{W_1} \varepsilon_0^2} z_2) \prod_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ (\nu, W_1) > 0}} \Big(1 - \mathbf{e}(\nu \varepsilon_0^2 z_1 + \overline{\nu \varepsilon_0^2} z_2)\Big)^{s(p\nu\overbrace{\varepsilon_0^2 \overline{\varepsilon_0}^2}^{=1}\overline{\nu}) a(p\nu\overbrace{\varepsilon_0^2 \overline{\varepsilon_0}^2}^{=1}\overline{\nu})} \\
&= \mathbf{e}(\rho_{W_2} z_1 + \overline{\rho_{W_2}} z_2) \prod_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ (\mu, W_2) > 0}} (1 - \mathbf{e}(\mu z_1 + \overline{\mu} z_2))^{s(p\mu\overline{\mu}) a(p\mu\overline{\mu})} = \Psi_2(z) = \Psi_2(D_{\varepsilon_0} \tau) \ .
\end{aligned}$$

Ψ_2 is a Hilbert modular form of weight k holding $\Psi_2(D_{\varepsilon_0} \tau) = \mu_{\Psi_2}(D_{\varepsilon_0}) \cdot N(\varepsilon_0^{-1})^k \cdot \Psi_2(\tau)$, so $\Psi_1 = \mu_{\Psi_2}(D_{\varepsilon_0}) (-1)^k \Psi_2$. $\square$

3.5 Hirzebruch-Zagier Divisors

All divisors of Borcherds products are Hirzebruch-Zagier divisors and vice versa (cf. Remark 3.1.6). In his book ([Ge88]) van der Geer describes for discriminant D the shape of some special sets of quadratic equations. A special case of this gives us the number of generating equations of $T(m)$ for given p and m. Then we just have to find sufficiently many independent equations, which works well in the cases we need.

We rewrite the Definition of $T(m)$ (cf. Definition 3.1.1) in an equivalent form:

$$T(m) := \bigcup_{\substack{(a,b,\lambda)\in\mathfrak{L}' \\ -q(a,b,\lambda)=ab-\mathrm{N}(\lambda)=m/p}} M(a,b,\lambda), \text{ where}$$
$$M(a,b,\lambda) := \left\{(\tau_1,\tau_2)\in\mathbb{H}\times\mathbb{H};\quad a\tau_1\tau_2+\lambda\tau_1+\overline{\lambda}\tau_2+b=0\right\}.$$

We want to investigate the operation of $\mathrm{SL}(2,\mathfrak{o})$ on $T(m)$. Since Hilbert modular forms are invariant under $\mathrm{SL}(2,\mathfrak{o})$ up to multiplier and $(c\tau+d)^k\neq 0$, their roots are invariant under $\mathrm{SL}(2,\mathfrak{o})$, i.e. $\mathrm{SL}(2,\mathfrak{o})T(m)=T(m)$. One easily checks that $\mathrm{SL}(2,\mathfrak{o})$ permutes the sets $M(a,b,\lambda)$. We need a representation

$$T(m)=\bigcup_{(a,b,\lambda)\in V}\mathrm{SL}(2,\mathfrak{o})M(a,b,\lambda)$$

with an appropriate minimal set of representatives V. We will not use van der Geer's set of representatives of the equations, but will instead simplify the vectors (a,b,λ) by the following rules. For all $G\in\mathrm{SL}(2,\mathfrak{o})$ we have:

$$\begin{aligned} GM(a,b,\lambda) &= \left\{G\tau \mid a\tau_1\tau_2+\lambda\tau_1+\overline{\lambda}\tau_2+b=0\right\}\\ &= \left\{\tau \mid aG^{-1}\tau_1\overline{G}^{-1}\tau_2+\lambda G^{-1}\tau_1+\overline{\lambda G}^{-1}\tau_2+b=0\right\}.\end{aligned}$$

After multiplication with the common divisor we get:

(*) Multiplicativity: Clearly we have $M(a,b,\lambda)=M(n\cdot a,n\cdot b,n\cdot\lambda)$ for all $n\in\mathbb{Z}\setminus\{0\}$, so we can assume that the triple (a,b,λ) is coprime over $\mathbb{Z}$, in the sense that there is no common divisor of a, b and λ in $\mathfrak{o}$ which is not a unit and contained in $\mathbb{Z}$. Especially we have $M(a,b,\lambda)=M(-a,-b,-\lambda)$, so we can choose a sign.

(J) J:
$$JM(a,b,\lambda)=M(b,a,-\overline{\lambda})$$
So we can interchange a and b.

(T) T:
$$TM(a,b,\lambda)=M(a,b+a-\lambda-\overline{\lambda},\lambda-a)$$
If $|2\lambda_1|=|\lambda+\overline{\lambda}|$ and $|a|$ are comparatively small compared to $|b|$, this allows the reduction of b with an appropriate power of T.

($\mathrm{T}_{\varepsilon_0}$) T_{ε_0}:

$$T_{\varepsilon_0} M(a,b,\lambda) = M(a, b - a - \lambda\varepsilon_0 - \overline{\lambda}\overline{\varepsilon_0}, \lambda - a\overline{\varepsilon_0})$$

This is a second possibility to reduce b.

(D_{ε_0}) $\mathrm{Diag}(\varepsilon_0, \varepsilon_0^{-1})$:

$$\begin{aligned}\mathrm{Diag}(\varepsilon_0, \varepsilon_0^{-1}) M(a,b,\lambda) &= M(-a, -b, -\lambda\varepsilon_0^2) \\ &= M(a, b, \lambda\varepsilon_0^2)\end{aligned}$$

So we can simplify λ without changing a and b. So, w.l.o.g. we have $\lambda = \lambda_1 + \sqrt{p}\lambda_2$ with $|\lambda_1|, |\lambda_2| \leq N$ with a constant N depending only on the norm of λ.

(T_μ) $T_\mu := \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$:

$$T_\mu M(a,b,\lambda) = M(a, b + \mathrm{N}(\mu)a - \lambda\mu - \overline{\lambda}\overline{\mu}, \lambda - a\overline{\mu})$$

Remark 3.5.1. *We are only interested in those $T(m)$ with $\chi_p(m) \geq 0$, also for example $M(0,1,0)$ is the empty set. This restriction seems to reduce the problems only by some easy cases.*

Corollary 3.5.2. *From the multiplicativity, $(*)$, we get for all natural numbers m and n that $T(m) \subset T(mn^2)$. If we had $T(m) = T(n^2 \cdot m)$ for some $n > 0$, then $\Psi_{n^2 \cdot m}/\Psi_m$ was a (holomorphic) Hilbert modular form without zeros. But there are no holomorphic Hilbert modular forms without zeros of positive weight, since the reciprocal was a (holomorphic) Hilbert modular form of negative weight (which was holomorphic in the cusps, since it was holomorphic in $\mathbb{H}^2$), so the case $T(m) = T(n^2 \cdot m)$ does not occur.*

Before we cite van der Geer, we will give a warning:

Remark 3.5.3. *There are triples (a_1, b_1, λ_1), (a_2, b_2, λ_2) in $\mathbb{Z}^2 \times \mathfrak{o}/\sqrt{13}$ with the properties $M(a_1, b_1, \lambda_1) \cap M(a_2, b_2, \lambda_2) \neq \emptyset$ and $\mathrm{SL}(2,\mathfrak{o}) M(a_1, b_1, \lambda_1) \neq \mathrm{SL}(2,\mathfrak{o}) M(a_2, b_2, \lambda_2)$. In the following we will understand that the components of a union $F_N = \cup_B F_B$ are the sets F_B no matter if they are connected components of F_N.*

Proof. We calculate (Maple™) that $M_1 := M\left(2, 4, 3 + 4\frac{\sqrt{13}}{13}\right)$ and $M_2 := M\left(0, 0, \frac{1}{2} + 5\frac{\sqrt{13}}{26}\right)$ have the following property:

$$M_1 \cap M_2 = \mathrm{SL}(2,\mathfrak{o}) M_1 \cap M_2 = \{p\},$$

where $p = \left(\frac{11}{24} - \frac{55}{312}\sqrt{13} + i\frac{19 - 5\sqrt{13}}{72}\sqrt{\frac{21413 + 5635\sqrt{13}}{26}}, -\frac{11}{24} - \frac{55}{312}\sqrt{13} + i\sqrt{\frac{21413 + 5635\sqrt{13}}{2^5 \cdot 3^2 \cdot 13}}\right)$. □

We need the following (cf. Definition [Ge88, I.2, Seite 6]):

Definition 3.5.4.

$$\mathrm{SL}(\mathfrak{o}_{\mathcal{K}} \oplus \mathfrak{b}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathcal{K}) : a,d \in \mathfrak{o}_{\mathcal{K}}, b \in \mathfrak{b}^{-1}, c \in \mathfrak{b} \right\} = \mathrm{SL}(2,\mathcal{K}) \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{b}^{-1} \\ \mathfrak{b} & \mathfrak{o} \end{pmatrix}$$

We will use [Ge88, V. Modular Curves on Modular Surfaces] and adjust it to our case. The short form of what we need is the following: "It was shown by Franke that for prime discriminants F_N has one or two components, the latter if and only if N is divisible by the square of the discriminant. Franke's results were extended to the general case by Haussmann. Both of them use the theory of hermitian lattices." ([Ge88, p. 93 et seqq.]).

Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are coprime resp. relative prime, if $\mathfrak{a}+\mathfrak{b} = \mathfrak{o}$ holds. In this case we investigate instead of D the ideal generated by D and get $\mathfrak{o} + (D) = \mathfrak{o}$ for $\mathfrak{a} = \mathfrak{o}$. We consider $\mathcal{K} = \mathbb{Q}(\sqrt{p})$, where $p \equiv 1 \pmod 4$, so the discriminant of $\mathcal{K}$ is $D = p$.

Definition 3.5.5. Let $\mathfrak{a}$ be an ideal in $\mathfrak{o}$ of norm $\mathrm{N}(\mathfrak{a}) = A$. A matrix $B \in M_2(\mathcal{K})$ is called **skew-hermitian** if

$${}^t\overline{B} = -B,$$

where $\overline{B}$ is the component wise conjugated matrix. A skew-hermitian matrix is called **integral** with respect to $\mathfrak{a}$, if it has the form

$$\begin{pmatrix} a\sqrt{D} & \lambda \\ -\overline{\lambda} & \frac{b}{A}\sqrt{D} \end{pmatrix}$$

with $a, b \in \mathbb{Z}, \lambda \in \mathfrak{a}^{-1}$. It is called **primitive** , if it is not divisible by a natural number $m > 1$.

Definition 3.5.6. Let B be an integral skew hermitian matrix and $N \in \mathbb{N}$. We define

$$\begin{aligned} F_B &:= {}_{\Gamma}\backslash\Gamma \left\{ z \in (\mathbb{H}^2)^* \cup P^1(\mathcal{K}) : \quad (z_2\ 1)B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\} \\ &= {}_{\Gamma}\backslash\Gamma \left\{ z \in (\mathbb{H}^2)^* \cup P^1(\mathcal{K}) : a\sqrt{p}z_1z_2 - \overline{\lambda}z_1 + \lambda z_2 + \frac{b}{A}\sqrt{p} = 0 \right\} \\ &= {}_{\Gamma}\backslash\Gamma \left\{ z \in (\mathbb{H}^2)^* \cup P^1(\mathcal{K}) : az_1z_2 + \overline{\left(\frac{\lambda}{\sqrt{p}}\right)} z_1 + \frac{\lambda}{\sqrt{p}} z_2 + \frac{b}{A} = 0 \right\} \end{aligned}$$

and

$$F_N := \bigcup_{\substack{B \text{ skew hermitian} \\ \text{integral, primitive} \\ \det B = N/A}} F_B$$

as well as

$$T_N = \bigcup_{t\geq 1, t^2|N} F_{N/t^2} = \bigcup_{\substack{B \text{ skew hermitian} \\ \text{integral} \\ \det B = N/A}} F_B.$$

We get from Definition 3.5.6 the obvious

Lemma 3.5.7. *Two matrices B_1 and B_2 define the same component of F_N, in the sense that $F_{B_1} = F_{B_2}$, if they there is an element $T \in \mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a})$ such that $B_1 = \pm{}^t\overline{T} B_2 T$.*

Definition 3.5.8.

$$\begin{aligned}
R_1 &:= \{q : q \text{ prime}, q|D,\ q \not| N\}\ , \\
R_2 &:= \{q : q \text{ prime}, q|D,\ \mathrm{ord}_p(N) \geq 2\,\mathrm{ord}_p(D)\}\ , \\
r_1 &:= |R_1|\ , \\
r_2 &:= |R_2|\ .
\end{aligned}$$

We just need the case $\mathfrak{a} = \mathfrak{o} = \mathfrak{o}_{\mathcal{K}}$. Then $\Gamma_{\mathcal{K}} = SL(2, \mathfrak{o}) = SL(\mathfrak{o} \oplus \mathfrak{a})$.

Theorem 3.5.9 (Theorem (3.2) in [Ge88]). *Two elements B_1 and B_2 in $F(N, \mathfrak{a})$ belong to the same $\mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a})$-orbit if and only if the r_1 invariants θ_q, $q \in R_1$ and the r_2 invariants η_q, $q \in R_2$, assume the same values for both of them. Moreover, if for none of the primes q dividing D one has $\chi_{D(q)}(N)(A, D)_q = -1$ then there are $2^{r_1+r_2}$ orbits.*

Definition 3.5.10.

$$\begin{aligned}
G_B &= \left\{T \in \mathrm{SL}(\mathfrak{o} \oplus \mathfrak{a}) : \quad (\overline{T})^{\mathrm{t}} B T = \pm B\right\} \\
E_B &= \left\{T \in G_B : \quad (\overline{T})^{\mathrm{t}} B T = B\right\}
\end{aligned}$$

For the formulation of the theorem about the number of components of F_N we need the Hilbert symbol, so we define:

Definition 3.5.11 (Hilbert symbol). Let q be a prime number, denote by $\mathbb{Q}_q$ the set of q-adic numbers and write $\mathbb{Q}_q^* = \mathbb{Q}_q \setminus \{q\}$. The **Hilbert symbol** $(\,,)_p : \mathbb{Q}_p^* \times \mathbb{Q}_p^* \to \{-1, 1\}$ is defined for all non-zero q-adic numbers a, b by

$$(a, b)_q := \begin{cases} 1, & \text{if there are } (x, y, z)^{\mathrm{tr}} \in \mathbb{Q}_q^3 \setminus \{0\} :\ z^2 = ax^2 + by^2, \\ -1, & \text{else.} \end{cases}$$

Lemma 3.5.12 (Some properties of the Hilbert symbol). *For all non-zero q-adic numbers a, b, c we have:*

i) $(a,b)_q = (b,a)_q$,

ii) $(a,b^2)_q = 1$,

iii) $(1,a)_q = 1$,

iv) $(a,-a)_q = 1$,

v) $(a,(1-a))_q = 1$,

Proof. i) trivial,

ii) for $x=0, y=1, z=b$ we have $z^2 = ax^2 + b^2y^2$,

iii) i) and ii) with $b=1$,

iv) for $x=1, y=1, z=0$ we have $z^2 = ax^2 - ay^2$,

v) for $x=y=z=1$ we have $z^2 = ax^2 + (1-a)y^2$,

□

Theorem 3.5.13 (Theorem (3.3) in [Ge88], also [Ha80]). *The curve F_N on Y_Γ, if non-empty, has $2^{r_1+r_2-1}[G_B : E_B]$ components. Moreover, if d is the square-free part of D (in our case $d = p = D$), then*

$$[G_B : E_B] = 2 \Leftrightarrow \begin{cases} (-1,D)_q = +1 \quad \textit{for all } q \in R_2 \textit{ and} \\ R_1 \textit{ contains no prime dividing } d. \end{cases}$$

Otherwise we have $[G_B : E_B] = 1$, since obviously $[G_B : E_B] \in \{1,2\}$. In this $(-1,D)_q$ is the Hilbert symbol (cf. Definition 3.5.11).

We restrict this to the case $\mathfrak{a} = (1) = \mathfrak{o}$. Then, for all $N \in \mathbb{N}$ with $\chi_p(N) \geq 0$, we get the following cases :

(i) $\mathbf{p} \not| \mathbf{N}$:

We have $R_1 = \{p\}$ and $R_2 = \{\}$, so $[G_B : E_B] = 1$ by Theorem 3.5.13. Thus $2^{r_1+r_2-1}[G_B : E_B] = 2^{1+0-1} \cdot 1 = 1$.

We get T_N as union of at most $|\{t \in \mathbb{N} : t^2|N\}|$ components $\mathrm{SL}(2,\mathfrak{o})F_B$

(ii) $\mathbf{p}|\mathbf{N}, \mathbf{p}^2 \not| \mathbf{N}$:

Then $R_1 = \{\}$ and $R_2 = \{\}$, so $[G_B : E_B] = 2$. Hence $2^{r_1+r_2-1}[G_B : E_B] = 2^{-1} \cdot 2 = 1$.

We get that T_N is a union of at most $|\{t \in \mathbb{N} : t^2|N\}|$ components $\mathrm{SL}(2,\mathfrak{o})F_B$.

(iii) $\mathbf{p^2|N}$:

In this case $R_1 = \{\}$ and $R_2 = \{p\}$. In order to calculate $[G_B : E_B]$, we need to calculate the Hilbert symbol. We skip this, for it will suffice for us to treat only Hirzebruch-Zagier divisors $T(m)$ and Borcherds products Ψ_m with $m < p^2$.

Definition 3.5.14. Let q be a prime, then $D(q) := \mathrm{disc}(\mathbb{Q}(\sqrt{q}))$ is called the **discriminant** of $\mathbb{Q}(\sqrt{q})$. By χ_D we denote the primitive Dirichlet character modulo $|D|$ with

$$\chi_D(2) = \begin{cases} 1 & D \equiv 1 \pmod 8 \\ -1 & D \equiv 5 \pmod 8 \end{cases}, \quad \chi_D(-1) = \operatorname{sign} D, \quad \chi_D(p) = \left(\frac{D}{p}\right)$$

for every prime p.

Lemma 3.5.15 (Lemma V(1.4) in [Ge88]). *The curve F_N on X_Γ is non-empty if and only if for each prime q dividing D and not dividing N we have*

$$\chi_{D(q)}(N) = (A, D)_q.$$

We only need the case $A = \mathrm{N}(\mathfrak{o}) = 1$ and $D = p \equiv 1 \pmod 4$ a prime. Then F_N on X_Γ is non-empty if and only if

$$p \nmid N \text{ and } \chi_p(N) \neq (1, p)_p = 1.$$

Some concrete results are given in table A.4 on page 147 and table A.5 on page 148. For the values of χ_p compare also table A.2 on page 145.

4 Properties of Hilbert Modular Forms

What are the possible multiplier systems for Hilbert modular forms defined in Definition 4.1.3? How do Hilbert modular forms behave under certain transformations not contained in the Hilbert modular group? We will give some answers and investigate how we can get new Hilbert modular forms by differentiation. The latter gives no new results in the cases $p \in \{5, 13, 17\}$, but might be useful in other cases.

4.1 Multiplier Systems

Gundlach presents in [Gu88] a program how to calculate the possible weights and multiplier systems of Hilbert modular forms. We adopt it to our case. Note that we have seen in Theorem 2.3.3, that for every multiplier system there is a not identically vanishing Hilbert modular form.

Gundlach gives the following

Definition 4.1.1 (w_{arg}, $a_{m,l}$, $b_{m,j}$, $c_{m,l}$, $d_{m,l}$). Let $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ and $M' = \left(\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right)$ be two matrices in $\mathrm{SL}(2,\mathbb{R})$ and τ an arbitrary point in $\mathbb{H}^2$. Write $\left(\begin{smallmatrix} * & * \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right) := MM'$. Define, using the argument function $\arg : \mathbb{C}^* \to (-\pi, \pi]$, the map $w_{\text{arg}} : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \{-1, 0, 1\}$:

$$w_{\text{arg}}(M, M') := \frac{1}{2\pi}(\arg(c \cdot M'\langle\tau\rangle + d) + \arg(c'\tau + d') - \arg(\tilde{c}\tau + \tilde{d}))$$

This is independent of τ, for calculations use $\tau = \varepsilon_0 i$. For any list $R = M_1, M_2, \ldots, M_k$ of matrices in $\mathrm{SL}(2,\mathbb{R})$ and a generating system $\mathfrak{E} := \{G_1, \ldots, G_g\}$ of $\mathrm{SL}(2,\mathfrak{o})$ we define:

$$w_{\text{arg}}(R) = \sum_{l=1}^{k-1} w_{\text{arg}} \left(\prod_{m=1}^{l} M_m, M_{l+1} \right)$$

and

$$\begin{aligned} c_{m,l} &= \left| \{1 \le j \le k : \ M_j = G_l\} \right| , \\ d_{m,l} &= \left| \left\{1 \le j \le k : \ M_j = G_l^{-1}\right\} \right| , \\ a_{m,l} &= c_{m,l} - d_{m,l} \ , \\ b_{m,j} &= w_{\text{arg}}(R) - \sum_{l=1} g d_{m,l} w_{arg}(G_l^{(j)}, (G_l^{(j)})^{-1}). \end{aligned}$$

Therein we denote by $G_l^{(j)}$ the image of G_l under the j^{th} imbedding of $\mathrm{SL}(2,\mathcal{K})$ in $\mathrm{SL}(2,\mathcal{K})^2$, so we have $G_l^{(1)} = G_l$ and $G_l^{(2)} = \overline{G_l}$.

Lemma 4.1.2. *The definition of $w_{\arg}$ is independent of τ and the image of $w_{\arg}$ is a subset of $\{-1, 0, 1\}$.*

Proof. For $\tau \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ we know that $M\tau$ is contained in $\mathbb{H}$ and we have $\arg(\tau) \in (0, \pi)$. Further on, we have $\arg(a \cdot b) - \arg(a) + \arg(b) \in 2\pi\mathbb{Z}$ for all $a, b \in C$. For $\tau \in \mathbb{H}$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ and $M' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ with $MM' = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ we get:

$$\begin{aligned} 2\pi w_{\arg}(M, M') &= \arg(cM'\tau + d) + \arg(c'\tau + d') - \arg(\tilde{c}\tau + \tilde{d}) \\ &= k_1 + \arg\left((cM'\tau + d) \cdot (c'\tau + d')\right) - \arg(\tilde{c}\tau + \tilde{d}) \\ &= k_1 + k_2 + \arg\left(\frac{(cM'\tau + d) \cdot (c'\tau + d')}{\tilde{c}\tau + \tilde{d}}\right) \\ &= k_1 + k_2 + \arg\left(\frac{c(a'\tau + b') + d(c'\tau + d')}{\tilde{c}\tau + \tilde{d}}\right) \\ &= k_1 + k_2 + \arg(1) = k_1 + k_2 \in 2\pi\{-1, 0, 1\} \end{aligned}$$

with $k_1 \in \{-2\pi, 0\}$ and $k_2 \in \{0, 2\pi\}$. Since the arguments of $\arg$ are contained in $\mathbb{H}$ and $\arg$ is continuous on $\mathbb{H}$, the function $w_{\arg}$ is continuous as function in τ with discrete image, hence constant. □

Definition 4.1.3 (Multiplier systems). A map $v : \mathrm{SL}(2, \mathfrak{o}) \to \mathbb{C} \setminus \{0\}$ is called **multiplier system of weight** (r_1, r_2) , if $v(-E) = 1$ and

$$v(L \cdot M) = v(L) \cdot v(M) \cdot \exp\left(2\pi i \left(w_{\arg}\left(L^{(1)}, M^{(1)}\right) r_1 + w_{\arg}\left(L^{(2)}, M^{(2)}\right) r_2\right)\right)$$

holds for all matrices $L, M \in \mathrm{SL}(2, \mathfrak{o})$. We will only need the case of homogeneous weights (r, r).

Remark 4.1.4. *By Lemma 4.1.2 every multiplier system of integral weight is a character. Additionally we get simplified rules of calculation for all multiplier systems from $w_{\arg}(T_\alpha, T_\beta) = 0$ for all $\alpha, \beta \in \mathcal{K}$ (For the definition of T_α and T_β compare Corollary and Definition 1.2.22). So for every multiplier system μ we have $\mu(T_\alpha)\mu(T_\beta) = \mu(T_\alpha T_\beta) = \mu(T_{\alpha+\beta})$ for all $\alpha, \beta \in \mathcal{K}$.*

Proof. If the weight $(r_1, r_2) \in \mathbb{Z}^2$ is integral, then $2\pi i(rw_{\arg} + rw_{\arg}) \in 2\pi i\mathbb{Z}$, so the multiplier system is commutative. □

A special case of Theorem 2.1 of [Gu88] is the following

Theorem 4.1.5. *Let $\mathfrak{E} = \{G_1, \ldots, G_g\}$ be a generating system of $\mathrm{SL}(2, \mathfrak{o})$ with the system of n defining relations $\tilde{R}_m = E$, where $\tilde{R}_m := \prod_{j=1}^{k_m}(R_{m,j})$ with $R_{m,j}$ or $R_{m,j}^{-1}$ in $\mathfrak{E}$ and $\tilde{R}_{n+1}$ denotes a relation $\tilde{R}_{n+1} = -E$ with the same notation. Write $R_m := R_{m,1}, \ldots R_{m,k_m}$ for $1 \leq m \leq n+1$. Then for $w : \mathfrak{E} \to \mathbb{C}^*$, $v(M) := \exp(2\pi i w(M))$ and (r_1, r_2) the following statements are equivalent:*

i) $v(G_1), \dots, v(G_g)$ *generate a multiplier system of weight* (r_1, r_2)*, i.e.* v *is the unique multiplier system given by its values on* $\mathfrak{E}$*.*

ii) For all $1 \leq m \leq n$ *we have*

$$\sum_{l=1}^{g} a_{m,l} w(G_l) + \sum_{j=1}^{2} b_{m,j} r_j \in \mathbb{Z}$$

and

$$\sum_{l=1}^{g} a_{n+1,l} w(G_l) + \sum_{j=1}^{2} \left(b_{n+1,j} + \frac{1}{2} \right) r_j \in \mathbb{Z} \, .$$

Remark 4.1.6. *The given linear restrictions for multiplier systems can be written in the form of an upper triangular matrix ([Gu88, Satz 2.2]) of full column rank with integral coefficients. One then easily finds all multiplier systems, since* v *only depends on* w *modulo* $\mathbb{Z}$*. Hence all possible multiplier systems can be determined from the upper triangular matrix by solving from* $j = g+2$ *down to* $j = 1$ *for* $e_j \in \mathbb{Q}^{g+2}$*. Therein the first* g *components are* $w(G_1)$ *to* $w(G_g)$ *and the* $g+1^{st}$ *and* $g+2^{nd}$ *component are* r_1 *and* r_2*.*

Remark 4.1.7 (Finite order of multiplier systems). *Since the coefficients of the matrix are integral,* w *and* r *are rationals. Hence every multiplier system has finite order, i.e. for every multiplier system* μ *there is a power* $k \in \mathbb{N}$ *such that* $\mu^k = 1$ *is the trivial multiplier system. Then Definition 4.1.3 is equivalent to Definition 1.2.8.*

Proof. The statements i) and ii) are exactly the statements (1.11) and (1.12) in [Gu88]. The greatest common divisor k of the denominators of the components of w holds $\mu^k = 1$. $\square$

Lemma 4.1.8 (Cusp forms for nontrivial multiplier systems). *If* f *is a Hilbert modular form of weight* k *with multiplier system* μ *and*

a) there is $\alpha \in \mathfrak{o}$ *such that* $\mu(T_\alpha) \neq 1$ *or*

b) $\mu(D_{\varepsilon_0}) \neq (-1)^k$*,*

then f *and its restriction to the diagonal are cusp forms.*

From table 5.1 and table 5.2 we get:

Remark 4.1.9. *In case* $p = 13$ *and* $p = 17$ *there is, for all multiplier systems* $\mu \not\equiv 1$*, an integer* $\alpha \in \mathfrak{o}$ *such that* $\mu(T_\alpha) \neq 1$ *holds. Hence every non-trivial Hilbert modular form for* $\mathbb{Q}(\sqrt{13})$ *and* $\mathbb{Q}(\sqrt{17})$ *with non-trivial multiplier system is a cusp form and for* $\mathbb{Q}(\sqrt{5})$ *there are no non-trivial multiplier systems.*

Proof of Lemma 4.1.8. By Remark 4.1.7 the subgroup $\mathfrak{t}_\mu := \{v \in \mathfrak{o} \mid \mu(T_v) = 1\}$ has finite index in $\mathfrak{o}$. By Lemma 1.1.11, f has a Fourier expansion of the form

$$f(\tau) = \sum_{g \in \mathfrak{t}_\mu^\#} a_g e^{2\pi i S(g\tau)},$$

where $a_g \neq 0$ only if $g \geq 0$ and $\overline{g} \geq 0$ hold. If we write $\tau = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{H}^2$, we get (cf. [Fr90, p. 49]):

$$\lim_{\substack{y_1 \cdot y_2 \to \infty \\ (x_1, x_2) \text{bounded}}} f(\tau) = a_0$$

and for all $v \in \mathfrak{o}$ we have

$$f(T_v\tau) = \mu(T_v) \underbrace{N(0\tau + 1)^k}_{=1} f(\tau),$$

so $a_0 = \mu(T_v)a_0$ follows. From

$$f(D_{\varepsilon_0}\tau) = \mu(D_{\varepsilon_0}) N({\varepsilon_0}^{-1})^k f(\tau)$$

and $D_{\varepsilon_0}\infty = \infty$ we get analogously $a_0 = \mu(D_{\varepsilon_0})(-1)^k a_0$. So in both cases i) and ii) we get $a_0 = 0$ and f is a cusp form. $\square$

We get the

Corollary 4.1.10. *If $p = 13$ or $p = 17$, then all Eisenstein series for nontrivial multiplier system vanish identically.*

Lemma 4.1.11 (Hilbert modular forms of odd weight, Bruinier). *Every Hilbert modular form of odd weight is a cusp form. Especially the restriction to the diagonal of a Hilbert modular form of weight 3 is trivial and there is no Hilbert modular form, whose restriction to the diagonal is the elliptic Eisenstein series of weight 6.*

Proof. Let f be a Hilbert modular form of odd weight k with multiplier system μ. In case $\mu(D_{\varepsilon_0}) \neq 1$, this follows from Lemma 4.1.8. Let $\mu(D_{\varepsilon_0}) = 1$, then

$$f(D_{\varepsilon_0}\tau) = \mu(D_{\varepsilon_0}) \cdot \mathrm{N}(0\tau + \varepsilon_0^{-1})^k f(\tau) = -\mu(D_{\varepsilon_0}) f(\tau) \overset{\mu(D_{\varepsilon_0})=1}{=} -f(\tau).$$

So we get from $\lim_{\tau \to 0} f(\tau) = \lim_{\tau \to 0} f(D_{\varepsilon_0}\tau) = -\lim_{\tau \to 0} f(\tau)$ that $\lim_{\tau \to 0} f(\tau) = 0$, which is equivalent to $\lim_{\tau \to \infty} f(\tau) = 0$, since $\mathbb{H}^2/\Gamma_K$ has exactly one cusp modulo equivalency and when we apply J, the contribution of the multiplier system is only a constant factor. Let f be a Hilbert modular form of weight 3 with trivial multiplier system. Since there are no elliptic cusp forms of weight 6 but 0 (cf. Theorem 2.4.7), we have $f(\tau, \tau) \equiv 0$ for $k = 3$. $\square$

Lemma 4.1.12. *All non-constant Hilbert modular forms have positive weight.*

Proof. If f is a Hilbert modular form of weight $k < 0$ with multiplier system μ. Then there is $m \in \mathbb{N}$ such that f^m has the trivial multiplier system and mk is an even number. Then $(E_2^H)^{-mk/2} \cdot f^m$ is a Hilbert modular form with trivial multiplier system of weight 0, so $f^m = c \cdot (E_2^H)^{mk/2}$ with a constant $c \in \mathbb{C}^*$ (cf. Lemma 1.1.21). Since $E_2^H(z,z) = E_4^*(z)$ for all $z \in \mathbb{H}$, the Eisenstein series E_2^H has zeros on $\mathbb{H}^2$ and f is no holomorphic function, hence f is not a Hilbert modular form. □

Remark 4.1.13. *Gundlach only considered subgroups of the Hilbert modular group. For the extended Hilbert modular group we can use Lemma 1.2.12, on the other hand, the reduction process in chapter 6 will give us a complete list of all Hilbert modular forms for symmetric multiplier systems, which are the only ones which can be continued to multiplier systems of the extended Hilbert modular group, and hence supply the complete list of multiplier systems for the extended group.*

4.2 Symmetry and Restriction to the Diagonal

The map $\mathbb{H}^2 \to \mathbb{H}^2$, $\tau \mapsto \overline{\tau} = (\tau_2, \tau_1)$ interchanges the two halfplanes. Even if it is not contained in $\mathrm{SL}(2,\mathfrak{o})$ and only contained in $\widehat{\mathrm{SL}(2,\mathfrak{o})}$, it defines a map from Hilbert modular forms to Hilbert modular forms. We will exploit this property.

We extend the principle that the non-trivial field automorphism of $\mathcal{K}$ corresponds to the interchange of components of $\mathbb{H}^2$ (cf. Remark 1.2.3) to matrices, multiplier systems, modular forms, divisors and Weyl chambers. Therefore we define

Definition 4.2.1 ($\overline{\tau}$, $\overline{M}$, $\overline{\mu}$, $\overline{f}$, $\overline{W}$). Consider a point $\tau \in \mathbb{H}^2$, a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathfrak{o})$, a multiplier system $\mu : \mathrm{SL}(2,\mathfrak{o}) \to \mathbb{C}$, a Hilbert modular form $f : \mathbb{H}^2 \to \mathbb{C}$ for $\mathbb{Q}(\sqrt{p})$ of weight k with multiplier system μ, a divisor T and a Weyl chamber W. We define the **reflection of the point** τ by

$$\overline{\tau} = \overline{(\tau_1, \tau_2)} = (\tau_2, \tau_1),$$

the **reflection of the matrix** M by

$$\overline{M} := \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix},$$

the **reflection of the multiplier system** μ by

$$\overline{\mu} : \begin{cases} \mathrm{SL}(2,\mathfrak{o}) & \longrightarrow & \mathbb{C}\,, \\ M & \longmapsto & \mu(\overline{M}) \end{cases},$$

the **reflection of the Hilbert modular form** f by

$$\overline{f} := \begin{cases} \mathbb{H}^2 & \longrightarrow & \mathbb{C}\,, \\ (\tau_1, \tau_2) & \longmapsto & f(\tau_2, \tau_1) \end{cases},$$

the **reflection of the divisor** T by

$$\overline{T} := \{(k,\tau) \in \mathbb{Z} \times \mathbb{H}^2 \mid (k,\overline{\tau}) \in T\}$$

and the **reflection of the Weyl chamber** W by

$$\overline{W} := \{\tau \in \mathbb{H}^2 \mid \overline{\tau} \in W\}.$$

If $\mu = \overline{\mu}$ then we call μ **symmetric** . In case $f = \overline{f}$ we call the Hilbert modular form f **symmetric** , in case $f = -\overline{f}$ we call it skew symmetric. That just means that f is an extended Hilbert modular form with multiplier system $\hat{\mu}$ with $\hat{\mu}|_{\mathrm{SL}(2,\mathfrak{o})} = \mu$ and $\hat{\mu}(\bar{\cdot}) = \pm 1$, depending on f being symmetric or skew symmetric.

Theorem 4.2.2 (Reflected Hilbert modular forms). *Let f be a Hilbert modular form for $\mathbb{Q}(\sqrt{p})$ of weight k with multiplier system μ. Then $\overline{f}$ is a Hilbert modular form for $\mathbb{Q}(\sqrt{p})$ of weight k with multiplier system $\overline{\mu}$. If f vanishes on the divisor T, then $\overline{f}$ vanishes (of same order) on the divisor $\overline{T}$. If $\Psi = f$ is a Borcherds product for the Weyl chamber W and $g \in \mathcal{A}_0^+(p,\chi_p)$, then $\overline{\Psi} = \overline{f}$ is the Borcherds product for the Weyl chamber $\overline{W}$ and g of same weight and reflected multiplier system.*

Proof. $\overline{\mu}$ is a multiplier system. From $M \in \mathrm{SL}(2,\mathfrak{o})$ we get $\overline{M} \in \mathrm{SL}(2,\mathfrak{o})$ and we obtain

$$\begin{aligned}
\overline{f}(M\tau) &= \overline{f}(M\tau_1, \overline{M}\tau_2)\\
&= f(\overline{M}\tau_2, M\tau_1)\\
&= f(\overline{M}\overline{\tau})\\
&= \mu(\overline{M})(\overline{c}\tau_2 + \overline{d})^k (c\tau_1 + d)^k f(\overline{\tau})\\
&= \overline{\mu}(M)(c\tau_1 + d)^k (\overline{c}\tau_2 + \overline{d})^k \overline{f}(\tau)\ .
\end{aligned}$$

Clearly $\bar{\cdot}$ maps equivalence classes of cusps to equivalence classes of cusp and $\overline{f}$ is a Hilbert modular form of the stated type.

Let $\Psi = f$ be the Borcherds product for $\sum_{k=-m}^{\infty} a(k)q^k \in A_0^+(p,\chi_p)$ for the Weyl chamber W. Consider an integer k with $-m \le k \le -1$. For every $\lambda \in \mathfrak{o}/\sqrt{p}$ with $-\mathrm{N}(\lambda) = k/p$, we have $\overline{\lambda} \in \mathfrak{o}/\sqrt{p}$ and $-\mathrm{N}(\overline{\lambda}) = -\mathrm{N}(\lambda) = k/p$. Additionally we get $M(\overline{\lambda}) = \{\tau \mid \overline{\tau} \in M(\lambda)\}$ from Definition 3.1.1. Thus $S(k) = \cup_{\lambda \in \mathfrak{o}/\sqrt{p},\, -\mathrm{N}(\lambda) = -k/p} M(\lambda)$ is symmetric for all $-1 \le k \le m$ and $\overline{W}$ is a component of $\mathbb{H}^2 \setminus \sum_{k=-m}^{-1} S(k)$, i.e. a Weyl chamber.

We check that $(\overline{\mu}, \overline{W}) > 0$ is equivalent to $(\mu, W) > 0$.

Let $\overline{\Psi}$ be the Borcherds product for $\sum_{k=-m}^{\infty} a(k)q^k$ and Weyl chamber $\overline{W}$. From the definition of ρ_W we get $\rho_{\overline{W}} = \overline{\rho_W}$. Then for all $\tau \in \overline{W}$ with $\mathrm{Im}\,(\tau_1)\,\mathrm{Im}\,(\tau_2) > |\min\{n; a(n) \neq 0\}|/p$ we have

$$\begin{aligned}\overline{\Psi}(\tau) &= \mathbf{e}(\rho_{\overline{W}}\tau_1 + \overline{\rho_{\overline{W}}}\tau_2) \prod_{\substack{\nu\in\mathfrak{o}/\sqrt{p} \\ (\nu,\overline{W})>0}} (1-\mathbf{e}(\nu\tau_1+\overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})} \\ &= \mathbf{e}(\rho_W\tau_2 + \overline{\rho_W}\tau_1) \prod_{\substack{\nu\in\mathfrak{o}/\sqrt{p} \\ (\nu,W)>0}} (1-\mathbf{e}(\nu\tau_2+\overline{\nu}\tau_1))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})} \\ &= \Psi(\overline{\tau}).\end{aligned}$$

The rest follows immediately from the previous results and Theorem 3.1.4. $\square$

Corollary 4.2.3 ($f+\overline{f}$). *If f is a Hilbert modular form for a symmetric multiplier system μ (i.e. $\mu(T_w)=\mu(\overline{T_w})$) of weight k, then $f+\overline{f}$ is a symmetric Hilbert modular form of weight k with multiplier system μ. Analogously $f-\overline{f}$ is a skew symmetric Hilbert modular form for the multiplier system μ of weight k, in other words $f+\overline{f}$ and $f-\overline{f}$ are extended Hilbert modular forms.*

Remark 4.2.4 (Symmetric multiplier systems). *Let μ be a multiplier system. Then $\overline{\mu}$ is the multiplier system given by*

i) $\overline{\mu}(J)=\mu(J)$,

ii) $\overline{\mu}(T)=\mu(T)$ *and*

iii) $\overline{\mu}(T_w):=\mu(\overline{T_w})=\mu(T\cdot T_w^{-1})=\frac{\mu(T)}{\mu(T_w)}$.

Additionally the equality

$$\mu(T_{\alpha+\beta})=\mu(T_\alpha\cdot T_\beta)=\mu(T_\alpha)\cdot\mu(T_\beta)\ . \tag{4.1}$$

holds for every two matrices $T_\alpha=\left(\begin{smallmatrix}1&\alpha\\0&1\end{smallmatrix}\right)$ and $T_\beta=\left(\begin{smallmatrix}1&\beta\\0&1\end{smallmatrix}\right)$, where $\alpha,\beta\in\mathcal{K}$. Especially we have, as we will see in Corollary 5.2.1, for $p=$

(5) exactly one multiplier system, the trivial multiplier system 1, and $\overline{1}=1$.

(13) exactly three different symmetric multiplier systems μ, namely the ones given by

$$(\mu(J),\mu(T),\mu(T_w))\in\{(1,1,1),(1,e^{2\pi i/3},e^{4\pi i/3}),(1,e^{4\pi i/3},e^{2\pi i/3})\},$$

as we easily get from the restriction $\mu(\overline{T_w})=\mu(T_w)=\frac{\mu(T)}{\mu(T_w)}$.

(17) exactly eight different symmetric multiplier systems, namely μ_0, $\mu_{1,2}$, $\mu_{2,2}$, $\mu_{2,3}$, $\mu_{3,3}$, $\mu_{3,4}=\mu_{17}$ (cf. Lemma 2.2.11), $\mu_{3,5}$, $\mu_{3,6}$. We have $\overline{\mu_{1,1}}=\mu_{1,3}$, $\overline{\mu_{2,1}}=\mu_{2,4}$, $\overline{\mu_{3,1}}=\mu_{3,2}$ and $\overline{\mu_{3,7}}=\mu_{3,8}$.

Proof. We have $\overline{T} = T$ and $\overline{J} = J$, so i) and ii) are trivial. For $T_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ and $T_\beta = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$, where $\alpha, \beta \in \mathcal{K}$, we clearly have $T_\alpha \cdot T_\beta = T_{\alpha+\beta}$ and

$$w_{\text{arg}}(T_\alpha, T_\beta) \overset{(4.1.1)}{=} \arg(0 \cdot \tau + 1) + \arg(0 \cdot \tau + 1) - \arg(0 \cdot \tau + 1) = 0 + 0 - 0 = 0$$

holds. We show iii): We have

$$\mu(\overline{T_w}) = \mu(T \cdot T_w^{-1}) \overset{(4.1.3)}{=} \mu(T) \cdot \mu(T_w^{-1}) \cdot \exp(2\pi i(w_{\text{arg}}(T, T_w^{-1})r + w_{\text{arg}}(\overline{T}, (\overline{T_w})^{-1})r)$$

and conclude for every multiplier system μ from (4.1): $\mu(T \cdot T_w^{-1}) = \mu(T) \cdot \mu(T_w^{-1})$ and $1 = \mu(E) = \mu(T_w \cdot T_w^{-1}) = \mu(T_w) \cdot \mu(T_w^{-1})$. We obtain the claimed statement. ☐

Corollary 4.2.5. *If f is a Hilbert modular form for $\mathbb{Q}(\sqrt{p})$, then in case $p =$*

(5) $f + \overline{f}$ is a Hilbert modular form,

(13) $f^3 + \overline{f^3}$ is a Hilbert modular form,

(17) $f^2 + \overline{f^2}$ is a Hilbert modular form.

Proof. In case $p = 5$ there is only the trivial multiplier system, which is symmetric. In case $p = 13$ the modular form f^3 has the trivial multiplier system, which is symmetric. In case $p = 17$ the square of every multiplier system is symmetric, as we can see from the table 5.3 on page 121. The statement follows from Corollary 4.2.3. ☐

Corollary 4.2.6 (Borcherds products have symmetric multiplier systems). *If the Hilbert modular form $\Psi \not\equiv 0$ is a Borcherds product with multiplier system μ then the multiplier system μ is symmetric and Ψ is an extended Hilbert modular form.*

Proof. By Theorem 4.2.2 (Reflected Hilbert modular forms) we know that $\overline{\Psi}$ is the Borcherds product for $\overline{W}$ of the same weight for the same modular form g. So by Lemma 3.4.4 (Change of Weyl chamber) there is $c \in \mathbb{C}^*$ such that $\Psi = c\overline{\Psi}$ and Ψ and $\overline{\Psi}$ share the same multiplier system. ☐

We give an obvious but useful property of Hilbert modular forms (cf. [Mü83, Lemma 2]):

Lemma and Definition 4.2.7 (Restriction to the diagonal). *Define the map $\delta : \mathbb{H} \to \mathbb{H}^2, z \mapsto (z, z)$. If $f : \mathbb{H}^2 \to \mathbb{C}$ is a Hilbert modular form of weight k with multiplier system μ for $\mathcal{K}_p = \mathbb{Q}(\sqrt{p})$, then $F = f \circ \delta : \mathbb{H} \to \mathbb{C},\ z \mapsto f(z, z)$ is an elliptic modular form of weight $2k$ with character $\mu\Big|_{\mathrm{SL}(2,\mathbb{Z})}$. If f is a cusp form, then F is a cusp form. We will say that F is the restriction of f to the diagonal and therefore implicitly identify $\mathbb{H}$ and $\mathrm{Diag} = \delta(\mathbb{H}) = \{(z, z);\ z \in \mathbb{H}\}$.*

Proof. For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_{\mathcal{K}}$ and $\tau \in \mathbb{H}^2$ we have

$$f(M\tau) = \nu(M)\,\mathrm{N}(c\tau + d)^k f(\tau).$$

Since c and d are rational integers we have $\overline{c} = c$ and $\overline{d} = d$. Hence we have for all $z \in \mathbb{H}$:

$$F(Mz) = f(M(z,z)) = \mu(M)\left((cz+d)^k\right)^2 f(z,z) = \mu(M)(cz+d)^{2k}F(z).$$

Additionally, by Remark 1.2.17 the Hilbert modular form f has an absolutely convergent Fourier expansion $f(\tau) = \sum_{g \in \mathbf{t}^\#} a_g e^{2\pi i\,\mathrm{S}(g\tau)}$ where $a_g \neq 0$ implies $g \geq 0$ and $\overline{g} \geq 0$. So F has the absolutely convergent Fourier expansion

$$F(z) = \sum_{m \in \frac{1}{n}\mathbb{Z}} A_m e^{2\pi i m z}$$

with

$$A_m = \sum_{g \in \mathbf{t}^\#, \mathrm{S}(g) = m} a_g$$

and n appropriate, such that $n\,\mathrm{S}(g) \in \mathbb{Z}$ for all $g \in \mathbf{t}^\#$. In case $A_m \neq 0$, we know that there is $g \in \mathbf{t}^\#$ with $a_g \neq 0$ and $\mathrm{S}\,g = m$. Hence $m = \mathrm{S}\,g = g + \overline{g} \geq 0$ and F is an elliptic modular form. If f is a cusp form, then $A_0 = a_0 = 0$ and F is a cusp form. □

4.3 Twisted Symmetry

Depending on the multiplier system and weight of a Hilbert modular form, we get the existence of certain roots of Hilbert modular forms by investigation of the map $\tau \mapsto D_{\varepsilon_0}\tau = (\varepsilon_0^2\tau_1, \overline{\varepsilon_0}^2\tau_2)$.

We define $\mathrm{Diag}_{\varepsilon_0} := M(0,0,-\overline{\varepsilon_0}) = \{\tau \in \mathbb{H}^2 \mid -\overline{\varepsilon_0}\tau_1 = \varepsilon_0\tau_2\}$ and investigate for $\tau \in \mathrm{Diag}_{\varepsilon_0}$:

$$\begin{aligned} D_{\varepsilon_0}\overline{\tau} &= D_{\varepsilon_0}(\tau_2, \tau_1) \\ &= \left(\varepsilon_0^2\tau_2, \overline{\varepsilon_0}^2\tau_1\right) \\ &= \left(\varepsilon_0^2 \frac{-\overline{\varepsilon_0}}{\varepsilon_0}\tau_1, \overline{\varepsilon_0}^2 \frac{\varepsilon_0}{-\overline{\varepsilon_0}}\tau_2\right) \\ &= (-\varepsilon_0\overline{\varepsilon_0}\tau_1, -\varepsilon_0\overline{\varepsilon_0}\tau_2) \\ &= \tau \end{aligned}$$

If f is a Hilbert modular form for $\mathrm{SL}(2,\mathfrak{o})$ of weight k with multiplier system μ, then ($\ln$ denotes the main branch of the complex logarithm)

$$\begin{aligned}
f(D_{\varepsilon_0}\overline{\tau}) &= \mu(D_{\varepsilon_0})(\varepsilon_0^{-1})^k(\overline{\varepsilon_0}^{-1})^k f(\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\ln(\varepsilon_0^{-1})+k\ln(-|\overline{\varepsilon_0}^{-1}|)} f(\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\ln(\varepsilon_0^{-1})+k\pi i+k\ln(|\overline{\varepsilon_0}^{-1}|)} f(\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\ln(\varepsilon_0^{-1}\cdot|\overline{\varepsilon_0}^{-1}|)+k\pi i} f(\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\ln((\overbrace{-\varepsilon_0\overline{\varepsilon_0}}^{=1})^{-1})+k\pi i} f(\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\pi i} f(\overline{\tau})
\end{aligned}$$

If μ is a symmetric multiplier system, then $f+\overline{f}$ and $f-\overline{f}$ are Hilbert modular forms of weight k, the fi rst one symmetric, the second one skew-symmetric and for $f \neq 0$ we have $f+\overline{f} \neq 0$ or $f-\overline{f} \neq 0$.

Lemma 4.3.1. *Let f be a Hilbert modular form of weight k with multiplier system μ.*

a) If f is symmetric and $\mu(D_{\varepsilon_0}) \neq e^{-k\pi i}$, then f vanishes on $\mathrm{Diag}_{\varepsilon_0}$.

b) If f is skew symmetric and $\mu(D_{\varepsilon_0}) \neq -e^{-k\pi i}$, then f vanishes on $\mathrm{Diag}_{\varepsilon_0}$.

Remark 4.3.2. *Note that $F_p = 1\,\mathrm{SL}(2,\mathfrak{o})\,\mathrm{Diag}_{\varepsilon_0}$.*

Remark 4.3.3. *For concrete values of $\mu(D_{\varepsilon_0})$ in the cases $p \in 5, 13, 17$ compare Corollary 5.2.1.*

Proof of Lemma 4.3.1.

- If f is symmetric, then for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$ we have:

$$\begin{aligned}
f(\tau) &= f(D_{\varepsilon_0}\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\pi i} f(\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\pi i} f(\tau)\ ,
\end{aligned}$$

  hence $f(\tau) = 0$ follows from $\mu(D_{\varepsilon_0})e^{k\pi i} \neq 1$.

- If f is skew symmetric, then for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$ we have:

$$\begin{aligned}
f(\tau) &= f(D_{\varepsilon_0}\overline{\tau}) \\
&= \mu(D_{\varepsilon_0})e^{k\pi i}\overline{f}(\tau) \\
&= -\mu(D_{\varepsilon_0})e^{k\pi i} f(\tau)\ ,
\end{aligned}$$

  thus $f(\tau) = 0$ follows from $\mu(D_{\varepsilon_0})e^{k\pi i} \neq -1$.

$\square$

4.4 Differentiation

We can get Hilbert modular forms by differentiation of other Hilbert modular forms. We will first introduce the differentiation procedure in the elliptic case and thus motivate two differentiation processes in the case of Hilbert modular forms. In the case of $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$, this method does not provide new Hilbert modular forms applied to a number of given generators, but the technique might be useful in other cases. Additionally it provides a method to calculate Hilbert modular forms of non-homogeneous weight $k = (k_1, k_2)$ with $k_1 \neq k_2$.

Similar to the elliptic case we can get new Hilbert modular forms by the way of differentiation. We start with the investigation, that for an elliptic modular form f of weight $k \in \mathbb{Z}$ with multiplier system μ with respect to the group Γ, the derivative $\mathrm{D}\, f$ is nearly a modular form, in the sense that for all $M \in \Gamma$ we have

$$\begin{aligned} \mathrm{D}\, f(Mz) &= \mathrm{D}(f \circ M)(z) \cdot (\mathrm{D}\, M)(z)^{-1} \\ &= \mathrm{D}(\mu_f(M)(cz+d)^k f(z))(cz+d)^2 \\ &= \mu_f(M)(cz+d)^{k+2}\, \mathrm{D}\, f(z) + \underline{kc\mu_f(M)(cz+d)^{k+1} f(z)}\ . \end{aligned} \tag{4.2}$$

If the underlined summand was 0 then $\mathrm{D}\, f$ was an elliptic modular form of weight $k+2$ with character μ_f. At least we have

Lemma 4.4.1 (Differentiation in the elliptic case). *Let f and g be elliptic modular forms for the discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$ and let f be of weight $k \in \mathbb{Z}$ with character μ_f and g of weight $l \in \mathbb{Z}$ with character μ_g. Then*

$$F := kf(\mathrm{D}\, g) - l(\mathrm{D}\, f)g$$

is an elliptic modular form for the group Γ with character $\mu_f\mu_g$ of weight $k + l + 2$.

Proof. By (4.2) we have for all $z \in \mathbb{H}$:

$$\begin{aligned} F(Mz) =& kf(Mz)(\mathrm{D}\, g)(Mz) - l(\mathrm{D}\, f)(Mz)g(Mz) \\ \overset{(4.2)}{=}& k\mu_f(M)(cz+d)^k f(z) \cdot \left(\mu_g(M)(cz+d)^{l+2}(\mathrm{D}\, g)(z) + lc\mu_g(M)(cz+d)^{l+1}g(z)\right) \\ &- l\left(\mu_f(M)(cz+d)^{k+2}(\mathrm{D}\, f)(z) + kc\mu_f(M)(cz+d)^{k+1}f(z)\right) \cdot \mu_g(M)(cz+d)^l g(z) \\ =& \mu_f(M)\mu_g(M)(cz+d)^{k+l+2}\underline{f(z)} \cdot \Big(k(\mathrm{D}\, g)(z) + \underline{klc(cz+d)^{-1}g(z)}\Big) \\ &- \mu_f(M)\mu_g(M)(cz+d)^{k+l+2}\Big(l(\mathrm{D}\, f)(z) + \underline{klc(cz+d)^{-1}f(z)}\Big) \cdot \underline{g(z)} \\ &= \mu_f(M)\mu_g(M)(cz+d)^{k+l+2}\left(kf(z)(\mathrm{D}\, g)(z) - l(\mathrm{D}\, f)(z)g(z)\right) \\ &= \mu_f(M)\mu_g(M)(cz+d)^{k+l+2}F(z)\ . \end{aligned}$$

□

The problems with Hilbert modular forms are that there are two differentiation operators D_{τ_1} and D_{τ_2} and the need to consider rational weights. We give two solutions, but first give the following

Lemma 4.4.2. *Let f be Hilbert modular form with respect to Γ of weight $k = (k_1, k_2) \in \mathbb{Q}^2$ with multiplier system μ_f and let g be a Hilbert modular form of weight $l = (l_1, l_2) \in \mathbb{Q}^2$ with multiplier system μ_g. Then*

$$[f, g]_1 := k_1 f(D_1 g) - l_1(D_1 f)g$$

is a Hilbert modular form with respect to Γ with multiplier system $\mu_f\mu_g$ of weight $(k_1 + l_1 + 2, k_2 + l_2)$ and

$$[f, g]_2 := k_2 f(D_2 g) - l_2(D_2 f)g$$

is a Hilbert modular form with respect to Γ with multiplier system $\mu_f\mu_g$ of weight $(k_1 + l_1, k_2 + l_2 + 2)$.

Additionally we have for all $\tau \in \mathbb{H}^2$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$:

$$(\mathrm{D}_1 f)(M\tau) = \mu(M)(c\tau_1 + d)^{k_1+2}(\overline{c}\tau_2 + \overline{d})^{k_2} \left(k_1 c(c\tau_1 + d)^{-1} f(\tau) + (\mathrm{D}_1 f)(\tau)\right)$$

and

$$(\mathrm{D}_2 f)(M\tau) = \mu(M)(c\tau_1 + d)^{k_1}(\overline{c}\tau_2 + \overline{d})^{k_2+2} \left(k_2\overline{c}(\overline{c}\tau_1 + \overline{d})^{-1} f(\tau) + (\mathrm{D}_2 f)(\tau)\right),$$

where we set $a^k := e^{k \ln a}$ with the main branch of the complex logarithm.

Remark 4.4.3. *All results of this section are formulated and proved for $\Gamma = \mathrm{SL}(2, \mathfrak{o})$, but are valid for all groups commensurable with the Hilbert modular group instead of Γ and the proves translate one to one. The extended case is more complicated, since $\overline{D_1 f} = D_2\overline{f}$ and $\overline{D_2 f} = D_1\overline{f}$ imply that for extended modular forms f and g we have $\overline{[f, g]_1} = [f, g]_2$. If both f and g are extended Hilbert modular forms, neither $[f, g]_1$ nor $[f, g]_2$ have to be extended hilbert modular forms.*

Remark 4.4.4. *In Lemma 4.4.2 we have $[f, g]_1 \equiv 0$ if and only if $f \equiv 0$, $g \equiv 0$ or $f^{l_1} \equiv c^* \cdot g^{k_1}$ with some constant $c^* \in \mathbb{C}^*$ and $[f, g]_2 \equiv 0$ if and only if $f \equiv 0$, $g \equiv 0$ or $f^{l_2} \equiv c^* \cdot g^{k_2}$ with a constant $c^* \in \mathbb{C}^*$.*

We get especially for every Hilbert modular form Ψ and all $a, b \in \mathbb{N}$:

$$[\Psi^a, \Psi^b]_1 = 0 = [\Psi^a, \Psi^b]_2$$

Proof of Lemma 4.4.2. For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathbb{H}^2$ we have

$$f(M\tau) = \mu(M)\,\mathrm{N}(c\tau + d)^k f(\tau) = \mu(M) e^{k_1 \ln(c\tau_1+d) + k_2 \ln(\overline{c}\tau_2+\overline{d})} f(\tau)$$

with the main branch $\ln : \mathbb{C}^* \to \mathbb{R} \times (-\pi, \pi]$ of the complex logarithm (cp. Definition 1.1.19). Additionally we have for $z \in \mathbb{H}$

$$\frac{d}{dz} e^{\alpha \ln(cz+d)} = \alpha \cdot e^{\alpha \ln(cz+d)} \cdot \frac{1}{cz + d} \cdot c = \alpha c \cdot e^{(\alpha-1) \ln(cz+d)}.$$

Mark that the main branch of the complex logarithm is not differentiable on the negative real axis. We could work around with a germ of the logarithm function, but our case is simpler, for $cz + d \in \mathbb{R}$ is equivalent to $c = 0$ and in this case the derivative vanishes, so the equation is still true. Hence we get for all $M \in \Gamma$ and $\tau \in \mathbb{H}^2$

$$\begin{aligned}(\mathrm{D}_1 f)(M\tau) &= (\mathrm{D}_1(f \circ M))(\tau) \cdot (\mathrm{D}_1 M(\tau))^{-1} \\ &= \mu(M)e^{k_1 \ln(c\tau_1+d)+k_2 \ln(\overline{c}\tau_2+\overline{d})} \left(k_1 c(c\tau_1 + d)^{-1} f(\tau) + (\mathrm{D}_1 f)(\tau)\right) \cdot (c\tau_1 + d)^2 \\ &= \mu(M)e^{(k_1+2) \ln(c\tau_1+d)+k_2 \ln(\overline{c}\tau_2+\overline{d})} \left(k_1 c(c\tau_1 + d)^{-1} f(\tau) + (\mathrm{D}_1 f)(\tau)\right) \ .\end{aligned}$$

Note that $e^{2\ln(c\tau_1+d)}$ is independent of the choice of the logarithm, so there do not arise any problems in

$$\begin{aligned}\exp\left(k_1 \ln(c\tau_1 + d) + k_2 \ln(\overline{c}\tau_2 + \overline{d})\right) &\cdot \exp\left(k_1 \ln(c\tau_1 + d)\right) \\ &= \exp\left((k_1 + 2) \ln(c\tau_1 + d) + k_2 \ln(\overline{c}\tau_2 + \overline{d})\right)\end{aligned}$$

by the specific choice of logarithmic branch (of course it is the same on both sides of the equation). Analogously we get

$$(\mathrm{D}_2 f)(M\tau) = \mu(M)e^{k_1 \ln(c\tau_1+d)+(k_2+2) \ln(\overline{c}\tau_2+\overline{d})} \left(k_2\overline{c}(\overline{c}\tau_2 + \overline{d})^{-1} f(\tau) + (\mathrm{D}_2 f)(\tau)\right).$$

If we insert the corresponding terms in the definitions of $[f, g]_1$ and $[f, g]_2$, we get the stated result by simple calculation as in the proof of Lemma 4.4.1. $\square$

Proof of Remark 4.4.4. If $f \not\equiv 0$ and $g \not\equiv 0$ we have locally, away from the zeros of f and g and the inverse images of the negative real line with respect to f, f_j^k, g and g_j^l, with the help of the main branch of the complex logarithm

$$\begin{aligned} & k_j f(D_j g) - l_j(D_j f)g = 0 \\ \iff \ & k_j f(D_j g) = l_j(D_j f)g \\ \iff \ & k_j \frac{D_j g}{g} = l_j \frac{D_j f}{f} \\ \iff \ & k_j D_j(\ln g) = l_j D_j(\ln f) \\ \iff \ & D_j(k_j \ln g) = D_j(l_j \ln f).\end{aligned}$$

Since $k \ln z = \ln z^k$ up to a (locally constant) integral multiple of $2\pi i$ we can further deduce

$$\begin{aligned} & D_j(k_j \ln g) = D_j(l_j \ln f) \\ \iff \ & D_j(\ln(g^{k_j})) = D_j(\ln(f^{l_j})) \\ \iff \ & D_j(\ln(g^{k_j}) - \ln(f^{l_j})) = 0 \\ \iff \ & \ln g^{k_j} - \ln f^{l_j} \text{ is constant.}\end{aligned}$$

Again we only change a constant and we get locally

$$\begin{aligned} & \ln g^{k_j} - \ln f^{l_j} \text{ is constant,} \\ \iff \quad & \ln \frac{g^{k_j}}{f^{l_j}} \text{ is constant,} \\ \iff \quad & \frac{g^{k_j}}{f^{l_j}} \text{ is constant.} \end{aligned}$$

The identity theorem then proves the global statement. $\square$

Lemma 4.4.5 (Differentiation of Hilbert modular forms (1)). *Let f be a Hilbert modular group with respect to Γ of weight k with multiplier system μ_f, let g be a Hilbert modular group with respect to Γ of weight l with multiplier system μ_g and let h be a Hilbert modular group with respect to Γ of weight m with multiplier system μ_h. Then*

$$\begin{aligned} F :=& klf(\mathrm{D}_1\, g)(\mathrm{D}_2\, h) + klfg(\mathrm{D}_1\, \mathrm{D}_2\, h) - kmf(\mathrm{D}_2\, g)(\mathrm{D}_1\, h) - kmf(\mathrm{D}_1\, \mathrm{D}_2\, g)h \\ & - (l^2 + lm)(\mathrm{D}_1\, f)g(\mathrm{D}_2\, h) + (lm + m^2)(\mathrm{D}_1\, f)(\mathrm{D}_2\, g)h \end{aligned}$$

is a Hilbert modular form with respect to Γ with multiplier system $\mu_f\mu_g\mu_h$ of weight $k+l+m+2$.

Remark 4.4.6. *If we have $0 \in \{f, g, h\}$ or $g \in \mathbb{C}^* h$, then F vanishes identically.*

Remark 4.4.7. *If we write*

$$\begin{aligned} [f, g]_1 &:= k_1 f(\mathrm{D}_1\, g) - l_1(\mathrm{D}_1\, f)g \text{ and} \\ [f, g]_2 &:= k_2 f(\mathrm{D}_2\, g) - l_2(\mathrm{D}_2\, f)g\ , \end{aligned}$$

for Hilbert modular forms f of weight k and g of weight l as in Lemma 4.4.2, we get

$$F = [f, [g, h]_2]_1.$$

This is not an intrinsic choice and other choices could be sensible. Especially we have $\overline{F} = \overline{f}, [\overline{g}, \overline{h}]_1]_2$. Even if f, g and h are extended Hilbert modular forms, in general F will not be an extended Hilbert modular form.

Proof of Lemma 4.4.5. By Lemma 4.4.2 the function

$$G := lg(\mathrm{D}_2\, h) - m(D_2 g)h$$

is a Hilbert modular form with respect to Γ of weight $(l+m, l+m+2)$ with multiplier system $\mu_g\mu_h$ and again with Lemma 4.4.2 the function

$$F = kf(D_1 G) - (l+m)(D_1 f)G$$

is a Hilbert modular form with respect to Γ with multiplier system $\mu_f\mu_g\mu_h$ of weight $(k+l+m+2, k+l+m+2)$. We easily see that both defi nitions of F are equivalent. $\square$

Proof of Remark 4.4.6. This follows directly from Remark 4.4.4. □

The second variant we take directly from [AI05], where it is formulated for Siegel modular forms (compare also [Ao06]):

Theorem 4.4.8 (Differentiation of Hilbert modular forms (2)). *Let f be a Hilbert modular group with respect to Γ of weight k with multiplier system μ_f, let g be a Hilbert modular group with respect to Γ of weight l with multiplier system μ_g and let h be a Hilbert modular group with respect to Γ of weight m with multiplier system μ_h. Then*

$$\langle f_1, f_2, f_3\rangle := \begin{vmatrix} k_1 f_1 & k_2 f_2 & k_3 f_3 \\ \mathrm{D}_1 f_1 & \mathrm{D}_1 f_2 & \mathrm{D}_1 f_3 \\ \mathrm{D}_2 f_1 & \mathrm{D}_2 f_2 & \mathrm{D}_2 f_3 \end{vmatrix}$$

is a Hilbert modular form of weight $k+l+m+2$ with multiplier system $\mu_f \cdot \mu_g \cdot \mu_h$. It vanishes identically if and only if f_1, f_2 and f_3 are algebraically dependent.

Proof. We expand by the last line and get

$$\langle f_1, f_2, f_3\rangle = (\mathrm{D}_2 f_1)[f_2, f_3]_1 - (\mathrm{D}_2 f_2)[f_1, f_3]_1 + (\mathrm{D}_2 f_3)[f_1, f_2]_1.$$

For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathbb{H}^2$ we get

$$\begin{aligned}
\langle f_1, f_2, f_3\rangle(M\tau) &= \left((\mathrm{D}_2 f_1)[f_2, f_3]_1 - (\mathrm{D}_2 f_2)[f_1, f_3]_1 + (\mathrm{D}_2 f_3)[f_1, f_2]_1\right)(M\tau) \\
&\overset{\text{Lemma}}{\underset{(4.4.2)}{=}} (\mu_1\mu_2\mu_3)(M)(c\tau_1 + d)^{k_1+l_1+m_1+2}(\overline{c}\tau_2 + \overline{d})^{k_2+m_2+l_2+2} \cdot \\
&\qquad \Big(\left((\mathrm{D}_2 f_1)[f_2, f_3]_1 - (\mathrm{D}_2 f_2)[f_1, f_3]_1 + (\mathrm{D}_2 f_3)[f_1, f_2]_1\right)(\tau) \\
&\qquad + \overline{c}(\overline{c}\tau_2 + \overline{d})^{-1}\left(k_1 f_1[f_2, f_3]_1 - k_2 f_2[f_1, f_3]_1 + k_3 f_3[f_1, f_2]_1\right)(\tau)\Big) \\
&= (\mu_1\mu_2\mu_3)(M)(c\tau_1 + d)^{k_1+m_1+l_1+2}(\overline{c}\tau_2 + \overline{d})^{k_2+m_2+l_2+2}\langle f_1, f_2, f_3\rangle(\tau) \\
&\qquad + \overline{c}(\mu_1\mu_2\mu_3)(M)(c\tau_1 + d)^{k_1+m_1+l_1+2}(\overline{c}\tau_2 + \overline{d})^{k_2+m_2+l_2+1} \cdot \begin{vmatrix} k_1 f_1 & k_2 f_2 & k_3 f_3 \\ \mathrm{D}_1 f_1 & \mathrm{D}_1 f_2 & \mathrm{D}_1 f_3 \\ k_1 f_1 & k_2 f_2 & k_3 f_3 \end{vmatrix} \\
&= (\mu_1\mu_2\mu_3)(M)(c\tau_1 + d)^{k_1+l_1+m_1+2}(\overline{c}\tau_2 + \overline{d})^{k_2+m_2+l_2+2} \cdot \langle f_1, f_2, f_3\rangle(\tau),
\end{aligned}$$

since the fi rst and the last line of the given matrix coincide.

It remains to proof that $\langle f_1, f_2, f_3\rangle$ vanishes identically if and only if f_1, f_2 and f_3 are algebraically dependent. The case that one of the functions f_1, f_2 or f_3 vanishes identically is trivial, we will assume that this is not the case. We use the idea given in [AI05] and rewrite

$$\langle f_1, f_2, f_3\rangle = \frac{f_1^{k_2+k_3} f_2^{1-k_1} f_3^{1-k_1}}{k_1^2} \begin{vmatrix} k_1 f_1 & k_1 k_2 f_2^{k_1} f_1^{-k_2} & k_1 k_3 f_3^{k_1} f_1^{-k_3} \\ \mathrm{D}_1\, f_1 & k_1 f_2^{k_1-1} f_1^{-k_2}\, \mathrm{D}_1\, f_2 & k_1 f_3^{k_1-1} f_1^{-k_3}\, \mathrm{D}_1\, f_3 \\ \mathrm{D}_2\, f_1 & k_1 f_2^{k_1-1} f_1^{-k_2}\, \mathrm{D}_2\, f_2 & k_1 f_3^{k_1-1} f_1^{-k_3}\, \mathrm{D}_2\, f_3 \end{vmatrix}.$$

Then substract from the second row $k_2 f_2^{k_1} f_1^{1-k_2}$ times the first row and substract from the third row $k_3 f_3^{k_1} f_1^{1-k_3}$ times the first row to get

$$\begin{aligned}\langle f_1, f_2, f_3\rangle &= \frac{f_1^{k_2+k_3} f_2^{1-k_1} f_3^{1-k_1}}{k_1^2} \begin{vmatrix} k_1 f_1 & 0 & 0 \\ \mathrm{D}_1\, f_1 & \mathrm{D}_1\left(\frac{f_2^{k_1}}{f_1^{k_2}}\right) & \mathrm{D}_1\left(\frac{f_3^{k_1}}{f_1^{k_3}}\right) \\ \mathrm{D}_2\, f_1 & \mathrm{D}_2\left(\frac{f_2^{k_1}}{f_1^{k_2}}\right) & \mathrm{D}_2\left(\frac{f_3^{k_1}}{f_1^{k_3}}\right) \end{vmatrix} \\ &= \frac{f_1^{1+k_2+k_3} f_2^{1-k_1} f_3^{1-k_1}}{k_1} \begin{vmatrix} \mathrm{D}_1\left(\frac{f_2^{k_1}}{f_1^{k_2}}\right) & \mathrm{D}_1\left(\frac{f_3^{k_1}}{f_1^{k_3}}\right) \\ \mathrm{D}_2\left(\frac{f_2^{k_1}}{f_1^{k_2}}\right) & \mathrm{D}_2\left(\frac{f_3^{k_1}}{f_1^{k_3}}\right) \end{vmatrix}.\end{aligned}$$

So the modular form $\langle f_1, f_2, f_3\rangle$ vanishes identically, if and only if the Jacobian of $(F_1, F_2) := \left(f_2^{k_1} f_1^{-k_2}, f_3^{k_1} f_1^{-k_3}\right)$ vanishes identically, so if and only if F_1 and F_2 are analytic dependent. Since F_1 and F_2 are meromorphic Hilbert modular forms of weight 0, they correspond to meromorphic functions on the compact space $(\mathbb{H}^2)^*/\Gamma$ (compare Remark 1.1.9) and we can use the result of Thimm (cf. [Th54, Hauptsatz II, p. 457] and [Re56, p. 278]) that in compact complex spaces, analytically dependent functions are algebraically dependent. Since we can treat different weights separately, F_1 and F_2 are algebraically dependent if and only if f_1, f_2 and f_3 are algebraically dependent, so we have proven the theorem. $\square$

Remark 4.4.9. *For f_1, f_2 and f_3 as in Theorem 4.4.8 we get*

$$\langle f_1, f_2, f_3\rangle = \begin{vmatrix} k_1\overline{f_1} & k_2\overline{f_2} & k_3\overline{f_3} \\ \mathrm{D}_2\,\overline{f_1} & \mathrm{D}_2\,\overline{f_2} & \mathrm{D}_2\,\overline{f_3} \\ \mathrm{D}_1\,\overline{f_1} & \mathrm{D}_1\,\overline{f_2} & \mathrm{D}_1\,\overline{f_3} \end{vmatrix} = -\left\langle \overline{f_1}, \overline{f_2}, \overline{f_3}\right\rangle,$$

so if f_1, f_2 and f_3 are extended Hilbert modular forms, F is an extended Hilbert modular form with multiplier system $\mu_1 \cdot \mu_2 \cdot \mu_3 \cdot \mu^*$ where $\mu^*|_{\mathrm{SL}(2,\mathbb{Z})} \equiv 0$ and $\mu^*(\bar{\cdot}) = -1$.

For calculations we need the following

Lemma 4.4.10 (Differentiation of the Fourier expansion). *Let f be a Hilbert modular form with Fourier expansion*

$$f(\tau) = \sum_{\nu\in\mathfrak{o}/\sqrt{p}} a_\nu e^{2\pi i(\nu\tau_1+\overline{\nu}\tau_2)}$$

for all $\tau \in \mathbb{H}^2$. *Then*

$$\frac{1}{2\pi i}\,\mathrm{D}_1\, f(\tau) = \sum_{\nu \in \mathfrak{o}/\sqrt{p}} \nu a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)}$$

and

$$\frac{1}{2\pi i}\,\mathrm{D}_2\, f(\tau) = \sum_{\nu \in \mathfrak{o}/\sqrt{p}} \overline{\nu} a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)}$$

and the given series converge absolutely for all $\tau \in \mathbb{H}^2$.

Proof. For all $\tau \in \mathbb{H}^2$ we have

$$\frac{d}{d\tau_1} e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} = 2\pi i \nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} \quad \text{and} \quad \frac{d}{d\tau_2} e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} = 2\pi i \overline{\nu} e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)}\ .$$

For all $\nu \in \mathfrak{o}/\sqrt{p}$ with $\nu \geq 0$ we have $\nu \leq e^\nu$ and for all $\nu \in \mathfrak{o}/\sqrt{p}$ with $\nu < 0$ we have $|\nu| \leq e^{-\nu}$ (compare the derivatives and $0 < 1 = e^0$). In addition to that we may rearrange the series due to the absolute convergence and obtain for all $\tau \in \mathbb{H}^2$ with $\mathrm{Im}\,(\tau_1) > \frac{1}{2\pi}$ and $\mathrm{Im}\,(\tau_2) > \frac{1}{2\pi}$:

$$\begin{aligned}
\mathrm{D}_1\, f(\tau) &= \mathrm{D}_1 \left(\sum_{\nu \in \mathfrak{o}/\sqrt{p}} a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} \right) \\
&= \mathrm{D}_1 \left(\sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu \geq 0}} a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} \right) + \mathrm{D}_1 \left(\sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu < 0}} a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} \right).
\end{aligned}$$

We abbreviate $f_\nu(\tau) := a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)}$ and get

$$\begin{aligned}
\left| \sum_{\nu \in \mathfrak{o}/\sqrt{p}} (D_1 f_\nu)(\tau) \right| &= \left| 2\pi i \left(\sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu \geq 0}} \nu a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} \right) + 2\pi i \left(\sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu < 0}} \nu a_\nu e^{2\pi i(\nu\tau_1 + \overline{\nu}\tau_2)} \right) \right| \\
&\leq 2\pi \left| \sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu \geq 0}} \nu e^{-\nu} a_\nu e^{2\pi i\left(\nu\left(\tau_1 - \frac{i}{2\pi}\right) + \overline{\nu}\tau_2\right)} \right| + 2\pi \left| \sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu < 0}} \nu e^{\nu} a_\nu e^{2\pi i\left(\nu\left(\tau_1 + \frac{i}{2\pi}\right) + \overline{\nu}\tau_2\right)} \right| \\
&\leq 2\pi \sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu \geq 0}} |a_\nu| e^{2\pi i\left(\nu\left(\mathrm{Im}(\tau_1) - \frac{1}{2\pi}\right) + \overline{\nu}\,\mathrm{Im}(\tau_2)\right)} + 2\pi \sum_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ \nu < 0}} |a_\nu| e^{2\pi i\left(\nu\left(\mathrm{Im}(\tau_1) + \frac{1}{2\pi}\right) + \overline{\nu}\,\mathrm{Im}(\tau_2)\right)} \\
&\leq 2\pi \sum_{\nu \in \mathfrak{o}/\sqrt{p}} |a_\nu| e^{2\pi i\left(\nu\left(\mathrm{Im}(\tau_1) - \frac{1}{2\pi}\right) + \overline{\nu}\,\mathrm{Im}(\tau_2)\right)} + 2\pi \sum_{\nu \in \mathfrak{o}/\sqrt{p}} |a_\nu| e^{2\pi i\left(\nu\left(\mathrm{Im}(\tau_1) + \frac{1}{2\pi}\right) + \overline{\nu}\,\mathrm{Im}(\tau_2)\right)}.
\end{aligned}$$

The right hand side converges since $f(\tau_1 - \frac{1}{2\pi}, \tau_2)$ and $f(\tau_1 + \frac{1}{2\pi}, \tau_2)$ converge. Especially the right hand side is independent of the real part of τ_1. Hence we can deduce due to the holomorphy of f

$$\begin{aligned} \mathrm{D}_1 f(\tau) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}^*}} \frac{f(\tau_1+h,\tau_2)-f(\tau)}{h} \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}^*}} \frac{\sum_{\nu\in\mathfrak{o}/\sqrt{p}} f_\nu(\tau_1+h,\tau_2) - \sum_{\nu\in\mathfrak{o}/\sqrt{p}} f_\nu(\tau)}{h}. \end{aligned}$$

Both sums converge absolutely, so

$$\mathrm{D}_1 f(x) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}^*}} \sum_{\nu\in\mathfrak{o}/\sqrt{p}} \frac{f_\nu(\tau_1+h,\tau_2)-f_\nu(\tau)}{h}.$$

For every $\nu \in \mathfrak{o}/\sqrt{p}$, suffi ciently small $\varepsilon > 0$ and $|h| \le \varepsilon$, the difference quotient $\frac{f_\nu(\tau_1+h,\tau_2)-f_\nu(\tau)}{h}$ equals $\mathrm{D}_1 f_\nu(\xi_n)$ with some appropriate ξ_ν, $|\xi_\nu - \tau| \le \varepsilon$, due to the mean value theorem. Because of $|\mathrm{D}_1 f_\nu(xi_\nu)| = |\mathrm{D}_1 f_\nu(\tau)|$ the sum $\sum_{\nu\in\mathfrak{o}/\sqrt{p}} |\mathrm{D}_1 f_\nu(\tau+h)|$ is locally bounded. Hence sum and limit can be interchanged and we get

$$\begin{aligned} \mathrm{D}_1 f(\tau) &= \sum_{\nu\in\mathfrak{o}/\sqrt{p}} \lim_{\substack{h\to 0\\ h\in\mathbb{R}^*}} \frac{f_\nu(\tau_1+h,\tau_2)-f_\nu(\tau)}{h} \\ &= \sum_{\nu\in\mathfrak{o}/\sqrt{p}} \mathrm{D}_1 f_\nu(\tau) \end{aligned}$$

for all $\tau \in \mathbb{H}^2$ with $\mathrm{Im}\,(\tau_1) > \frac{1}{2\pi}$ and $\mathrm{Im}\,(\tau_2) > \frac{1}{2\pi}$. The derivative $D_1 f : \mathbb{H}^2 \to \mathbb{C}$ is invariant under the transformations T and T_ω, so it has an unique Fourier expansion on $\mathbb{H}^2$. So the calculated coeffi cients are correct and the expansion converges for all $\tau \in \mathbb{H}^2$. Along with f also $\overline{f} : \tau \mapsto f(\tau_2, \tau_1)$ meets the restrictions and we get the result for $D_1 \tilde{f}(\tau_2, \tau_1) = D_2 f(\tau)$ as well as for $D_1 f(\tau)$. □

5 Calculation of Borcherds Products

We investigate the remaining tasks for the concrete calculation of Borcherds products as described in Theorem 3.1.4. We calculate a basis of $A_0^+(p,\chi_p)$ via Eisenstein series in some space $M_k(p,1)$ and rational function in η and $\eta^{(p)}$ and determine the multiplier system of a Borcherds product from the Weyl vector. At last we describe a way to calculate the Fourier expansion of a Borcherds product up to some degree.

5.1 A Basis for the Plus Space

We calculate the basis $\{f_m;\ m \in \mathbb{N}, \chi_p(m) \neq -1\}$ of $A_0^+(p,\chi_p)$, $p \in \{5,13,17\}$, defined in Definition 2.5.37, i.e. we give an algorithm capable of calculating each of the infinitely many elements of the basis up to every desired precision.

Let $p \in \{5,13,17\}$. Note that the modular form

$$H^{(1)} : \mathbb{H} \to \mathbb{C}, z \mapsto \frac{\eta(z)^p}{\eta(pz)}$$

is contained in $M_{\frac{p-1}{2}}(p,\chi_p)$ by Theorem 2.5.12 and, since $24|(p^2-1)$ holds for all p in $\{5,13,17\}$ ($5^2-1=24$, $13^2-1=7\cdot 24$ and $17^2-1=12\cdot 24$), the modular form

$$H^{(q)} = \Delta^{\frac{p^2-1}{24}} \cdot \left(H^{(1)}\right)^{-p} : \mathbb{H} \to \mathbb{C}, z \mapsto \frac{\eta(pz)^p}{\eta(z)}$$

is contained in $M_{\frac{p-1}{2}}(p,\chi_p)$ by Theorem 2.5.13. The Fourier expansion of $H^{(1)}$ starts with 1, the Fourier expansion of $H^{(q)}$ starts with $q^{(p^2-1)/24} = e^{2\pi i z(p^2-1)/24}$. So given a basis of $M_{\frac{p-1}{2}}(p,1)$, we can calculate the Elements $f_1 \cdot H^{(q)}, \ldots, f_{\frac{p^2-1}{24}} \cdot H^{(q)} \in M_{\frac{p-1}{2}}(p,1)$ by comparison of Fourier expansions. By Remark 2.5.6 we know that we only need to compare the first $\frac{p^2-1}{24}$ coefficients and can easily check whether a given set of linear independent modular forms is a basis of $M_{\frac{p-1}{2}}(p,1)$. In our cases, it happens that a basis can be obtained by multiplication of Hecke's Eisenstein series of Haupttypus, E_k (the elliptic Eisenstein series for $SL(2,\mathbb{Z})$) and $E_k^{(p)} = z \mapsto E_k(pz)$ (cf. Theorem 2.5.24) and the Eisenstein series of Nebentypus, $G_k = 1 + \ldots$ and $H_k = q + \ldots$, (cf. Definition 2.5.30), where one takes care of character and weight.

There are three more useful functions. First

$$\tilde{H} = \eta^k/(\eta^{(p)})^k, \qquad (k = 24/\gcd(p-1,24))$$

is contained in $A_0(p,1)$ by Corollary 2.5.14. Second, by Definition 2.5.30, the function

$$E_0 : z \mapsto E_2^+(z) \cdot \frac{E_4 E_6}{\Delta}(pz)$$

is contained in $A_0(p,\chi_p)$. Finally

$$j^{(p)} : z \mapsto j(pz) = \frac{E_4^3}{\Delta}(pz) = q^{-p} + 744 + 196884 q^p + O\left(q^{2p}\right)$$

is an element of $A_0(p,1)$. The modular forms $\tilde{H}$ and $j^{(p)}$ operate on $A_0(p,\chi_p)$ by multiplication and E_0 works as a first good gess for f_p. For an easy algorithm, we will also calculate the $f_k = q^{-k} + O(1) \in A_0^-(p,\chi_p)$. Assume that we have already calculated $f_1, \dots f_{n-1}$. Then define

$$\tilde{f}_n = \begin{cases} f_1^n, & \text{if } n \text{ is odd and } n < p, \\ f_2 f_1^{n-2}, & \text{if } 1 < n \leq \frac{p-1}{\gcd(p-1,24)} \text{ and } n \text{ is even,} \\ f_{n-\frac{p-1}{\gcd(p-1,24)}} \cdot \tilde{H}, & \text{if } \frac{p-1}{\gcd(p-1,24)} < n < p \text{ and } n \text{ is even,} \\ E_0, & \text{if } p = n, \\ f_{n-p} \cdot j^{(p)}, & \text{if } n > p. \end{cases} \tag{5.1}$$

Alternatively we can write $\tilde{f}_n = f^{n-2} f_2$ for even $n < p$, $n \geq 4$. If we write $\tilde{f} = \sum_{m=-n}^{\infty} a(m)s(m)q^n$ then

$$f = \frac{1}{s(m)} \left(\tilde{f} - \sum_{m=1-n}^{1} a(m) f(m) \right)$$

is the desired basis element. In case $(p-1)k/24 > 1$ (where $k = 24/\gcd(p-1,24)$) we can simplify the algorithm by setting $\tilde{f}_n = f_1^{n-2} f_2$ for even $n < p$.

Remark 5.1.1. *We can show by calculation of Fourier exponents that E_2^+ is a Theta Nullwert in case $p \in \{5, 13, 17\}$. Especially we set*

$$M_5 := \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 4 & 5 \\ & & 5 & 10 \end{pmatrix}, \quad M_{13} := \begin{pmatrix} 2 & 1 & & \\ 1 & 4 & 3 & \\ & 3 & 10 & 13 \\ & & 13 & 26 \end{pmatrix} \textit{ and } M_{17} := \begin{pmatrix} 2 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 10 & 17 \\ & & 17 & 34 \end{pmatrix}.$$

The inverse matrices M_p^{-1} are each contained in $(\mathbb{Z}/p)^{4\times 4}$, so by Theorem 2.5.16, the functions

$$z \mapsto \sum_{g \in \mathbb{Z}^4} e^{\pi i g^t M_p g z}$$

are modular forms for $\Gamma_0(p)$. Then we can compare Fourier coefficients and get

$$E_2^+(z) = \sum_{g\in\mathbb{Z}^4} e^{\pi i g^t M_p g z} \quad \textit{for all } p \in \{5, 13, 17\}\,.$$

Remark 5.1.2 (Precision invariant under multiplication and division). *For concrete calculations it is necessary to truncate each expansion $\sum_n a(n)q^n$ to get a finite sum $\sum_{n\le N} a(n)q^n$. If $f_{(1)} = q^k \left(\sum_{n=0}^M a(n)q^n + O(q^{M+1})\right)$ and $f_{(2)} = q^l \left(\sum_{n=0}^N b(n)q^n + O(q^{N+1})\right)$ are given with $k, l \in \mathbb{Z}$, $a(0), b(0) \neq 0$ and $M, N \in \mathbb{N}$, then*

$$f_{(1)} f_{(2)} = q^{k+l} \left(\sum_{n=0}^{\min(M,N)} \left(\sum_{m=0}^{n} a(m)b(n-m) \right) q^n + O\left(q^{\min(M,N)}\right) \right)$$

and

$$\frac{1}{f_{(1)}} = q^{-k} \left(\sum_{n=0}^{M} c(n)q^n + O\left(q^{M+1}\right) \right)$$

with appropriate coefficients $c(n)$ (multiply with $f_{(2)}$ and compare the coefficients in front of q^n, the resulting linear equation system has upper triangular form and hence can easily be solved for the first $M+1$ variables $c(0), \ldots, c(M)$). So, in order to determine the first N coefficients of a product or quotient of Fourier expansions, for each of the factors the first N coefficients have to be determined.

5.1.1 A Basis in the case $\mathbb{Q}(\sqrt{5})$

In case $\mathbb{Q}(\sqrt{5})$ we can directly apply the methods worked out above to calculate the fi rst $p+1$ elements of a basis of $A_k^+(5, \chi_5)$. All further elements can easily be obtained by repeated multiplication with $j^{(p)}$, as described in (5.1).

We calculate

$$\begin{aligned}
H^{(1)} &= 1 - 5q + 5q^2 + 10q^3 - 15q^4 - 5q^5 - 10q^6 + 30q^7 + O\left(q^8\right), \\
H^{(q)} &= \Delta \cdot \left(H^{(1)}\right)^{-5} = q + q^2 + 2q^3 + 3q^4 + 5q^5 + 2q^6 + 6q^7 + O\left(q^8\right), \\
\tilde{H} &= \eta^6/(\eta^{(5)})^6 = H^{(1)}/H^{(q)} = q^{-1} - 6 + 9q + 10q^2 - 30q^3 + 6q^4 - 25q^5 + O\left(q^6\right), \\
j^{(5)} &= q^{-5} + 744 + 196884q^5 + 21493760q^{10} + 864299970q^{15} + O(q^{20})
\end{aligned}$$

and get

$$\begin{aligned}
f_1 = \frac{E_2^+}{H^{(q)}} &= q^{-1} + 5 + 11q - 54q^4 + O(q^5) \in A_0^+(5,\chi_5),\\
f_2 = \tilde{H}\cdot f_1 + f_1 &= q^{-2} + O(1) \in A_0^-(5,\chi_5),\\
f_3 = f_1^3 - 15f_2 - 108f_1 &= q^{-3} + O(1) \in A_0^-(5,\chi_5),\\
f_4 = \tilde{H}\cdot f_1^3 - 9f_3 - 27f_2 + 48f_1 &= q^{-4} + 15 - 216q + 4959q^4 + O(q^5) \in A_0^+(5,\chi_5),\\
f_5 = \frac{1}{2}E_0 + 10f_4 + 30f_1 &= \frac{1}{2}q^{-5} + 15 + 275q + 27550q^4 + O(q^5) \in A_0^+(5,\chi_5),\\
f_6 = j^{(5)}\cdot f_1 - 10f_5 - 11f_4 - 690f_1 &= q^{-6} + 10 + 264q - 136476q^4 + O\left(q^5\right) \in A_0^+(5,\chi_5),\\
&\vdots
\end{aligned}$$

In this case the η-quotient $H^{(q)}$ is equal to the Eisenstein series H_2.

5.1.2 A Basis in the case $\mathbb{Q}(\sqrt{13})$

$H^{(q)}$ is contained in $A_6(13,\chi_{13})$ and its Fourier expansion starts with q^7, so we need to find an element g of $M_6(13,\chi_{13})$ with Fourier expansion starting with q^6, since then $f_1 = g/H^{(q)}$.

Lemma 5.1.3 (Calculation of f_1). *The modular forms E_6, $E_6^{(13)}$, $E_2^{(13)}E_4$, $E_2^{(13)}E_4^{(13)}$, H_4H_2, H_4G_2 and G_4H_2 form a basis of the vector space $M_6(13,1)$ of holomorphic modular forms for $\Gamma_0(13)$ of weight 6 with trivial character (for notations compare Lemma 2.5.24 and Lemma 2.5.30). We get*

$$f_1(z) = \frac{g}{H^{(q)}}(z) = \frac{g(z)\eta(z)}{\eta(13z)^{13}} \qquad \text{for all } z \in \mathbb{H}$$

where

$$\begin{aligned}
g = \frac{1}{90720}E_6 - \frac{253}{90720}E_6^{(13)} - \frac{13}{191520}E_2^{(13)}E_4 + \frac{109}{38304}E_2^{(13)}E_4^{(13)} + \frac{4}{19}H_4H_2\\
-\frac{457}{19152}H_4G_2 - \frac{5}{19152}G_4H_2
\end{aligned}$$

and surprisingly find $f_1 = E_2^+/H_2$ as in case $p = 5$, even if now $H_2 \neq H^{(q)}$ and therefore might have zeros.

Sketch of proof. We divide the given modular forms in $M_6(13,1)$ by $H^{(q)}$ and obtain a set M of $(13^2-1)/24 = 7$ linear independent functions in $A_0(p,\chi_p)$ with Fourier expansions $\sum_{k\geq -7} a_k q^k$. By Remark 2.5.6, each modular form is uniquely determined by its principal part, so the given modular forms form a basis of $M_6(13,1)$ and f_1 is a linear combination of M, which can be easily determined with a computer. $\square$

We calculate

$$
\begin{aligned}
H^{(1)} &= 1 - 13q + 65q^2 - 130q^3 - 65q^4 + 728q^5 - 871q^6 - 715q^7 + O\left(q^8\right), \\
H^{(q)} &= \Delta^7 \cdot \left(H^{(1)}\right)^{-13} = q^7 + q^8 + 2q^9 + 3q^{10} + 5q^{11} + 7q^{12} + 11q^{13} + O\left(q^{14}\right), \\
\tilde{H} &= \eta^2/(\eta^{(13)})^2 = q^{-1} - 2 - q + 2q^2 + q^3 + 2q^4 - 2q^5 - 2q^7 + O\left(q^8\right), \\
j^{(13)} &= q^{-13} + 744 + 196884q^{13} + 21493760q^{26} + 864299970q^{39} + O(q^{52})
\end{aligned}
$$

and get

$$
\begin{aligned}
f_1 =& g/H^{(q)} = q^{-1} + 1 + q + 3q^3 - 2q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_2 =& \tilde{H} f_1 + f_1 = q^{-2} + O\left(1\right) \in A_0^-(13, \chi_{13}) \\
f_3 =& f_1^3 - 3f_2 - 6f_1 = q^{-3} + 4 + 9q - 2q^3 + 12q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_4 =& \tilde{H} f_1^3 - f_3 + f_2 + 6f_1 = q^{-4} + 3 - 8q + 16q^3 + 29q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_5 =& f_1^5 - 5f_4 - 15f_3 - 30f_2 - 60f_1 = q^{-5} + O\left(1\right) \in A_0^-(13, \chi_{13}) \\
f_6 =& \tilde{H} f_1^5 - 3f_5 - 4f_4 + 3f_3 + 4f_2 + 12f_1 = q^{-6} + O\left(1\right) \in A_0^-(13, \chi_{13}) \\
f_7 =& f_1^7 - 7f_6 - 28f_5 - 77f_4 - 182f_3 - 378f_2 - 714f_1 = q^{-7} + O\left(1\right) \in A_0^-(13, \chi_{13}) \\
f_8 =& \tilde{H} f_1^7 - 5f_7 - 13f_6 - 16f_5 - 15f_4 - 2f_3 + 30f_2 + 174f_1 = q^{-8} + O\left(1\right) \in A_0^-(13, \chi_{13}) \\
f_9 =& f_1^9 - 9f_8 - 45f_7 - 156f_6 - 441f_5 - 1080f_4 - 2382f_3 - 4680f_2 - 8397f_1 \\
=& q^{-9} + 13 - 9q + 36q^3 - 198q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_{10} =& \tilde{H} f_1^9 - 7f_9 - 26f_8 - 59f_7 - 103f_6 - 143f_5 - 154f_4 + 54f_3 + 524f_2 + 1285f_1 \\
=& q^{-10} + 4 - 40q - 200q^3 + 60q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_{11} =& f_1^{11} - 11f_{10} - 66f_9 - 275f_8 - 913f_7 - 2585f_6 - 6512f_5 - 14762f_4 - 30525f_3 \\
& - 58036f_2 - 102718f_1 = q^{-11} + O\left(1\right) \in A_0^-(13, \chi_{13}) \\
f_{12} =& \tilde{H} f_1^{11} - 9f_{11} - 43f_{10} - 134f_9 - 320f_8 - 629f_7 - 1065f_6 - 1364f_5 - 988f_4 \\
& + 915f_3 + 4652f_2 + 11758f_1 \\
& = q^{-12} + 12 + 48q - 272q^3 - 255q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_{13} =& \frac{1}{2}E_0 + 2f_{12} + 8f_{10} + 6f_9 + 26f_4 + 8f_3 + 24f_1 \\
& = \frac{1}{2}q^{-13} + 7 + 39q + 221q^3 + 494q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
f_{14} =& j^{(13)} f_1 - 2f_{13} - f_{12} - 3f_{10} + 2f_9 + f_4 + 4f_3 - 748f_1 \\
& = q^{-14} + 6 + 504q^3 - 1232q^4 + O\left(q^5\right) \in A_0^+(13, \chi_{13}) \\
& \vdots
\end{aligned}
$$

5.1.3 A Basis in the case $\mathbb{Q}(\sqrt{17})$

Since the Fourier expansion of $\tilde{H}$ starts with q^2, we do not have $\tilde{f}_2 = \tilde{H} f_1$ as for $p = 5$ and $p = 13$, so we need to calculate both f_1 and f_2. Therefore we determine $f_1 \cdot H^{(q)}$ and $f_2 \cdot H^{(q)}$ in the space $M_4(17,1)$ as a linear combination of a basis. Then we continue by writing $\tilde{f}_3 = f_1^3$, $\tilde{f}_4 = f_1^2 f_2, \ldots$.

Lemma 5.1.4 (Calculation of f_1 and f_2). *The modular forms*

$$E_4,\ E_4^{(17)},\ (E_2^{(17)})^2,\ H_2^2,\ H_2G_2 \text{ and } G_2^2$$

form a basis of the vector space $M_4(17,1)$ of holomorphic modular forms for $\Gamma_0(17)$ of weight 4 with trivial character (for notation compare Lemma 2.5.24 and Lemma 2.5.30). We calculate

$$f_1 = \frac{g_1(\tau)}{\eta(\tau)\eta(17\tau)^7} \quad \text{and} \quad f_2 = \frac{g_2(\tau)}{\eta(\tau)\eta(17\tau)^7},$$

where

$$\begin{aligned} g_1 &= \frac{1}{960}E_4 + \frac{119}{960}E_4^{(17)} + \frac{3}{8}\left(E_2^{(17)}\right)^2 - \frac{245}{32}H_2^2 + \frac{51}{16}G_2H_2 - \frac{13}{32}G_2^2 \\ g_2 &= \frac{1}{2880}E_4 - \frac{1241}{2880}E_4^{(17)} + \frac{1}{72}\left(E_2^{(17)}\right)^2 - \frac{21}{32}H_2^2 - \frac{1}{16}G_2H_2 + \frac{3}{32}G_2^2. \end{aligned}$$

The proof is analogous to the proof of Lemma 5.1.3. We calculate

$$\begin{aligned} H^{(1)} &= 1 - 17q + 119q^2 - 408q^3 + 476q^4 + 1309q^5 - 5236q^6 + 4233q^7 + O\left(q^8\right), \\ H^{(q)} &= \Delta^{12} \cdot \left(H^{(1)}\right)^{-17} = q^{12} + q^{13} + 2q^{14} + 3q^{15} + 5q^{16} + 7q^{17} + 11q^{18} + O\left(q^{19}\right), \\ \tilde{H} &= \eta^3/(\eta^{(17)})^3 = q^{-2} - 3q^{-1} + 5q - 7q^4 + 9q^8 - 11q^{13} + 3q^{15} - 9q^{16} + O\left(q^{18}\right), \\ j^{(17)} &= q^{-17} + 744 + 196884q^{17} + 21493760q^{34} + 864299970q^{51} + O(q^{68}) \end{aligned}$$

and obtain

$$\begin{aligned} f_1 =& \left(\frac{1}{960}E_4 + \frac{119}{960}E_4^{(17)} + \frac{3}{8}\left(E_2^{(17)}\right)^2 - \frac{245}{32}H_2^2 + \frac{51}{16}H_2G_2 - \frac{13}{32}G_2^2\right)\frac{\eta}{\left(\eta^{(17)}\right)^7} \\ =& q^{-1} + \frac{1}{2} - q + q^2 + 2q^4 - q^8 - 2q^9 + q^{13} - q^{15} + 2q^{16} + O\left(q^{17}\right) \in A_0^+(17,\chi_{17}) \\ f_2 =& \left(\frac{1}{2880}E_4 - \frac{1241}{2880}E_4^{(17)} + \frac{1}{72}\left(E_2^{(17)}\right)^2 - \frac{21}{32}H_2^2 - \frac{1}{16}H_2G_2 + \frac{3}{32}G_2^2\right)\frac{\eta}{\left(\eta^{(17)}\right)^7} \\ =& 7q^{-2} + \frac{3}{2} + 2q + 3q^2 - q^4 + 6q^8 - 6q^9 - 8q^{13} - 3q^{16} + O\left(q^{17}\right) \in A_0^+(17,\chi_{17}) \\ f_3 =& f_1^3 - \frac{3}{2}f_2 + \frac{9}{4}f_1 = q^{-3} + O\left(1\right) \in A_0^-(17,\chi_{17}) \\ f_4 =& f_2f_1^2 - f_3 + \frac{1}{4}f_2 - \frac{9}{2}f_1 \\ =& q^{-4} + \frac{7}{2} + 8q - 2q^2 + 11q^4 - 5q^8 + 16q^9 - 56q^{13} + O\left(q^{15}\right) \in A_0^+(17,\chi_{17}) \end{aligned}$$

$$
\begin{aligned}
f_5 =& f_1^5 - \frac{5}{2}f_4 + \frac{5}{2}f_3 + \frac{15}{4}f_2 - \frac{205}{16}f_1 = q^{-5} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_6 =& f_2f_1^4 - 2f_5 + f_4 - \frac{7}{2}f_3 - \frac{197}{16}f_2 - \frac{13}{4}f_1 = q^{-6} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_7 =& f_1^7 - \frac{7}{2}f_6 + \frac{7}{4}f_5 + \frac{77}{8}f_4 - \frac{287}{16}f_3 - \frac{1085}{32}f_2 + \frac{861}{64}f_1 = q^{-7} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_8 =& f_2f_1^6 - 3f_7 + \frac{3}{4}f_6 - \frac{345}{16}f_4 - \frac{231}{16}f_3 + \frac{237}{64}f_2 - \frac{1059}{32}f_1\\
=& q^{-8} + \frac{15}{2} - 8q + 24q^2 - 10q^4 + 27q^8 + 216q^9 + 288q^{13} + O\left(q^{15}\right) \in A_0^+(17,\chi_{17})\\
f_9 =& f_1^9 - \frac{9}{2}f_8 + \frac{33}{2}f_6 - \frac{135}{8}f_5 - \frac{1215}{16}f_4 + \frac{225}{16}f_3 + \frac{2835}{32}f_2 - \frac{11673}{256}f_1\\
=& q^{-9} + \frac{7}{2} - 18q - 27q^2 + 36q^4 + 243q^8 + 41q^9 - 279q^{13} + O\left(q^{15}\right) \in A_0^+(17,\chi_{17})\\
f_{10} =& f_2f_1^8 - 4f_9 - \frac{1}{2}f_8 + 5f_7 - \frac{223}{8}f_6 - \frac{173}{4}f_5 + \frac{39}{2}f_4 - \frac{467}{16}f_3 - \frac{40873}{256}f_2 - \frac{7045}{32}f_1\\
=& q^{-10} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_{11} =& f_1^{11} - \frac{11}{2}f_{10} - \frac{11}{4}f_9 + \frac{187}{8}f_8 - \frac{55}{8}f_7 - \frac{2123}{16}f_6 - \frac{891}{32}f_5 + \frac{15499}{64}f_4 - \frac{11341}{256}f_3\\
&- \frac{287463}{512}f_2 - \frac{630223}{1024}f_1 = q^{-11} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_{12} =& f_2f_1^{10} - 5f_{11} - \frac{11}{4}f_{10} + \frac{21}{2}f_9 - 28f_8 - \frac{723}{8}f_7 + \frac{577}{32}f_6 + \frac{313}{8}f_5 - \frac{77177}{256}f_4\\
&- \frac{114813}{256}f_3 - \frac{228167}{1024}f_2 + \frac{121379}{512}f_1 = q^{-12} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_{13} =& f_1^{13} - \frac{13}{2}f_{12} - \frac{13}{2}f_{11} + \frac{117}{4}f_{10} + \frac{221}{16}f_9 - \frac{6279}{32}f_8 - \frac{1157}{8}f_7 + \frac{14989}{32}f_6 + \frac{40105}{256}f_5\\
&- \frac{653003}{512}f_4 - \frac{661193}{512}f_3 + \frac{927173}{1024}f_2 + \frac{12277915}{4096}f_1\\
=& q^{-13} + 7 + 13q - 52q^2 - 182q^4 + 468q^8 - 403q^9 + 4172q^{13} + O\left(q^{15}\right) \in A_0^+(17,\chi_{17})\\
f_{14} =& f_2f_1^{12} - 6f_{13} - 6f_{12} + \frac{31}{2}f_{11} - \frac{315}{16}f_{10} - \frac{303}{2}f_9 - \frac{1003}{32}f_8 + \frac{3069}{16}f_7 - \frac{105351}{256}f_6\\
&- \frac{129619}{128}f_5 - \frac{96075}{256}f_4 + \frac{258723}{512}f_3 - \frac{3568301}{4096}f_2 - \frac{6432951}{1024}f_1\\
&= q^{-14} + O\left(1\right) \in A_0^-(17,\chi_{17})\\
f_{15} =& f_1^{15} - \frac{15}{2}f_{14} - \frac{45}{4}f_{13} + \frac{265}{8}f_{12} + \frac{735}{16}f_{11} - \frac{8163}{32}f_{10} - \frac{23345}{64}f_9 + \frac{91005}{128}f_8 + \frac{197265}{256}f_7\\
&- \frac{1207365}{512}f_6 - \frac{3217191}{1024}f_5 + \frac{4388475}{2048}f_4 + \frac{26582645}{4096}f_3 - \frac{33676545}{8192}f_2 - \frac{397673595}{16384}f_1\\
=& q^{-15} + 4 - 15q - 240q^4 - 1800q^8 - 2060q^9 - 7140q^{13} + O\left(q^{15}\right) \in A_0^+(17,\chi_{17})
\end{aligned}
$$

$$\begin{aligned}
f_{16} =& f_2 f_1^{14} - 7f_{15} - \frac{41}{4} f_{14} + 19 f_{13} - \frac{27}{16} f_{12} - \frac{3493}{16} f_{11} - \frac{10233}{64} f_{10} + \frac{13149}{32} f_9 - \frac{82621}{256} f_8 \\
& - \frac{533777}{256} f_7 - \frac{997075}{1024} f_6 + \frac{194025}{128} f_5 - \frac{3092257}{4096} f_4 - \frac{45672899}{4096} f_3 - \frac{214903451}{16384} f_2 \\
& + \frac{83440457}{8192} f_1 \\
=& q^{-16} + \frac{31}{2} + 32q - 24q^2 + 56q^4 - 2074q^8 - 2240q^9 + 15904q^{13} + O\left(q^{15}\right) \in A_0^+(17, \chi_{17}) \\
f_{17} =& \frac{1}{2} E_0 + f_{16} + 3f_{15} + 7f_{13} + 15f_9 + 7f_8 + 14f_4 + 8f_2 + 31f_1 \\
=& \frac{1}{2} q^{-17} + \frac{9}{2} + 34q + 51q^2 + 204q^4 + 1581q^8 + 2499q^9 + 12019q^{13} + O\left(q^{15}\right) \in A_0^+(17, \chi_{17}) \\
f_{18} =& j^{(p)} f_1 - f_{17} + f_{16} - f_{15} - 2f_{13} + f_9 + 2f_8 - f_4 + f_2 - 746 f_1 \\
=& q^{-18} + \frac{21}{2} - 54q + 54q^2 + 459q^4 - 2484q^8 - 5542q^9 + 3024q^{13} + O\left(q^{15}\right) \in A_0^+(17, \chi_{17}) \\
& \vdots
\end{aligned}$$

5.2 Weight and Multiplier Systems

We investigate the weights and multiplier systems possible for Hilbert modular forms following the work of Gundlach [Gu88]. In the cases $p \in \{5, 13, 17\}$ we present a way to determine the multiplier system of a Borcherds product depending only on the Weyl vector.

From Theorem 4.1.5 we calculate

Corollary 5.2.1 (Multiplier systems and weights of Hilbert modular forms for $p = 5, 13, 17$).

(5) $f \in M_k^5(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0$ *and* $\mu \equiv 1$*:*

All weights of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ *are integral and there is no multiplier system but the trivial one.*

(13) $f \in M_k^{13}(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0$, $\mu(J) = \mu(T)^3 = \mu(T_w)^3 = 1$*:*

All weights of Hilbert modular forms for $\mathbb{Q}(\sqrt{13})$ *are integral. All multiplier systems* μ *can be obtained by the choice of* $a, b \in \left\{e^{2\pi i/3}, e^{4\pi i/3}, 1\right\}$ *as the extension of* $\mu(J) = 1$, $\mu(T) = a$ *and* $\mu(T_w) = b$*. In all these cases we have* $\mu(D_{\varepsilon_0}) = \mu(JT_{\varepsilon_0^{-1}} JT_{\varepsilon_0} JT_{\varepsilon_0^{-1}}) = \mu(J)^3 \mu(T_{\varepsilon_0})^{-1} = \mu(TT_w)^{-1} = \mu(T)^{-1} \mu(T_w)^{-1}$ *and all multiplier systems are characters.*

(17) $f \in M_k^{17}(\mu) \setminus \{0\} \Rightarrow k \in \mathbb{N}_0/2$, $\mu^2(T) = \mu(T_w)^4 = (-1)^{2k}$ *and* $\mu(J) = \mu(T)^3$*:*

All weights of Hilbert modular forms for $\mathbb{Q}(\sqrt{17})$ *are half integral. Write* $a := 1$ *for integral weight,* $a = e^{\pi i/4}$ *else. Then we get all multiplier systems by the choices of*

μ	$\mu(J)$	$\mu(T)$	$\mu(T_w)$	$\mu(D_{\varepsilon_0})$	μ^2	symmetry
$\mu_{0,0}$	1	1	1	1	$\mu_{0,0}$	symmetric
$\mu_{1,2}$	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$	1	$\mu_{2,1}$	symmetric
$\mu_{2,1}$	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$	1	$\mu_{1,2}$	symmetric
$\mu_{0,1}$	1	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$\mu_{0,2}$	$\overline{\mu_{0,1}} = \mu_{0,2}$
$\mu_{0,2}$	1	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$	$\mu_{0,1}$	$\overline{\mu_{0,2}} = \mu_{0,1}$
$\mu_{1,0}$	1	$e^{2\pi i/3}$	1	$e^{4\pi i/3}$	$\mu_{2,0}$	$\overline{\mu_{1,0}} = \mu_{1,1}$
$\mu_{1,1}$	1	$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$\mu_{2,2}$	$\overline{\mu_{1,1}} = \mu_{1,0}$
$\mu_{2,0}$	1	$e^{4\pi i/3}$	1	$e^{2\pi i/3}$	$\mu_{1,0}$	$\overline{\mu_{2,0}} = \mu_{2,2}$
$\mu_{2,2}$	1	$e^{4\pi i/3}$	$e^{4\pi i/3}$	$e^{2\pi i/3}$	$\mu_{1,1}$	$\overline{\mu_{2,2}} = \mu_{2,0}$

Table 5.1: Multiplier systems for $p = 13$.

$b \in \{-1, 1\}$ and $c \in \{1, i, -1, -i\}$ as extensions μ of $\mu(J) = b \cdot a^2$, $\mu(T) = \mu(J)^3 = b \cdot a^{-2}$ and $\mu(T_w) = a \cdot c$. For D_{ε_0} we then have $\mu(D_{\varepsilon_0}) = -\mu(J)^3 \mu(T)^{-7} \mu(T_w)^6$. If the weight is integral, μ is a character.

Proof. [Gu88, §5] and [MWS].

For calculations note, that each multiplier system, restricted to the subgroup of translations T_x, is a character. In case $p = 17$ it is $T_{\varepsilon_0} = T_{4+\sqrt{17}} = T^3 (T_w)^2$ and $T_{\varepsilon_0^{-1}} = T_{-4+\sqrt{17}} = T^{-5}(T_w)^{-2}$. If we apply Definition 4.1.3 to the equation $D_{\varepsilon_0} = JT_{-\overline{\varepsilon_0}} JT_{\varepsilon_0} JT_{-\overline{\varepsilon_0}}$, we get for half integral weight $(p = 17)$ and every multiplier system μ

$$\mu(D_{\varepsilon_0}) = -\mu(J)^3 \mu(T)^{-7} \mu(T_w)^6 = -\mu(J)\mu(T)\mu(T_w)^2$$

by calculations and $\mu(J)^2 = -1$, $\mu(T)^2 = -1$ and $\mu(T_w)^4 = -1$. □

Table 5.1 consist of a collection of all characters in case $p = 13$. We get the tables 5.2 and 5.3 for the multiplier systems for $p = 17$ from the corollary. The multiplier systems, whose restrictions to the diagonal are trivial, are underlined in Table 5.3. We have seen in Remark 4.2.4 that a multiplier system is symmetric if and only if $\mu(T_w) = \mu(T)/\mu(T_w)$.

Corollary 5.2.2 (Half integral weight implies eight power). *If f is a modular form for $\mathbb{Q}(\sqrt{17})$ of half integral weight k, then*

(i) f is a modular form with one of the multiplier systems $\mu_{3,1}, \ldots, \mu_{3,8}$.

character	$\mu(J)$	$\mu(T)$	$\mu(T_w)$	$\mu(D_{\varepsilon_0})$	b	c	symmetry	diagonal	square	weight
μ_0	1	1	1	1	1	1	symmetric	1	μ_0	$\mathbb{Z}$
$\mu_{1,1}$	-1	-1	1	1	-1	1	$\overline{\mu_{1,1}} = \mu_{1,3}$	μ_η^{12}	μ_0	$\mathbb{Z}$
$\mu_{1,2}$	1	1	-1	1	1	-1	symmetric	1	μ_0	$\mathbb{Z}$
$\mu_{1,3}$	-1	-1	-1	1	-1	-1	$\overline{\mu_{1,3}} = \mu_{1,1}$	μ_η^{12}	μ_0	$\mathbb{Z}$
$\mu_{2,1}$	1	1	i	-1	1	i	$\overline{\mu_{2,1}} = \mu_{2,4}$	1	$\mu_{1,2}$	$\mathbb{Z}$
$\mu_{2,2}$	-1	-1	i	-1	-1	i	symmetric	μ_η^{12}	$\mu_{1,2}$	$\mathbb{Z}$
$\mu_{2,3}$	-1	-1	$-i$	-1	-1	$-i$	symmetric	μ_η^{12}	$\mu_{1,2}$	$\mathbb{Z}$
$\mu_{2,4}$	1	1	$-i$	-1	1	$-i$	$\overline{\mu_{2,4}} = \mu_{2,1}$	1	$\mu_{1,2}$	$\mathbb{Z}$
$\mu_{3,1}$	i	$-i$	$\sqrt{i}$	i	1	1	$\overline{\mu_{3,1}} = \mu_{3,2}$	μ_η^{18}	$\mu_{2,2}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,2}$	i	$-i$	$-\sqrt{i}$	i	1	-1	$\overline{\mu_{3,2}} = \mu_{3,1}$	μ_η^{18}	$\mu_{2,2}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,3}$	$-i$	i	$\sqrt{i}$	i	-1	1	symmetric	μ_η^{6}	$\mu_{2,2}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,4}$	$-i$	i	$-\sqrt{i}$	i	-1	-1	symmetric	μ_η^{6}	$\mu_{2,2}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,5}$	i	$-i$	$i\sqrt{i}$	$-i$	1	i	symmetric	μ_η^{18}	$\mu_{2,3}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,6}$	i	$-i$	$-i\sqrt{i}$	$-i$	1	$-i$	symmetric	μ_η^{18}	$\mu_{2,3}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,7}$	$-i$	i	$i\sqrt{i}$	$-i$	-1	i	$\overline{\mu_{3,7}} = \mu_{3,8}$	μ_η^{6}	$\mu_{2,3}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
$\mu_{3,8}$	$-i$	i	$-i\sqrt{i}$	$-i$	-1	$-i$	$\overline{\mu_{3,8}} = \mu_{3,7}$	μ_η^{6}	$\mu_{2,3}$	$\mathbb{Z}/2 \setminus \mathbb{Z}$
μ_η	$e^{\pi i/4}$	$e^{\pi i/12}$	compare with the character of the Dedekind η-function							$\frac{1}{2}$

Table 5.2: Multiplier systems for $p = 17$.

$$
\begin{array}{c}
\underline{\mu_0}\\
\uparrow\\
\overbrace{\mu_{1,1}\qquad \underline{\mu_{1,2}}\qquad \mu_{1,3}}\\
\uparrow\\
\overbrace{\underline{\mu_{2,1}}\qquad\quad \mu_{2,2}\qquad\qquad \mu_{2,3}\qquad\quad \underline{\mu_{2,4}}}\\
\nearrow \qquad\qquad\qquad \nwarrow\\
\overbrace{\mu_{3,1}\ \ \mu_{3,2}\ \ \mu_{3,3}\ \ \mu_{3,4}}\qquad \overbrace{\mu_{3,5}\ \ \mu_{3,6}\ \ \mu_{3,7}\ \ \mu_{3,8}}
\end{array}
$$

Table 5.3: Taking squares of multiplier systems for $p = 17$. The multiplier systems with trivial restriction to the diagonal are underlined.

(ii) f^4 *is a Hilbert modular form of weight* $4 \cdot k$ *with multiplier systems* $\mu_{1,2}$.

(iii) f^8 *is a Hilbert modular form of weight* $8 \cdot k$ *with trivial multiplier system* μ_0.

The following question arises naturally for a given Borcherds product: What is its multiplier system? There is a simple answer in case $p \in \{5, 13, 17\}$. We follow a suggestion of Bruinier, that all necessary information should be given by the Weyl vector:

Theorem 5.2.3 (Multiplier systems of Borcherds products). *The multiplier system of a Borcherds product for* $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ *and* $\mathbb{Q}(\sqrt{17})$ *can be read from its Weyl vector. Especially we have* $\mu(T_\lambda) = \mathbf{e}((\mathrm{S}(\rho_W \lambda))$.

Proof. Let Ψ be a Borcherds product with multiplier system μ. Ψ has the Fourier expansion

$$\Psi(\tau_1, \tau_2) = \mathbf{e}(\rho_W \tau_1 + \overline{\rho_W} \tau_2) \underbrace{\prod_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ (\nu, W) > 0}} (1 - \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})}}_{H} .$$

Let $\nu = \alpha + \beta\frac{\sqrt{p}}{p} \in \mathfrak{o}\,/\sqrt{p}$ and $\lambda = a + b\sqrt{p} \in \mathfrak{o}$ with $a, b, \alpha, \beta \in \mathbb{Z}/2$ and $b - a, \alpha + \beta \in \mathbb{Z}$. Then we have for all $\tau \in \mathbb{H}^2$:

$$\begin{aligned}
\mathbf{e}(\nu(\tau_1 + \lambda) + \overline{\nu}(\tau_2 + \overline{\lambda})) &= \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2) \cdot \mathbf{e}(\mathrm{S}(\nu\lambda)) \\
&= \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2) \cdot \mathbf{e}(2(a\alpha + b\beta)) \\
&= \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2) \cdot \mathbf{e}(2(a\alpha + a\beta + b\beta - a\beta)) \\
&= \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2) \cdot \mathbf{e}(\underbrace{2a}_{\in\mathbb{Z}} \cdot \underbrace{(\alpha + \beta)}_{\in\mathbb{Z}} + \underbrace{2\beta}_{\in\mathbb{Z}} \cdot \underbrace{(b - a)}_{\in\mathbb{Z}})) \\
&= \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2) .
\end{aligned}$$

Hence H is invariant under the operation of T and T_w and we have $\mu(T) = \mathbf{e}(\mathrm{S}(\rho_W))$ and $\mu(T_w) = \mathbf{e}(\mathrm{S}(\rho_W w))$. So by Corollary 5.2.1 the multiplier system μ is uniquely determined by the Weyl vector in the cases $p = 5$, $p = 13$ and $p = 17$. $\square$

Remark 5.2.4.

- *For $p = 13$ we calculate*

$$\mathbf{e}\left(\mathrm{S}\left(\frac{1}{6} + \frac{\sqrt{13}}{26}\right)\right) = e^{2\pi i/3}$$

and

$$\begin{aligned} \mathbf{e}\left(\mathrm{S}\left(\left(\frac{1}{6} + \frac{\sqrt{13}}{26}\right) \cdot \left(\frac{1}{2} + \frac{1}{2}\sqrt{13}\right)\right)\right) &= \mathbf{e}\left(2\left(\frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right)\right) \\ &= \mathbf{e}\left(\frac{2}{3}\right) = e^{4\pi i/3}. \end{aligned}$$

Thus we can determine the multiplier systems (here: characters) of some Borcherds products ($p = 13$), for their Weyl vectors are often a power of $\frac{1}{6} + \frac{\sqrt{13}}{26}$. Note that the characters are symmetric by Remark 4.2.4.

- *For $p = 17$ we calculate*

$$\mathbf{e}\left(\mathrm{S}\left(\frac{1}{8} + \frac{\sqrt{17}}{34}\right)\right) = i$$

and

$$\begin{aligned} \mathbf{e}\left(\mathrm{S}\left(\left(\frac{1}{8} + \frac{\sqrt{17}}{34}\right) \cdot \left(\frac{1}{2} + \frac{1}{2}\sqrt{17}\right)\right)\right) &= \mathbf{e}\left(2\left(\frac{1}{8} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right)\right) \\ &= \mathbf{e}\left(\frac{5}{8}\right) = -\sqrt{i}, \end{aligned}$$

where we write $\sqrt{i} := e^{\pi i/4}$. Hence we can calculate the multiplier systems of some Borcherds products ($p = 17$), for their Weyl vectors are often a power of $\frac{1}{8} + \frac{\sqrt{17}}{34}$. Additionally we have $\mathbf{e}\left(\mathrm{S}\left(\frac{1}{8}\right)\right) = i$ and $\mathbf{e}\left(\mathrm{S}\,\frac{1}{8}\left(\frac{1}{2} + \frac{1}{2}\sqrt{17}\right)\right) = \mathbf{e}\left(\frac{1}{8}\right) = \sqrt{i}$, so Ψ_{17} has the multiplier system $\mu_{3,3}$ (compare Table A.6 and Table 5.2).

5.3 Fourier Expansion of Borcherds Products

This section describes a method to calculate Fourier coeffi cients of Borcherds products and introduces our concept of the realization of Fourier coeffi cients of Hilbert modular forms on a computer.

The Fourier expansion of a Hilbert modular form is a sum of $c(\lambda)e^{2\pi i \,\mathrm{S}(\lambda\tau)}$, where $\tau \in \mathbb{H}^2$ and $\lambda \in \mathcal{K}$ (cf. Remark 1.2.17) and $c(\lambda)$ is some complex constant. We rewrite for $\lambda = \lambda_1 + \lambda_2\sqrt{p} \in K$:

$$e^{2\pi i \,\mathrm{S}(\lambda\tau)} = e^{2\pi i(\lambda\tau_1+\lambda'\tau_2)} = e^{2\pi i\left(\lambda_1(\tau_1+\tau_2)+\lambda_2(\tau_1-\tau_2)\sqrt{p}\right)} = e^{\pi i(2\lambda_1)(\tau_1+\tau_2)}e^{\pi i(2p\lambda_2)(\tau_1-\tau_2)/\sqrt{p})}.$$

We write $\hat{\lambda}_1 := 2\lambda_1$ and $\hat{\lambda}_2 := 2p\lambda_2$ and

$$g := e^{\pi i(\tau_1+\tau_2)} \qquad \text{and} \qquad h := e^{\pi i(\tau_1-\tau_2)/\sqrt{p})}$$

to simplify

$$e^{2\pi i \,\mathrm{S}(\lambda\tau)} = g^{\hat{\lambda}_1}h^{\hat{\lambda}_2}$$

and get an easy criterion whether $\lambda = \hat{\lambda}_1/2 + \hat{\lambda}_2\sqrt{p}/(2p)$ is contained in $\mathfrak{o}^{\#}$: If and only if $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathbb{Z}$ and $\hat{\lambda}_1 + \hat{\lambda}_2 \in 2\mathbb{Z}$, then $\lambda \in \mathfrak{o}^{\#} = \mathfrak{o}\,/\sqrt{p}$. We can read three properties of such a Fourier expansion:

Remark 5.3.1. *Let f be a Hilbert modular form with multiplier system μ and with Fourier expansion $f(\tau) = \sum_{a,b} c(a,b)g^a h^b$, then*

a) if $c(a,b) = 0$ for all $(a,b) \notin \mathbb{Z}^2$ and all $a+b \notin 2\mathbb{Z}$, then $\mu(T_\lambda) = 1$ for all $\lambda \in \mathfrak{o}$,

b) if $p \in \{5,13,17\}$ and $c(a,b) = 0$ for all $(a,b) \notin \mathbb{Z}^2$ and all $a+b \notin 2\mathbb{Z}$, then μ is the trivial multiplier system,

c) if $c(a,-b) = c(a,b)$ for all a,b, then f is symmetric, if $c(a,-b) = -c(a,b)$ for all a,b, then f is skew symmetric and

d) $c(a,b) \neq 0$ only for $a \geq 0$ and $|b| \leq a\sqrt{p}$.

Proof. a) In this case, the Fourier expansion is invariant under all transformations $\tau \mapsto \tau + \lambda$ with $\lambda \in \mathfrak{o}$.

b) a) and Corollary 5.2.1

c) Trivial.

d) Remark 1.2.17.

□

Now we can easily calculate the product of two Fourier expansions up to a given accuracy: Given two Hilbert modular forms

$$f_{(1)}(\tau) = \sum_{\substack{a,b\in\mathcal{Z}\\ a\geq 0\\ |b|\leq\sqrt{p}a}} c(a,b)g^a h^b \qquad \text{and} \qquad f_{(2)}(\tau) = \sum_{\substack{a,b\in\mathcal{Z}\\ a\geq 0\\ |b|\leq\sqrt{p}a}} d(a,b)g^a h^b$$

we get

$$f_{(1)}f_{(2)}(\tau) = \sum_{\substack{a,b\in\mathcal{Z} \\ a\geq 0 \\ |b|\leq\sqrt{p}a}} \left(\sum_{\substack{0\leq\alpha\leq a \\ \alpha\in\mathcal{Z}}} \sum_{\substack{\beta\in\mathcal{Z} \\ |\beta|\leq\sqrt{p}\alpha \\ |b-\beta|\leq a-\alpha}} c(\alpha,\beta)d(a-\alpha,b-\beta) \right) g^a h^b$$

where $\mathcal{Z} = \mathbb{Z}$ and $a+b \in 2\mathbb{Z}$ in the case of trivial multiplier system and $\mathcal{Z}$ is a rational ideal in $\mathbb{Q}$ otherwise. This motivates the defi nition

Definition 5.3.2. If f is a Hilbert modular form with Fourier expansion

$$f(\tau) = \sum_{\substack{a,b\in\mathcal{Z} \\ a\geq 0 \\ |b|\leq\sqrt{p}a}} c(a,b)g^a h^b$$

and there is $N \in \mathbb{N}$ such that $c(a,b)$ is known for all $a \leq N$, $|b| \leq \sqrt{p}a$, then f respectively

$$\sum_{\substack{a,b\in\mathcal{Z} \\ 0\leq a\leq N \\ |b|\leq\sqrt{p}a}} c(a,b)g^a h^b$$

are said to be given with precision g^N.

Hence we get

Lemma 5.3.3. *If $f_{(1)}$ and $f_{(2)}$ are Hilbert modular forms given with precision g^N, then their product $f_{(1)}f_{(2)}$ is given with precision g^N.*

and

Lemma 5.3.4 (Calculation of Borcherds products with given precision). *Let $p \equiv 1 \pmod 4$ be a prime, $m \in \mathbb{N}$ with $\chi_p(m) \geq 0$, denote by f_m the unique basis element of $A_0^+(p,\chi_p)$ with Fourier expansion $s(-m)^{-1}q^{-m} + \sum_{k\geq 0} a(k)q^k$, let W be a Weyl chamber attached to f_m and $\tau \in W$ with $y_1 = \mathrm{Im}\,(\tau_1)$ and $y_2 = \mathrm{Im}\,(\tau_2)$. Define $a(-m) = s(-m)^{-1}$ and $a(-k) = 0$ for all $k \in \mathbb{N}_0 \setminus \{m\}$. Then for every $N \in \mathbb{N}$, Ψ_m, given by*

$$\Psi_m(\tau_1,\tau_2) = \mathbf{e}(\rho_W\tau_1 + \overline{\rho_W}\tau_2) \prod_{\substack{\nu\in\mathfrak{o}/\sqrt{p} \\ (\nu,W)>0}} (1-\mathbf{e}(\nu\tau_1+\overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})} \qquad \textit{for all } \tau \in W,$$

can be calculated with precision g^N by the following algorithm:

Step 1: Calculate the leading coefficient $a(h,-k)g^{-k}$ *(with respect to* $g = \mathbf{e}((\tau_1+\tau_2)/2)$*) of*

$$\prod_{\substack{\nu=\nu_1+\nu_2\sqrt{p}\in\mathfrak{o}/\sqrt{p} \\ -\sqrt{(y_1-y_2)^2 m/(4py_1y_2)}<\nu_1<0 \\ \nu_2^2\le\nu_1^2/p+m/p^2}} (1-\mathbf{e}(\nu\tau_1+\overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})},$$

and in the case that $\mathrm{S}(\rho_W)$ *is negative, rewrite* $k := k - \mathrm{S}(\rho_W)$*.*

Step 2: Expand

$$R = g^{\mathrm{S}(\rho_W)} h^{(\rho_W-\overline{\rho_W})\sqrt{p}} \prod_{\substack{\nu=\nu_1+\nu_2\sqrt{p}\in\mathfrak{o}/\sqrt{p} \\ -\sqrt{(y_1-y_2)^2 m/(4py_1y_2)}<\nu_1\le(N+k)/2 \\ \nu_2^2\le\nu_1^2/p+m/p^2}} \left(1-g^{2\nu_1}h^{2p\nu_2}\right)^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})},$$

where we expand each factor $(1-g^{2\nu_1}h^{2p\nu_2})^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})}$ *with precision* g^{k+N} *and neglect higher order terms. For negative exponents use the geometric series*

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n \qquad \textit{for } |x|<1.$$

Then Ψ_m *is given by* R *with precision of* g^N*.*

Proof. Let $\mu = \mu_1 + \mu_2\sqrt{p} \in \mathfrak{o}/\sqrt{p}$. Then $\mathbf{e}(\mathrm{S}(\nu\tau)) = g^{2\nu_1}h^{2p\nu_2}$ and the factor $(1-g^{2\nu_1}h^{2p\nu_2})$ has a negative power of g if and only if $\nu_1 < 0$. In this case we get from $(W,\nu) > 0$:

$$\begin{aligned} & \nu_1(y_1+y_2) + \nu_2(y_1-y_2)\sqrt{p} > 0 \\ \iff\ & \nu_2(y_1-y_2)\sqrt{p} > \underbrace{-\nu_1}_{>0}\underbrace{(y_1+y_2)}_{>0} \\ \iff\ & |\nu_2||y_1-y_2|\sqrt{p} > -\nu_1(y_1+y_2) \\ \iff\ & |\nu_2| > |\nu_1|\frac{y_1+y_2}{|y_1-y_2|\sqrt{p}} \end{aligned}$$

Furtheron for $\mathrm{N}(\nu) < -m/p$ we have $a(p\,\mathrm{N}(\nu)) = 0$, so we can skip

$$(1-\mathbf{e}(\nu_1\tau_1+\nu_2\tau_2))^{s(p\,\mathrm{N}(\nu))a(p\,\mathrm{N}(\nu))} = 1$$

in the product expansion of Ψ_m whenever $\mathrm{N}(\nu) < -m/p$. So negative exponents only may derive from the factor $\mathbf{e}(\rho_W\tau_1 + \overline{\rho_W}\tau_2)$ and $\nu \in \mathfrak{o}/\sqrt{p}$ with $(W,\nu) > 0$ and

$$\begin{aligned} & \mathrm{N}(\nu) = \nu_1^2 - p\nu_2^2 \ge -m/p \\ \iff\ & \nu_2^2 \le \frac{\nu_1^2}{p} + \frac{m}{p^2}. \end{aligned}$$

The combination of both conditions gives

$$\begin{aligned}
&|\nu_1|^2 \frac{(y_1+y_2)^2}{|y_1-y_2|^2 p} < \nu_2^2 \leq \frac{\nu_1^2}{p} + \frac{m}{p^2} \\
\Longrightarrow\ & \frac{|\nu_1|^2}{p}\left(\frac{(y_1+y_2)^2}{|y_1-y_2|^2} - 1\right) < \frac{m}{p^2} \\
\Longrightarrow\ & |\nu_1|^2 \frac{(y_1+y_2)^2-(y_1-y_2)^2}{(y_1-y_2)^2} < \frac{m}{p} \\
\Longrightarrow\ & |\nu_1|^2 < \frac{m}{p}\frac{(y_1-y_2)^2}{4y_1y_2}
\end{aligned}$$

Since $s(-m)a(-m) = 1$ every factor $(1-q^{2\nu_1}h^{2p\nu_2})$ with negative q-exponent occurs once, so by Lemma 5.3.3 we need every factor in the product expansion of Ψ_m with precision g^{N+k}. It remains to show that the geometric series can be applied for negative exponents. Since $\nu \operatorname{Im}(\tau_1) + \overline{\nu} \operatorname{Im}(\tau_2) > 0$ by $(W, \nu) > 0$ and

$$|\mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2)| = e^{-2\pi(\nu \operatorname{Im}(\tau_1) + \overline{\nu} \operatorname{Im}(\tau_2))} < 1$$

the geometric series converges. $\square$

Remark 5.3.5. *Some results of these calculations can be found in the Tables A.7, A.9 and A.11 in the appendix. The full data and the corresponding Maple™-worksheets can be found at* `http://www.matha.rwth-aachen.de`.

6 Rings of Hilbert Modular Forms

We determine the rings of extended Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$*,* $\mathbb{Q}(\sqrt{13})$ *and* $\mathbb{Q}(\sqrt{17})$ *and compare them to other results by various authors.*

6.1 Reduction process

We describe a method to calculate the Ring of extended Hilbert modular forms.

At least in the cases $p \in \{5, 13, 17\}$ we have calculated a sufficient number of Hilbert modular forms to get the Ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{p})$ with symmetric multiplier systems. In detail, Chapter 5 is about calculating Hilbert modular forms as Borcherds products (cf. Table 6.1) with given divisors (Section 3.5), Section 2.1 explains how to calculate Hilbert Eisenstein series, the computation of theta series is described in Section 2.2. In all the cases $p \equiv 1 \pmod 4$ prime, there is a Hilbert modular form Ψ_1 vanishing of first order on the diagonal Diag and – modulo Γ – only on the diagonal. Consider some Hilbert modular form f of weight $k > 0$ with multiplier system μ. Its restriction to the diagonal is an elliptic modular form $F = f \circ \delta$ of weight $2k$ with character $\mu|_{\mathrm{SL}(2,\mathbb{Z})}$ (cf. Lemma and Definition 4.2.7). Assume that we know a Hilbert modular form g with multiplier system μ, such that the restriction of g to the diagonal is F. Then f has weight k and $f - g$ is a Hilbert modular form of weight k with multiplier system μ vanishing on the diagonal. Hence it is a multiple of Ψ_1 and $(f - g)/\Psi_1$ is a Hilbert modular form of weight less than k. Supposed there can always be constructed a Hilbert modular form with same restriction to the diagonal, it is possible to reduce the weight iteratively until one reaches weight 0 (constant modular forms). But some elliptic modular forms F of weight $k/2$ with character $\mu|_{\mathrm{SL}(2,\mathbb{Z})}$ do not have a Hilbert modular form f of weight k with multiplier system

$f \in A_0^+(p)$	$\longmapsto \Psi$	divisor	Hilbert modular form g
$f_1 = q^{-1} + O(1)$	$\longmapsto \Psi_1$	$\Gamma \cdot \mathrm{Diag}$	$g\|_{\mathrm{Diag}} \equiv 0 \Rightarrow \Psi_1 \| g$
$f_p = \frac{1}{2}q^{-p} + O(1)$	$\longmapsto \Psi_p$	$\Gamma \cdot \mathrm{Diag}_{\varepsilon_0}$	$g\|_{\mathrm{Diag}_{\varepsilon_0}} \equiv 0 \Rightarrow \Psi_p \| g$
$f_j = \frac{1}{s(n)}q^{-j} + O(1)$	$\longmapsto \Psi_j$		

Table 6.1: Borcherds products in the reduction process

μ such that F is the restriction of f to the diagonal. In the cases $p \in \{5, 13, 17\}$ we will see that these exceptions can be determined by Remark 4.1.9, Lemma 4.1.11 and Lemma 4.3.1. For this the following definition will be useful:

Definition 6.1.1. Let $f : \mathbb{H}^2 \to \mathbb{C}$ be a Hilbert modular form. We define the symmetric part $f^+ : \mathbb{H}^2 \to \mathbb{C}$ and the skew-symmetric part $f^- : \mathbb{H}^2 \to \mathbb{C}$ of f by

$$f^+(\tau_1, \tau_2) = \frac{1}{2}\left(f(\tau_1, \tau_2) + f(\tau_2, \tau_1)\right) \quad \text{and} \quad f^-(\tau_1, \tau_2) = \frac{1}{2}\left(f(\tau_1, \tau_2) - f(\tau_2, \tau_1)\right).$$

It will be possible to show that f^+ vanishes on $\mathrm{Diag}_{\varepsilon_0}$ in a number of cases and clearly f^- vanishes on the diagonal Diag.

The graded ring of elliptic modular forms is generated by E_4, E_6 and, in case of nontrivial character, the Dedekind-η-function. For the first weights and trivial character we have

k	0	2	4	6	8	10	12	odd
$[\mathrm{SL}(2,\mathbb{Z}), k]$	$\mathbb{C}$	$\{0\}$	$\mathbb{C}E_4$	$\mathbb{C}E_6$	$\mathbb{C}\underbrace{E_4^2}_{E_8}$	$\mathbb{C}\underbrace{E_4E_6}_{E_{10}}$	$\mathbb{C}E_4^3 + \mathbb{C}E_6^2$	$\{0\}$

For all $z \in \mathbb{H}$ we get

$$E_2^H(z, z) = E_4(z)$$

with the Hilbert Eisenstein series E_2^H of weight 2, since the Fourier expansion of both modular forms starts with 1 and the weight of the restriction of E_2^H to the diagonal is 4. Analogously

$$(E_2^H)^2(z, z) = E_4^H(z, z) = E_8(z).$$

This does not imply $(E_2^H)^2 = E_4^H$ which is true for $p = 5$ and false for $p = 17$. Since all Hilbert Eisenstein series have even weight, none of them is a lift of E_6. What is more, by Lemma 4.1.11 odd weight Hilbert modular forms are cusp forms, so there can be no such Hilbert modular form at all.

In the cases $p \in \{5, 13, 17\}$ these methods suffice to determine the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$ and symmetric characters.

6.2 State of Art

Some rings of Hilbert modular forms are already known. We will give some examples and sketch the methods used to obtain each result. The completely determined ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ will be handled in the next subsection.

The first one to publish on Hilbert modular forms was Blumenthal. He proves in his Habilitationsschrift [Bl04a] that the field of meromorphic Hilbert modular forms of weight 0 with trivial multiplier system for a totally real number field of degree n and the group $\mathrm{GL}(2,\mathfrak{o})$ is generated by n algebraically independent modular forms (cf. [Bl03], [Bl04a] and the introduction of this work).

Every element of $\mathrm{GL}(2,\mathfrak{o})$ can be written as $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & 1\end{smallmatrix}\right) M$ with $\varepsilon \in \mathfrak{o}^*$ and $M \in \mathrm{SL}(2,\mathfrak{o})$ and since $\left(\begin{smallmatrix}\varepsilon^2 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right)$ induce the same map on $\mathbb{H}^n$, the group $\mathrm{SL}(2,\mathfrak{o})$ has finite index $|\,\mathfrak{o}^*/(\mathfrak{o}^*)^2|$ in $\mathrm{GL}(2,\mathfrak{o})$, where $(\mathfrak{o}^*)^2$ is the subgroup of squares in $\mathfrak{o}^*$.

Assume that there are $n+1$ algebraically independent meromorphic Hilbert modular forms $f_1,\ldots,f_{n+1}$ of weight 0 with trivial multiplier system for a totally real number field of degree n and the group $\mathrm{SL}(2,\mathfrak{o})$. Then append the set $\{f_1,\ldots,f_{n+1}\}$ with all translates $f_j(\varepsilon\tau)$, $\varepsilon \in \mathfrak{o}^*/(\mathfrak{o}^*)^2$, $1 \leq j \leq n+1$ and get a set of at most $(n+1)|\,\mathfrak{o}^*/(\mathfrak{o}^*)^2|$ elements of which at least $n+1$ are algebraically independent. The elementary symmetric polynomials form an isomorphism from the polynomial ring generated freely by this set to the subset of symmetric polynomials, so the image of our set gives us at least $n+1$ algebraically independent symmetric polynomials. The symmetry forces their invariance under the transformations in $\mathrm{GL}(2,\mathfrak{o})$, so this is a contradiction to [Bl04a] and there are n algebraically independent meromorphic Hilbert modular forms of weight 0 with trivial multiplier system for $\mathrm{SL}(2,\mathfrak{o})$.

Hence the maximal number of algebraically independent (holomorphic) Hilbert modular forms is $n+1$ (compare also [Re56, p. 277, 278] and [Th54, Hauptsatz II, p. 457]) and Freitag gave an existence theorem of $n+1$ algebraically independent Poincaré series in his book [Fr90] (also see [Bl03, Part II]).

Another general result on Hilbert modular forms is the formula of Shimizu (e.g. cf. [TV83, Theorem 2.16]). It gives the dimension of the space of Hilbert cuspforms. For even weights, by adding the number of cusps, we obtain the dimension of the space of Hilbert modular forms of fixed weight for trivial multiplier system (cf. [Fr90, Corollary I.5.10$_1$]).

There have been a number of works on rings of Hilbert modular forms for small discriminant, where generators and relations in between the generators have been determined. By $M^p_{\text{even}}(1)$ we denote the subring of Hilbert modular forms of even weight with trivial multiplier system for $\mathbb{Q}(\sqrt{p})$. A good overview is given in [TV83], where we find

$$\begin{aligned}
M^5_{\text{even}}(1) &= \mathbb{C}[X_1, X_3, X_5, X_{10}]/(R_{20})\\
M^2_{\text{even}}(1) &= \mathbb{C}[X_1, X_2, X_3, X_7]/(R_{14})\\
M^{13}_{\text{even}}(1) &= \mathbb{C}[X_1, X_2, X_3, Y_3, X_4]/(R_6, R_8)
\end{aligned}$$

with generators X_j, Y_j (depending on p) and relations R_j of weight $2j$ next to a reference to van der Geer ([Ge78]) who investigated $M^6_{\text{even}}(1)$. Also some results on Hilbert modular forms for congruence subgroups of Γ are given.

The spaces of Hilbert modular forms for $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ have been investigated by Nagaoka (cf. [Na82] and [Na83a]) in 1982 and 1983 and Nagaoka determined the $\mathbb{Z}$-module of Hilbert modular forms for $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ with integral Fourier coefficients shortly after (cf. [Na83b] and [Na86]). Müller constructed the rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ (cf. [Mü83] and [Mü85]) in 1983 and 1985 in terms of theta series and Hammond's modular embedding. Müller also repeats the results of Resnikoff [Re74], Hirzebruch [Hi76] and Gundlach [Gu63] for Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ and compares them to his own.

Resnikoff (cf. [Re74]) writes that after [Ha66a], Hammond's modular embedding cannot give complete results for the space of Hilbert modular forms for quadratic number fi elds, if the discriminant is larger than 8. In case $p = 2$ (then the discriminant equals 8) one solely gets the symmetric modular forms of even weight.

Hermann calculated the rings of symmetric Hilbert modular forms with trivial multiplier system for $\mathbb{Q}(\sqrt{17})$ and $\mathbb{Q}(\sqrt{65})$ in 1981 (cf. [He81] and [He83]). His main result is

Theorem 6.2.1 (Satz 5 in [He81]). *The ring of symmetric Hilbert modular forms of even weight for $\mathbb{Q}(\sqrt{17})$ is generated by G_2, G_4, H_4, G_6 and H_6, where $G_2 := (\eta_1^2 - 4\eta_2^2)/\Theta_{(1,1,1,1)}$, $G_4 := \eta_1\eta_2\Theta_{(1,1,1,1)}$, $H_4 = \Theta^4_{(1,1,1,1)}$, $G_6 = \eta_1^2\Theta^3_{(1,1,1,1)}$ and $H_6 = \eta_1^3\eta_2$. The definition of the function $\Theta_{(1,1,1,1)}$ can be found in Definition 2.2.8; It it a Hilbert modular form of weight 1 with multiplier system μ_{17}^2. The Hilbert modular form η_2 was defined in Definition 2.2.11 and η_1 is an homogeneous polynomial in Θ_m like η_2 with multiplier system μ_{17}^5.*

This result was refi ned in 1985 by [Ch85], who determined the ring of symmetric Hilbert modular forms with trivial multiplier system for $\mathbb{Q}(\sqrt{17})$ (not necessarily even weight) as

$$M^{17}_{\text{symm}}(1) = \langle A_2, B_4, B_6, C_4, C_6, D_9, F_7, F_9\rangle$$

and the ring of Hilbert modular forms with trivial multiplier system as

$$M^{17}(1) = \langle A_2, B_3, B_4, B_5, C_4, C_5, C_6, D_6, D_8, F_7, F_9\rangle\,.$$

6.3 The Ring of Hilbert Modular Forms for $\mathbb{Q}(\sqrt{5})$

This is a benchmark of the reduction process, since the ring is already known. We will compare the known results described in the last subsection to our results. Both coincide and we give some Borcherds product expansions for two of the generators. There is only one multiplier system.

As done by Gundlach, Resnikoff and others before, we calculate the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ and get:

Theorem 6.3.1. *M^5 is generated by the Eisenstein series E_2^H and E_6^H and the Borcherds products Ψ_1 and Ψ_5 (cf. table 6.2) and all relations in between the given generators are induced by the relation R_{30}:*

$$\begin{aligned}\Psi_5^2 - &\left(\frac{67}{25}E_6^H - \frac{42}{25}\left(E_2^H\right)^3\right)\left(\frac{67}{43200}\left(\left(E_2^H\right)^3 - E_6^H\right)\right)^4\\ &= \Psi_1^2\Big(3125\,\Psi_1^4 + \frac{1}{1728}\Psi_1^2\left(335\left(E_2^H\right)^2 E_6^H - 227\left(E_2^H\right)^5\right)\\ &+ \frac{4486}{89579520000}\left(43\left(E_2^H\right)^{10} - 153\left(E_2^H\right)^7 E_6^H + 177\left(E_2^H\right)^4\left(E_6^H\right)^2 - 67E_2^H\left(E_6^H\right)^3\right)\Big)\end{aligned}$$

f	E_2^H	Ψ_1	$e_6 := \frac{67}{25}E_6^H - \frac{42}{25}\left(E_2^H\right)^3$	Ψ_5
$f \circ \delta$	E_4	0	E_6^2	$\Delta^2 E_6$
weight of f	2	5	6	15

Table 6.2: Minimal generating set for M_5

In other words if we write $X_2 = E_2^H$, $X_5 = \Psi_1$, $X_6 = e_6$ *and* $X_{15} = \Psi_5$ *we get*

$$M^5 = \mathbb{C}[X_2, X_5, X_6, X_{15}]/\left\langle R_{30}\right\rangle .$$

Proof by induction. By Corollary 4.1.12 every non-constant Hilbert modular form has positive weight. So we can start induction by $M_0^{(5)}(1) = \mathbb{C}$.

Let $k \in \mathbb{N}$. Assume that every Hilbert modular form of weight at most $k-1$ was contained in the subring R of M^5 generated by $\left\{E_2^H, E_6^H, \Psi_1, \Psi_5\right\}$.

Let $f \in M_k^5(1)$. Consider the two cases

- k is odd: Since $\tau = D_{\varepsilon_0}(\tau_2, \tau_1)$ for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$, we get

$$\begin{aligned} f^+(\tau) &= f^+(D_{\varepsilon_0}(\tau_2,\tau_1)) \\ &\overset{\mu\equiv 1}{=} \mathrm{N}(\varepsilon_0^{-1})^k f^+(\overline{\tau}) \\ &\overset{f^+ \text{ symm.}}{=} -f^+(\tau) \end{aligned}$$

  for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$. So f^+ vanishes on $\mathrm{Diag}_{\varepsilon_0}$ and f^- vanishes on Diag by construction. Since Ψ_5 vanishes on $\mathrm{Diag}_{\varepsilon_0}$ of first order and only vanishes on $\Gamma\,\mathrm{Diag}_{\varepsilon_0}$ and Ψ_1 vanishes on $\Gamma\,\mathrm{Diag}$ of first order and only there, we have $\Psi_5 | f^+$ and $\Psi_1 | f^-$. Then f^+/Ψ_5 has weight $k-15$ and is contained in R by the induction hypothesis, as is the modular form f^-/Ψ_1 of weight $k-5$. So $f = \Psi_5\left(f^+/\Psi_5\right) + \Psi_1\left(f^-/\Psi_1\right)$ is contained in R, too.

- k is even: $f \circ \delta$ is an elliptic modular form of weight $2k$ for $SL(2,\mathbb{Z})$, so there is a polynomial q with $f\circ\delta - q(g_2, g_3^2) \equiv 0$. Hence $f - q(E_2^H, e_6)|_{\mathrm{Diag}} = 0$ and $\Psi_1 | \left(f - q(E_2^H, e_6)\right)$. We conclude as before that f is contained in R by the induction hypothesis.

Since $\Psi_1 \not\equiv 0$, it is clear from the restriction to the diagonal, that Ψ_1, E_2^H and e_6 are algebraically independent. So the left hand side of the given relation follows immediately from the elliptic case, the right hand side can be easily calculated by a computer. $\square$

In order to confirm this result, have a look at Müller (cf. [Mü85]). He introduces the cusp forms s_5 and s_6, where $s_5 = \Theta$ (p. 245) and $s_6 = \frac{67}{2^5 \cdot 3^3 \cdot 5^2}\left(\left(E_2^H\right)^3 - E_6^H\right)$ (p. 242).

Theorem 6.3.2 (Satz 1 in [Mü85]). *The ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ is generated by the modular forms E_2^H, s_5, s_6 and s_{15} of weights 2, 5, 6 and 15. They form a minimal generating set and can be represented by the 10 Thetanullwerte $\Theta_i, i = 0, 1, \ldots, 9$. The skew-symmetric cusp form s_5 and the symmetric cusp form s_{15} satisfy the relations*

$$\begin{aligned} s_5^2 &= s_{10} \\ s_{15}^2 &= 5^5 s_{10}^3 - 2^{-1} \cdot 5^3 \left(E_2^H\right)^2 s_6 s_{10}^2 + 2^{-4} \left(E_2^H\right)^5 s_{10}^2 + 2^{-1} \cdot 3^2 \cdot 5^2 E_2^H s_6^3 s_{10} \\ &\qquad - 2^{-3} \left(E_2^H\right)^4 s_6^2 s_{10} - 2 \cdot 3^3 s_6^5 + s^{-4} \left(E_2^H\right)^3 s_6^4 \end{aligned}$$

in the ring of symmetric Hilbert modular forms of even weight for $\mathbb{Q}(\sqrt{5})$. So every modular form f of weight k is given uniquely in the form

$$f = \begin{cases} p_1(g_2, s_6, s_{10}) + s_5 \cdot s_{15} \cdot p_2(E_2^H, s_6, s_{10}) & \textit{for } k \equiv 0 \pmod 2 \\ s_{15} \cdot p_1(E_2^H, s_6, s_{10}) + s_5 \cdot p_2(E_2^H, s_6, s_{10}) & \textit{for } k \equiv 1 \pmod 2, \end{cases}$$

where p_1 and p_2 are appropriate isobaric polynomials in E_2^H, s_6 and s_{10}.

Müller gives some Fourier exponents, so we can easily compare the generators and get $s_5 = \Psi_1$, $s_6 = \frac{1}{864}\left(\left(E_2^H\right)^3 - e_6\right)$ and $s_{15} = \Psi_5$.

6.4 The Ring of Hilbert Modular Forms for $\mathbb{Q}(\sqrt{13})$

We give the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{13})$ with symmetric characters and the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{13})$ with trivial character.

Theorem 6.4.1. *M^{13} is generated by Ψ_1, $\frac{\Psi_4}{2\Psi_1}$, E_2^H and Ψ_{13} (cf. table 6.3) and the relations in between the given generators are induced by*

$$\begin{aligned} R_{14}:\ \Psi_{13}^2 - \left(\frac{\Psi_4}{2\Psi_1}\right)^4 \left(\left(E_2^H\right)^3 - 2^6 3^3 \left(\frac{\Psi_4}{2\Psi_1}\right)^3\right) &= -108\Psi_1^{12}\Psi_2 - \frac{27}{16}\Psi_1^{10}\left(E_2^H\right)^2 \\ &+ \frac{495}{8}\Psi_1^8\Psi_2^2 E_2^H - \frac{1459}{16}\Psi_1^6\Psi_2^4 + \frac{41}{8}\Psi_1^6\Psi_2 E_2^H - 512\Psi_1^6\left(\frac{\Psi_4}{2\Psi_1}\right)^4 + \frac{1}{16}\Psi_1^4\left(E_2^H\right)^5 \\ &- \frac{97}{4}\Psi_1^4\Psi_2^3\left(E_2^H\right)^2 - \frac{1}{8}\Psi_1^2\Psi_2^2\left(E_2^H\right)^4 - 144\Psi_1^2\left(\frac{\Psi_4}{2\Psi_1}\right)^5 E_2^H + \frac{189}{8}\Psi_1^2\Psi_2^5 E_2^H. \end{aligned}$$

In other words if we write $X_1 = \Psi_1$, $X_2 = \frac{\Psi_4}{2\Psi_1}$, $Y_2 = E_2^H$ and $X_7 = \Psi_{17}$ we get

$$M^{13} = \mathbb{C}[X_1, X_2, Y_2, X_7] / \langle R_{14} \rangle .$$

f	Ψ_1	$\frac{\Psi_4}{2\Psi_1}$	E_2^H	Ψ_{13}
$f \circ \delta$	0	η^8	E_4	$\eta^{16}E_6$
weight of f	1	2	2	7
multiplier system	μ_{13}	μ_{13}	1	μ_{13}^2

Table 6.3: Minimal generating set for M_{13}. The multiplier system μ_{13} is given by $\mu_{13}(J) = 1$, $\mu_{13}(T) = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, $\mu_{13}(T_w) = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$.

Proof. We can prove Theorem 6.4.1 quite similar to Theorem 6.3.1. Now there is more than one multiplier system, but every symmetric multiplier system of a Hilbert modular form f is already determined by $f \circ \delta$ (cf. table 5.1). To start induction, note that by Corollary 4.1.12 all non-constant Hilbert modular forms have positive weight. We write $R = \left\langle \Psi_1, \frac{\Psi_4}{2\Psi_1}, E_2^H, \Psi_{13} \right\rangle$. Let $k \in \mathbb{N}$. Assume that all Hilbert modular forms of weight at most $k-1$ are contained in R. Let $f \in M_k^{13}(\mu)$ with some symmetric multiplier system μ. Consider the two cases:

- If k is odd, we get (it is $\tau = D_{\varepsilon_0}(\tau_2, \tau_1)$ for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$):

$$\begin{aligned} f^+(\tau) &= f^+(D_{\varepsilon_0}(\tau_2, \tau_1)) \\ &= \mu(D_{\varepsilon_0})\, \mathrm{N}(\varepsilon_0^{-1})^k f^+(\overline{\tau}) \\ &\overset{f^+ \text{ symm.}}{=} -\mu(D_{\varepsilon_0}) f^+(\tau) \end{aligned}$$

  for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$. Since $\mu(D_{\varepsilon_0}) \in e^{2\pi i \mathbb{Z}/3}$ it is $-\mu(D_{\varepsilon_0}) \neq 1$ and we obtain $f^+|_{\mathrm{Diag}_{\varepsilon_0}} \equiv 0$. Analogously to the case of M^5 we get $\Psi_{13}|f^+$ and $\Psi_1|f^-$. Then f^+/Ψ_{13} has weight $k-7$ and is contained in R by the induction hypothesis, as is the modular form f^-/Ψ_1 of weight $k-1$. So $f = \Psi_{13}\,(f^+/\Psi_{13}) + \Psi_1\,(f^-/\Psi_1)$ is contained in R, too.

- If k is even, then $f \circ \delta$ is an elliptic modular form of weight $2k$ for $SL(2,\mathbb{Z})$ and there is a polynomial q with $f \circ \delta - q(\eta^8, E_4) \equiv 0$. Hence $f - q\left(\frac{\Psi_4}{2\Psi_1}, E_2^H\right)|_{\mathrm{Diag}} = 0$ and $\Psi_1 | \left(f - q\left(\frac{\Psi_4}{2\Psi_1}, E_2^H\right)\right)$. Then f is contained in R by the induction hypothesis.

So we have shown $R = M^{13}$.

Since $\Psi_1 \not\equiv 0$, it is clear from the restriction to the diagonal, that Ψ_1, $\frac{\Psi_4}{2\Psi_1}$ and E_2^H are algebraically independent. So the left hand side of the given relation follows immediately from the elliptic case:

$$(\eta^{16}E_6)^2 - (\eta^8)^4 E_6^2 = 0,$$

so simple computations yield the given relation between the Hilbert modular forms. $\square$

	X_4	X_{18}	X_{12}	X_{10}	X_{16}	X_8	X_6
f	E_2^H	$\frac{\Psi_4}{2\Psi_1}\Psi_{13}$	$\left(\frac{\Psi_4}{2\Psi_1}\right)^3$	$\Psi_1\left(\frac{\Psi_4}{2\Psi_1}\right)^2$	$\Psi_1\Psi_{13}$	$\Psi_1^2\frac{\Psi_4}{2\Psi_1}$	Ψ_1^3
$f\circ\delta$	E_4	ΔE_6	Δ	0	0	0	0
weight of f	4	18	12	10	16	8	6

Table 6.4: Minimal generating set for $M^{13}(1)$

Corollary 6.4.2. *We write* $X_4 = E_2^H$, $X_6 = \Psi_1^3$, $X_8 = \Psi_1^2\frac{\Psi_4}{2\Psi_1}$, $X_{10} = \Psi_1\left(\frac{\Psi_4}{2\Psi_1}\right)^2$, $X_{12} = \left(\frac{\Psi_4}{2\Psi_1}\right)^3$, $X_{16} = \Psi_1\Psi_{13}$ *and* $X_{18} = \frac{\Psi_4}{2\Psi_1}\Psi_{13}$ *and define the relations*

$$R_{18}: \quad X_{10}X_8 = X_{12}X_6, \qquad R_{20}: \quad X_{10}^2 = X_{12}X_8,$$

$$R_{24}: \quad X_{16}X_8 = X_6X_{18},$$

$$\begin{aligned} R_{36}: \quad X_{18}^2 = {} & X_{12}^2X_4^3 - 1728X_{12}^3 - 108X_3X_6^4 + \tfrac{1}{16}X_8^2X_4^5 + \tfrac{41}{8}X_{12}X_6^2X_4^3 - \tfrac{1459}{16}X_{12}^2X_6^2 \\ & + \tfrac{495}{8}X_{10}^2X_6^2X_4 - \tfrac{97}{4}X_8X_4^2X_{10}^2 - \tfrac{27}{16}X_{10}X_6^3X_4^2 - \tfrac{1}{8}X_{10}^2X_4^4 + \tfrac{189}{8}X_4X_{12}^2X_8. \end{aligned}$$

Then

$$M^{13}(1) = \mathbb{C}[X_4, X_6, X_8, X_{10}, X_{12}, X_{16}, X_{18}] \,/\, (R_{18}, R_{20}, R_{24}, R_{36}).$$

6.5 The Ring of Hilbert Modular Forms for $\mathbb{Q}(\sqrt{17})$

We give the ring of Hilbert modular forms with symmetric multiplier systems and the ring of Hilbert modular forms for $\mathbb{Q}(\sqrt{17})$ *with trivial character.*

Theorem 6.5.1. M^{17} *is generated by* $X_{\frac{1}{2}} = \Psi_1$, $X_{\frac{3}{2}} = -\Psi_2$, $Y_{\frac{3}{2}} = \eta_2$, $X_2 = E_2^H$ *and* $X_{\frac{9}{2}} = \Psi_{17}$. *Together with the two relations of weight 3 and 9,*

$$R_3 : \eta_2^2 - 64\Psi_2^2 = 16\Psi_1^2E_2^H$$

and

$$\begin{aligned} R_9 : \Psi_{17}^2 - \Psi_2^2\left(E_2^H\right)^3 + 216\Psi_2^5\eta_2 = {} & -256\Psi_1^{18} \\ & - 176\Psi_1^{12}\Psi_2\eta_2 - \frac{2671}{4096}\Psi_1^6\eta_2^4 + \frac{103}{8}\Psi_1^4\left(E_2^H\right)^2\Psi_2\eta_2 \\ & - \frac{87}{16}\Psi_1^{10}\left(E_2^H\right)^2 - \frac{99}{128}\Psi_1^2E_2^H\Psi_2\eta_2^3 + \frac{1387}{128}\Psi_1^8E_2^H\eta_2^2, \end{aligned}$$

we have $M^{17} = \mathbb{C}[X_{\frac{1}{2}}, X_{\frac{3}{2}}, Y_{\frac{3}{2}}, X_2, X_{\frac{9}{2}}]/(R_3, R_9)$.

f	Ψ_1	E_2^H	$-\Psi_2$	$\eta_2/8$	Ψ_{17}
$f \circ \varphi$	0	E_4	η^6	η^6	$\eta^6 E_6$
weight of f	$\frac{1}{2}$	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$
multiplier system	μ_{17}	1	μ_{17}^5	μ_{17}	μ_{17}^5

Table 6.5: Minimal generating set for M^{17}

Proof. The proof is similar to the ones for M^5 and M^{13}, with two differences. First, the restriction of a symmetric multiplier system to the diagonal loses information about the multiplier system, second, there are half integral weights and the symmetric Hilbert modular forms of odd weight are not in general divisible by Ψ_{17}.

Again every non-constant Hilbert modular form has positive weight by Corollary 4.1.12. Let $k \in \mathbb{Z}/2$. We write $R = \left\langle \Psi_1, E_2^H, -\Psi_2, \eta_2, \Psi_{17} \right\rangle$ and assume that every modular form of weight at most $k - \frac{1}{2}$ is contained in R.

Let $f \in M_k^{17}(\mu)$ be a Hilbert modular form with symmetric multiplier system. Its restriction to the diagonal $F = f|_{\mathrm{Diag}}$ is then contained in $\langle \eta^6, E_4, E_6 \rangle$. If we compare weights and multiplier systems of η^6, E_4 and E_6, we fi nd that $\langle \eta^6, E_4, E_6 \rangle = \langle \eta^6, E_4 \rangle + E_6 \langle \eta^6, E_4 \rangle$. So we distinguish the cases:

- $F \in E_6 \langle \eta^6, E_4 \rangle$:

  The symmetric part f^+ of f holds

$$\begin{aligned} f^+(\tau) = f^+(D_{\varepsilon_0}\overline{\tau}) &= \mu(D_{\varepsilon_0}) N(\varepsilon_0^{-1})^k f^+(\overline{\tau}) \\ &= e^{k\pi i} \mu(D_{\varepsilon_0}) f^+(\tau), \qquad (\overline{\tau} := (\tau_2, \tau_1)) \end{aligned}$$

  for all τ in the twisted diagonal $\mathrm{Diag}_{\varepsilon_0}$. We get from table 5.2 that $\mu(D_{\varepsilon_0})$ only depends on $\mu|_{\mathrm{SL}(2,\mathbb{Z})}$, so it only depends on F. For the three special cases $F \in \{\eta^6, E_4, E_6\}$ we get

F	$2k =$ weight of F	$\mu\vert_{\mathrm{SL}(2,\mathbb{Z})}$	$\mu(D_{\varepsilon_0})$	$e^{-k\pi i} = (-i)^{2k}$
η^6	3	μ_η^6	i	i
E_4	4	1	1	1
E_6	6	1	1	-1

  Hence it is $f^+(\tau) = -f^+(\tau)$ for all $\tau \in \mathrm{Diag}_{\varepsilon_0}$ and therefore f^+ is divisible by Ψ_{17}. Since f^- vanishes on Diag, it is divisible by Ψ_1, so we can reduce the weight and use induction.

- $F \in \langle \eta^6, E_4 \rangle$ is not a cusp form.

In this case f has trivial multiplier system by Lemma 4.1.8 and there is a complex polynomial q in two variables such that $F = f \circ \varphi = q(E_4, \eta^{24})$, so $f - q(E_2^H, (-\Psi_2)^3 \cdot \eta_2/8)$ vanishes on the diagonal and is a Hilbert modular form of weight k with trivial multiplier system. Hence we can divide it by Ψ_1 reducing its weight, showing that f is contained in R.

- $F \in \eta^6 \langle \eta^6, E_4 \rangle$ (is a cusp form)

  Then there is a complex polynomial q in two variables such that $F/\eta^6 = q(\eta^6, E_4)$. We write $q(X,Y) = \sum_m a_m X^{a(m)} Y^{b(m)}$ and define $g_n = a_m(-\Psi_2)^{a(m)+1} E_4^{b(m)}$, if its multiplier system equals μ, and $g_m = a_m \eta_2 (-\Psi_2)^{a(m)} E_4^{b(m)}/8$ otherwise. Then $\sum_m g_m$ and $f - \sum_m g_m$ are Hilbert modular forms of weight k with multiplier system μ and the latter vanishes on the diagonal. Hence $f - \sum_m g_m$ is divisible by Ψ_1 and we get $f \in R$ by induction.

So we have shown $M^{17} = R$.

Since four of the five generators do not vanish on the diagonal, every non-constant complex polynomial q in 5 variables with $q(\Psi_1, E_2^H, -\Psi_2, \eta_2, \Psi_{17}) = 0$ defines a non-constant polynomial $r := q(0, \cdot, \cdot, \cdot, \cdot)$ with $r(E_4, \eta^6, \eta^6, \eta^6 E_6) = 0$. The simple solution $r(x_1, x_2, x_3, x_4) = x_2 - x_3$ has no correspondence as relation between the generators, since $-\Psi_2$ and η_2 have different multiplier systems, but $r(x_1, x_2, x_3, x_4) = x_2^2 - x_3^2$ comes from the identity R_3. All relations in between E_4, η^6 and $\eta^6 E_6$ are induced by $\eta^{12}(E_6^2 - E_4^3) = 1728\eta^{12}\Delta$, so the stated result follows from the elliptic case. □

Corollary 6.5.2. *We write*

$$\begin{aligned}
&X_2 = E_2^H, \quad X_6 = -\Psi_2^3\, \eta_2/8, \quad X_9 = \Psi_2^2 \Psi_{17}\, \eta_2/8, \quad X_5 = -\Psi_1 \Psi_2^3, \\
&X_8 = \Psi_1 \Psi_2^2 \Psi_{17}, \quad X_4 = -\Psi_1^2 \Psi_2\, \eta_2/8, \quad X_7 = \Psi_1^2 \Psi_{17}\, \eta_2/8, \quad X_3 = -\Psi_1^3 \Psi_2, \\
&Y_6 = \Psi_1^3 \Psi_{17}, \quad Y_5 = \Psi_1^7\, \eta_2/8, \quad Y_4 = \Psi_1^8,
\end{aligned}$$

and define the relations

$$\begin{aligned}
R_9 :& \quad X_4 X_5 = X_3 X_6, & R_{10} :& \quad Y_4 X_6 = X_3^2 X_4, \\
R_{11} :& \quad Y_5 X_6 = X_3 X_4^2, & R_{12} :& \quad X_4 X_8 = X_5 X_7, \\
R'_{12} :& \quad X_6 Y_6 = X_5 X_7, & R_{13} :& \quad X_6 X_7 = X_9 X_4, \\
R_{14} :& \quad X_5 X_9 = X_6 X_8, & &
\end{aligned}$$

$$\begin{aligned}
R_{18} : \quad X_9^2 = {}& X_3^2(X_3 + X_2^3) - 256 X_4 Y_4^2 X_6 - 1408 X_3^2 X_4^3 - \tfrac{2671}{4} X_2 X_4^4 \\
&-2671 X_4^2 X_5^2 + \tfrac{2671}{4} X_2 X_3 X_4^2 X_5 - 103 X_2^2 X_4^2 X_6 - \tfrac{87}{16} X_2^2 X_4 Y_4 X_6 \\
&+\tfrac{99}{128} X_2 X_4 X_6^2 + \tfrac{99}{512} X_2^2 X_4^2 X_6 + \tfrac{1387}{2} X_2 X_4^4.
\end{aligned}$$

Then

$$M^{17}(1) = \mathbb{C}[X_2, X_3, X_4, Y_4, X_5, Y_5, X_6, Y_6, X_7, X_8, X_9]/(R_{18}, R_{14}, R_{13}, R_{12}, R_9, R'_{12}, R_{11}, R_{10}).$$

	X_2	X_6	X_9	X_5	X_8
f	E_2^H	$-\Psi_2^3\,\eta_2/8$	$\Psi_2^2\Psi_{17}\,\eta_2/8$	$-\Psi_1\Psi_2^3$	$\Psi_1\Psi_2^2\Psi_{17}$
j = zero order of f on diagonal	0	0	0	1	1
$f/\Psi_1^j \circ \delta$	E_4	Δ	ΔE_6	η^{18}	$\eta^{18}E_6$
weight of f	2	6	9	5	8

	X_4	X_7	X_3	Y_6	Y_5	Y_4
f	$-\Psi_1^2\Psi_2\,\eta_2/8$	$\Psi_1^2\Psi_{17}\,\eta_2/8$	$-\Psi_1^3\Psi_2$	$\Psi_1^3\Psi_{17}$	$\Psi_1^7\,\eta_2/8$	Ψ_1^8
j	2	2	3	3	7	8
$f/\Psi_1^j \circ \delta$	η^{12}	$\eta^{12}E_6$	η^6	η^6E_6	η^6	1
weight of f	4	7	3	6	5	4

Table 6.6: Minimal generating set for $M^{17}(1)$

Proof. This is a corollary of Theorem 6.5.1. Note that Ψ_1 is only generator given in the theorem vanishing on the diagonal and that the restriction of a Hilbert modular form with some multiplier system to the diagonal is contained in $\eta^{6m}\,\langle E_4, \Delta, E_6\rangle$, where $0 \le m < 4$ depends on the multiplier system. In case $m = 0$, it is even contained in the subset $\langle E_4, \Delta, \Delta E_6\rangle$. We can give the generators of the ring for trivial multiplier system some structure by the order of which they vanish on the diagonal. The rest, including the relations, is a simple bookkeeping argument (we can sort the generators by the multiplicity of the divisor F_1, i.e. by the zero order on the diagonal). □

7 Perspectives

This work presents a method to calculate rings of Hilbert modular forms and applies it to the case of extended Hilbert modular forms of homogeneous weights for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. Some of the results and parts of the method can be used for further investigation. We will discuss some further questions and give partial answers to how to solve them.

The easiest question deals with Hilbert modular forms for slightly different groups.

Question 7.1: *How can we get rings of Hilbert modular forms for groups different from the (extended) modular group?*

If the group Γ contains $\mathrm{SL}(2,\mathfrak{o})$ or $\widehat{\mathrm{SL}(2,\mathfrak{o})}$, then the corresponding ring of Hilbert modular forms clearly is a subring of the ring $\mathrm{SL}(2,\mathfrak{o})$ resp. of $\widehat{\mathrm{SL}(2,\mathfrak{o})}$ and a construction as in section 6.2, where we symmetrized modular forms by application of the elementary symmetric polynomials to a Hilbert modular form and all of its translates with respect to the representation of $\Gamma/\mathrm{SL}(2,\mathfrak{o})$.

Therefore we have to restrict our investigation to the action of the group, not the group itself and it will suffi ce to consider fi nite extensions, since the Hilbert modular group acts discrete on $\mathbb{H}^2$.

Subgroups are a different task, since less restrictions lead to more modular forms. In this case we can apply the construction principle for Eisenstein series for the smaller group and might get along with these Eisenstein series and the subring of modular forms invariant under the full modular group.

Question 7.2: *Are there Hilbert modular forms for non-symmetric multiplier systems?*

Eisenstein series, theta series and Borcherds products are extended Hilbert modular forms (cf. Proposition 2.1.2, Lemma 2.2.9 and Corollary 4.2.6), so by Lemma 1.2.12 all Hilbert modular forms constructed in this work have symmetric multiplier systems. On the other hand, note that by Proposition 2.3.3 about Poincaré-series, there are non-trivial Hilbert modular forms for non-symmetric multiplier systems, but the proof thereof is not constructive. The only constructive information we have so far, is that the ring M^p of Hilbert modular forms with symmetric multiplier systems operates on the set of Hilbert modular forms with non-symmetric multiplier systems. Next we pose two structural questions.

Question 7.3: *What is the subring generated by Theta series?*

Question 7.4: *How does differentiation operate on the generators?*

Both are an easy task, since a sufficient number of Fourier coefficients is known and merely calculations remain.

Question 7.5: *Which problems occur in the case $p \equiv 3 \pmod 4$?*

In case $p \equiv 3 \pmod 4$, there is no modular embedding by Theorem 2.2.5, hence there are no theta series. Borcherds products can be constructed as in the case $p \equiv 1 \pmod 4$, only some details of the lift, some constants, change compared to the case $p \equiv 1 \pmod 4$.

Question 7.6: *What obstacles are to be expected in the calculation of rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{p})$ if p is a "large" prime number?*

We have calculated the rings of Hilbert modular forms with symmetric multiplier systems for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. For larger p we get some additional problems:

- $p > 17$: The obstruction space gets nontrivial and it becomes more and more complicated to construct a sufficient number of Borcherds products. But in our case, we only needed few products, so there is hope that in many cases many Borcherds products of small weight can be found nevertheless.

- Resnikoff [Re74] writes, that the modular embedding cannot give complete results for large discriminant, so with growing p the tool of theta series gets less and less useful for the calculation of Hilbert modular forms.

- The calculations get more and more involved, for example for increasing p it gets more difficult to calculate in $\mathbb{Q}(\sqrt{p})$, but still should be not to complicated. An example are Fourier expansions, where we have quite moderate growth: we need to know all elements of $\mathfrak{o}$ with fixed norm m. Since $\mathrm{N}(\varepsilon_0) = \pm 1$ for the fundamental unit $\varepsilon_0 = x_0 + \sqrt{p}y_0$ of $\mathfrak{o}$ we have $\pm 1 + x_0^2 = py_0^2$ and for large p we get ($x_0, y_0 > 0$ by Lemma 3.2.1) $\sqrt{p}/2 - 1 \leq x_0 \approx \sqrt{p}y_0$. Hence we have $\alpha_p = 2x_0y_0/(x_0^2 + py_0^2) \approx \frac{1}{p} - \frac{1}{4p^2y_0^4}$ as $p \to \infty$ and by Lemma 3.2.2 we have, for large values of p, to calculate approximately $4\sqrt{p}y_0^4 m$ elements of $\mathfrak{o}$ to get those of fixed norm m. We can reduce this effort down to approximately $2y_0^2\sqrt{m}$ by going through all x_0 in the range described in Lemma 3.2.2 and checking whether $y_0 := \pm 2\sqrt{(x_0^2 \mp 1)/p}$ is an integer and $x_0 + \sqrt{p}y_0$ is in $\mathfrak{o}$.

- For larger class numbers we have more than one cusp and we should suspect to find more multiplier systems, so we will most likely get a more complicated ring of Hilbert modular forms. Of course this will make calculations harder.

The given reasons show that increasing values of p complicate further calculations, but at least some subring of Hilbert modular forms should be possible to calculate for some primes $p > 17$.

Question 7.7: *Can we calculate the rings of Hilbert modular forms for inhomogeneous weight?*

In [Gu85], Gundlach describes an algorithm to calculate all possible (inhomogeneous) weights with corresponding multiplier systems. We can at least try to calculate the rings of Hilbert modular forms of weight in $(k, k+2\mathbb{Z})$, $k \in \mathbb{Q}$. This can be done by differentiation (compare 4.4.2), since we can start with Hilbert modular forms of homogeneous weight k and map into but probably not onto the subspace of Hilbert modular forms of weight $(k+2, k)$ respectively $(k, k+2)$. Then we can differentiate again to obtain Hilbert modular forms of homogeneous weight $k+2$, where we already know all modular forms. We can try to integrate the last step (at least find some restraints on the Fourier coefficients) while fixing one of the functions of inhomogeneous weight and hence might be able to calculate the ring of Hilbert modular forms of weight $(k, k+2)$ and $(k+2, k)$. But we have to be cautious: the functions obtained by integration might not be Hilbert modular forms, which is for example the case for most of them if the differentiation procedure is not surjective. If we iterate this and additionally multiply Hilbert modular forms of various weights, homogeneous and inhomogeneous, we might be able to calculate some rings of Hilbert modular forms of inhomogeneous weights.

Question 7.8: *How can we calculate Hilbert modular forms for non-quadratic totally real number fields?*

Consider two different prime numbers p, q and the associated rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{p})$, $\mathbb{Q}(\sqrt{q})$, $\mathbb{Q}(\sqrt{pq})$ and $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ (maybe the case $p = 5$, $p = 13$ is a good choice, since then the rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and the ring of symmetric Hilbert modular forms for trivial multiplier system for $\mathbb{Q}(\sqrt{65})$ (cf. [He83]) are known. We reformulate Question 7.8 into

Question 7.9: *Is there a relation between Hilbert modular forms for $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ and Hilbert modular forms for $\mathbb{Q}(\sqrt{p})$, $\mathbb{Q}(\sqrt{q})$ and $\mathbb{Q}(\sqrt{pq})$?*

We have the following diagram for the fields $\mathbb{Q}(\sqrt{p})$, $\mathbb{Q}(\sqrt{q})$, $\mathbb{Q}(\sqrt{pq})$ and $\mathbb{Q}(\sqrt{p}, \sqrt{q})$:

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{p}, \sqrt{q}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{p}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{q}) \arrow[d, dash] & \mathbb{Q}(\sqrt{pq}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

This gives us the following

Lemma 7.10. *If $f : \mathbb{H}^4 \to \mathbb{C}$ is a Hilbert modular form of weight $k = (k_0, k_1, k_2, k_3)$ with multiplier system μ for the group $\Gamma = \mathrm{SL}(2, \mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers for $\mathcal{K} = \mathbb{Q}(\sqrt{p}, \sqrt{q})$,* *and we fix the field automorphisms $\pi_1 = id$, π_2, π_3, π_4 of $\mathcal{K}$ with*

	$\mathrm{sign}(\pi_1)$	$\mathrm{sign}(\pi_2)$	$\mathrm{sign}(\pi_3)$	$\mathrm{sign}(\pi_4)$
$\sqrt{p}$	$+$	$-$	$+$	$-$
$\sqrt{q}$	$+$	$+$	$-$	$-$
$\sqrt{pq}$	$+$	$-$	$-$	$+$

then the functions

$$\begin{aligned} f_1 &: \mathbb{H}^2 \to \mathbb{C}, f_1(\tau) = f(\tau_1, \tau_2, \tau_1, \tau_2) \\ f_2 &: \mathbb{H}^2 \to \mathbb{C}, f_2(\tau) = f(\tau_1, \tau_1, \tau_2, \tau_2) \\ f_3 &: \mathbb{H}^2 \to \mathbb{C}, f_3(\tau) = f(\tau_1, \tau_2, \tau_2, \tau_1) \end{aligned}$$

are Hilbert modular forms, more precisely f_1 is a Hilbert modular form for $\mathbb{Q}(\sqrt{p})$ of weight $(k_1 + k_3, k_2 + k_4)$ with multiplier system $\mu|_{\mathrm{SL}(2,\mathfrak{o}_{\sqrt{p}})}$, f_2 is a Hilbert modular form for $\mathbb{Q}(\sqrt{q})$ of weight $(k_1 + k_2, k_3 + k_4)$ with multiplier system $\mu|_{\mathrm{SL}(2,\mathfrak{o}_{\sqrt{q}})}$ and f_3 is a Hilbert modular form for $\mathbb{Q}(\sqrt{pq})$ of weight $(k_1 + k_4, k_2 + k_3)$ with multiplier system $\mu|_{\mathrm{SL}(2,\mathfrak{o}_{\sqrt{pq}})}$ (where $\mathfrak{o}_{\sqrt{m}}$ is the ring of integers of $\mathbb{Q}(\sqrt{m})$).

And in terms of Fourier expansions we get

Lemma 7.11. *If f is a Hilbert modular form for $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ with Fourier expansion*

$$f(\tau) = \sum_{\alpha} c(\alpha) e^{\mathrm{S}(\alpha)},$$

then the Fourier expansions of f_1, f_2 and f_3 are given by

$$\begin{aligned} f_1(\tau) &= \sum_{\beta} \left(\sum_{\substack{\alpha \\ \alpha_1+\alpha_3=\beta_1 \\ \alpha_2+\alpha_4=\beta_2}} c(\alpha) \right) e^{\mathrm{S}(\beta\tau)}, \\ f_2(\tau) &= \sum_{\beta} \left(\sum_{\substack{\alpha \\ \alpha_1+\alpha_2=\beta_1 \\ \alpha_3+\alpha_4=\beta_2}} c(\alpha) \right) e^{\mathrm{S}(\beta\tau)}, \\ f_3(\tau) &= \sum_{\beta} \left(\sum_{\substack{\alpha \\ \alpha_1+\alpha_4=\beta_1 \\ \alpha_2+\alpha_3=\beta_2}} c(\alpha) \right) e^{\mathrm{S}(\beta\tau)}. \end{aligned}$$

Additionally we immediately get $f_1(\infty, \infty) = f_2(\infty, \infty) = f_3(\infty, \infty) = f(\infty, \infty, \infty, \infty)$.

The other direction is more complicated. At least we have some conditions on Fourier coefficients from Lemma 7.11, since we can get constraints for (and calculate some of) the coefficients of the Fourier expansion of f (having determined the rings for $\mathbb{Q}(\sqrt{p})$, $\mathbb{Q}(\sqrt{q})$ and $\mathbb{Q}(\sqrt{pq})$). Additionally it should be possible to give some dimensions of the space of Hilbert modular forms for $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ of fixed weight. Last but not least a somewhat different question:

Question 7.12: *What kind of applications are there for Hilbert modular forms?*

For one thing Hilbert modular forms play a central role in solving the generalized Fermat equation $x^p + y^q = z^r$ for p, q, r primes (cf. [Da00]). For another, they can be used for Ramanujan graphs and the construction of communication networks (cf. [Li01]) and a variant of the Serre conjecture claims that certain Galois representations connected to algebraic number fields can be constructed with Hilbert modular forms (cf. [De06] for all three applications and cf. [Ta89] and [BDJ] for the construction of Galois representations from Hilbert modular forms)

We may add, that another kind of application of Hilbert Blumenthal modular forms is given by what Blumenthal wrote in his Habilschrift, that Hilbert Blumenthal modular forms are "eine neue Funktionsklasse [...], deren Untersuchung sich in ausgedehntem Maße durchführen läßt, und die daher bei dem Ausbau der allgemeinen Theorie [der komplexen Funktionen in mehreren Variablen] gute Dienste wird leisten können." (a new class of functions [...], whose Investigation can be achieved to a large extend and which hence will be quite useful in the extension of the general theory [of complex functions in several variables]). This problem still remains, more than a hundred years after Hilbert gave his sketches on a new type of modular functions to his doctoral student Ludwig Otto Blumenthal.

A Tables

The following pages contain informations about weights and divisors of Borcherds products and some Fourier coefficients of Borcherds products and Hilbert Eisenstein series. Further data and the corresponding algorithms are accessible at `http://www.matha.rwth-aachen.de`.

p	$B(1)$	$B(2)$	$B(3)$	$B(4)$	$B(5)$	$B(6)$	$B(7)$	$B(8)$	$B(9)$	$B(10)$	$B(11)$	$B(12)$	$B(13)$
5	-10			-30	-30	-20			-70	-20	-120		
13	-2		-8	-6					-26	-8		-24	-14
17	-1	-3		-7				-15	-7				-14

Table A.1: Fourier coefficients of $E_2^+ = 1 + \sum_{n\in\mathbb{N}} B(n)q^n$

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\chi_5(m)$	1	-1	-1	1		1	-1	-1	1		1	-1	-1	1		1
$\chi_{13}(m)$	1	-1	1	1	-1	-1	-1	-1	1	1	-1	1		1	-1	1
$\chi_{17}(m)$	1	1	-1	1	-1	-1	-1	1	1	-1	-1	-1	1	-1	1	1

Table A.2: $\chi_p(m)$ for $m \leq 16$ and $p = 5, 13, 17$.

Weights of the Borcherds products Ψ_j which are lifts of $f_j \in A_0(p,\chi_p)$													
p	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	Ψ_8	Ψ_9	Ψ_{10}	Ψ_{11}	Ψ_{12}	Ψ_{13}
5	5			15	15	10			35	10	60		
13	1		4	3					13	4		12	7
17	$1/2$	$3/2$		$7/2$				$15/2$	$7/2$				7
p	Ψ_{14}	Ψ_{15}	Ψ_{16}	Ψ_{17}	Ψ_{18}	Ψ_{19}	Ψ_{20}	Ψ_{21}	Ψ_{22}	Ψ_{23}	Ψ_{24}	Ψ_{25}	Ψ_{26}
5	30	20	55			100	45	60			50	65	60
13	6		11	18					10	24		21	6
17		4	$31/2$	$9/2$	$21/2$	10		6				$21/2$	21
p	Ψ_{27}	Ψ_{28}	Ψ_{29}	Ψ_{30}	Ψ_{31}	Ψ_{32}	Ψ_{33}	Ψ_{34}	Ψ_{35}	Ψ_{36}	Ψ_{37}	Ψ_{38}	Ψ_{39}
5			150	30	160			80	60	105			120
13	40		30	16					24	39		18	28
17				12		$63/2$	10	$27/2$	12	$49/2$		30	
p	$\frac{\Psi_4}{\Psi_1}$	$\frac{\Psi_8}{\Psi_2}$	$\frac{\Psi_9}{\Psi_1}$	$\frac{\Psi_{12}}{\Psi_3}$	$\frac{\Psi_{16}}{\Psi_1}$	$\frac{\Psi_{16}}{\Psi_4}$	$\frac{\Psi_{18}}{\Psi_2}$	$\frac{\Psi_{20}}{\Psi_5}$	$\frac{\Psi_{24}}{\Psi_6}$	$\frac{\Psi_{25}}{\Psi_1}$	$\frac{\Psi_{27}}{\Psi_3}$	$\frac{\Psi_{32}}{\Psi_8}$	$\frac{\Psi_{36}}{\Psi_9}$
5	10		30		50	40		30	40	60			70
13	2		12	8	10	8				20	36		26
17	3	6	3		15	12	9			21		24	21

Table A.3: The weights of Borcherds products and of some of their holomorphic quotients

p	5	13	17
F_1	$\Gamma M(0,0,\frac{1}{5}\sqrt{5})$	$\Gamma M(0,0,\frac{1}{13}\sqrt{13})$	$\Gamma M(0,0,\frac{1}{17}\sqrt{17})$
F_2	0	0	$\Gamma M(0,0,\frac{1}{2}+\frac{5}{34}\sqrt{17})$
F_3	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{5}{26}\sqrt{13})$	0
F_4	$\Gamma M(0,-1,\frac{2}{5}\sqrt{5})$	$\Gamma M(0,-1,\frac{2}{13}\sqrt{13})$	$\Gamma M(0,0,\frac{-3}{2}+\frac{13}{34}\sqrt{17})$
F_5	$\Gamma M(0,0,\frac{1}{2}+\frac{1}{2}\sqrt{5})$	0	0
F_6	$\Gamma M(1,-1,\frac{-1}{2}+\frac{7}{10}\sqrt{5})$	0	0
F_7	0	0	0
F_8	0	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{7}{34}\sqrt{17})$
F_9	$\Gamma M(0,1,\frac{3}{5}\sqrt{5})$	$\Gamma M(0,0,\frac{-1}{2}+\frac{7}{26}\sqrt{13})$	$\Gamma M(0,1,\frac{3}{17}\sqrt{17})$
F_{10}	$\Gamma M(1,1,\frac{1}{2}+\frac{1}{2}\sqrt{5})$	$\Gamma M(-1,-1,\frac{1}{2}+\frac{1}{26}\sqrt{13})$	0
F_{11}	$\Gamma M(0,0,\frac{-1}{2}+\frac{7}{10}\sqrt{5})$	0	0
F_{12}	0	$\Gamma M(0,-1,-1+\frac{5}{13}\sqrt{13})$	0
F_{13}	0	$\Gamma M(0,0,\frac{3}{2}+\frac{1}{2}\sqrt{13})$	$\Gamma M(0,0,-2+\frac{9}{17}\sqrt{17})$
F_{14}	$\Gamma M(1,-1,\frac{1}{2}+\frac{9}{10}\sqrt{5})$	$\Gamma M(1,-1,\frac{-1}{2}+\frac{11}{26}\sqrt{13})$	0
F_{15}	$\Gamma M(1,-1,1+1\sqrt{5})$	0	$\Gamma M(1,-1,1+\frac{7}{17}\sqrt{17})$
F_{16}	$\Gamma M(0,-1,\frac{4}{5}\sqrt{5})$	$\Gamma M(0,-1,\frac{4}{13}\sqrt{13})$	$\Gamma M(0,0,\frac{-1}{2}+\frac{9}{34}\sqrt{17})$
F_{17}	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{9}{26}\sqrt{13})$	$\Gamma M(0,0,4+\sqrt{17})$
F_{18}	0	0	$\Gamma M(0,1,\frac{-3}{2}+\frac{15}{34}\sqrt{17})$
F_{19}	$\Gamma M(0,0,\frac{1}{2}+\frac{9}{10}\sqrt{5})$	0	$\Gamma M(0,0,-1+\frac{6}{17}\sqrt{17})$
F_{20}	$\Gamma M(0,-1,1+1\sqrt{5})$	0	0
F_{21}	$\Gamma M(1,1,\frac{4}{5}\sqrt{5})$	0	$\Gamma M(1,-1,\frac{-1}{2}+\frac{13}{34}\sqrt{17})$
F_{22}	0	$\Gamma M(1,1,\frac{1}{2}+\frac{7}{26}\sqrt{13})$	0
F_{23}	0	$\Gamma M(0,0,-1+\frac{6}{13}\sqrt{13})$	0
F_{24}	$\Gamma M(1,-1,\frac{-1}{2}+\frac{11}{10}\sqrt{5})$	0	0

Table A.4: F_N for $p=5,13,17$ and $N\leq 24$.

p	5	13	17
T_1	$\Gamma M(0,0,\frac{1}{5}\sqrt{5})$	$\Gamma M(0,0,\frac{1}{13}\sqrt{13})$	$\Gamma M(0,0,\frac{1}{17}\sqrt{17})$
T_2	0	0	$\Gamma M(0,0,\frac{1}{2}+\frac{5}{34}\sqrt{17})$
T_3	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{5}{26}\sqrt{13})$	0
T_4	$F_1+\Gamma M(0,-1,\frac{2}{5}\sqrt{5})$	$F_1+\Gamma M(0,-1,\frac{2}{13}\sqrt{13})$	$F_1+\Gamma M(0,0,\frac{-3}{2}+\frac{13}{34}\sqrt{17})$
T_5	$\Gamma M(0,0,\frac{1}{2}+\frac{1}{2}\sqrt{5})$	0	0
T_6	$\Gamma M(1,-1,\frac{-1}{2}+\frac{7}{10}\sqrt{5})$	0	0
T_7	0	0	0
T_8	0	0	$F_2+\Gamma M(0,0,\frac{-1}{2}+\frac{7}{34}\sqrt{17})$
T_9	$F_1+\Gamma M(0,1,\frac{3}{5}\sqrt{5})$	$F_1+\Gamma M(0,0,\frac{-1}{2}+\frac{7}{26}\sqrt{13})$	$F_1+\Gamma M(0,1,\frac{3}{17}\sqrt{17})$
T_{10}	$\Gamma M(1,1,\frac{1}{2}+\frac{1}{2}\sqrt{5})$	$\Gamma M(-1,-1,\frac{1}{2}+\frac{1}{26}\sqrt{13})$	0
T_{11}	$\Gamma M(0,0,\frac{-1}{2}+\frac{7}{10}\sqrt{5})$	0	0
T_{12}	0	$F_3+\Gamma M(0,-1,-1+\frac{5}{13}\sqrt{13})$	0
T_{13}	0	$\Gamma M(0,0,\frac{3}{2}+\frac{1}{2}\sqrt{13})$	$\Gamma M(0,0,-2+\frac{9}{17}\sqrt{17})$
T_{14}	$\Gamma M(1,-1,\frac{1}{2}+\frac{9}{10}\sqrt{5})$	$\Gamma M(1,-1,\frac{-1}{2}+\frac{11}{26}\sqrt{13})$	0
T_{15}	$\Gamma M(1,-1,1+1\sqrt{5})$	0	$\Gamma M(1,-1,1+\frac{7}{17}\sqrt{17})$
T_{16}	$F_1+F_4+\Gamma M(0,-1,\frac{4}{5}\sqrt{5})$	$F_1+F_4+\Gamma M(0,-1,\frac{4}{13}\sqrt{13})$	$F_1+F_4+\Gamma M(0,0,\frac{-1}{2}+\frac{9}{34}\sqrt{17})$
T_{17}	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{9}{26}\sqrt{13})$	$\Gamma M(0,0,4+\sqrt{17})$
T_{18}	0	0	$F_2+\Gamma M(0,1,\frac{-3}{2}+\frac{15}{34}\sqrt{17})$
T_{19}	$\Gamma M(0,0,\frac{1}{2}+\frac{9}{10}\sqrt{5})$	0	$\Gamma M(0,0,-1+\frac{6}{17}\sqrt{17})$
T_{20}	$F_5+\Gamma M(0,-1,1+1\sqrt{5})$	0	0
T_{21}	$\Gamma M(1,1,\frac{4}{5}\sqrt{5})$	0	$\Gamma M(1,-1,\frac{-1}{2}+\frac{13}{34}\sqrt{17})$
T_{22}	0	$\Gamma M(1,1,\frac{1}{2}+\frac{7}{26}\sqrt{13})$	0
T_{23}	0	$\Gamma M(0,0,-1+\frac{6}{13}\sqrt{13})$	0
T_{24}	$F_6+\Gamma M(1,-1,\frac{-1}{2}+\frac{11}{10}\sqrt{5})$	0	0

Table A.5: Divisors of the Borcherds products

n	$p=5$	$p=13$	$p=17$
1	$\left\{\frac{1}{5}\sqrt{5}\right\}$	$\left\{\frac{1}{13}\sqrt{13}\right\}$	$\left\{\frac{1}{17}\sqrt{17}\right\}$
2	$\{\}$	$\{\}$	$\left\{-\frac{1}{2}+\frac{5}{34}\sqrt{17}, \frac{1}{2}+\frac{5}{34}\sqrt{17}\right\}$
3	$\{\}$	$\left\{-\frac{1}{2}+\frac{5}{26}\sqrt{13}, \frac{1}{2}+\frac{5}{26}\sqrt{13}\right\}$	$\{\}$
4	$\left\{\frac{2}{5}\sqrt{5}\right\}$	$\left\{\frac{2}{13}\sqrt{13}\right\}$	$\left\{-\frac{3}{2}+\frac{13}{34}\sqrt{17}, \frac{3}{2}+\frac{13}{34}\sqrt{17}, \frac{2}{17}\sqrt{17}\right\}$
5	$\left\{\frac{1}{2}\sqrt{5}-\frac{1}{2}\right\}$	$\{\}$	$\{\}$
8	$\{\}$	$\{\}$	$\left\{\begin{matrix}\frac{1}{2}+\frac{7}{34}\sqrt{17}, 1+\frac{5}{17}\sqrt{17}, \\ -1+\frac{5}{17}\sqrt{17}, -\frac{1}{2}+\frac{7}{34}\sqrt{17}\end{matrix}\right\}$
9	$\left\{\frac{3}{5}\sqrt{5}\right\}$	$\left\{\begin{matrix}\frac{1}{2}+\frac{7}{26}\sqrt{13}, \frac{3}{13}\sqrt{13}, \\ -\frac{1}{2}+\frac{7}{26}\sqrt{13}\end{matrix}\right\}$	$\left\{\frac{3}{17}\sqrt{17}\right\}$
10	$\{\}$	$\{\}$	$\{\}$
11	$\left\{\begin{matrix}\frac{1}{2}+\frac{7}{10}\sqrt{5}, \\ -\frac{1}{2}+\frac{7}{10}\sqrt{5}\end{matrix}\right\}$	$\{\}$	$\{\}$
12	$\{\}$	$\left\{1+\frac{5}{13}\sqrt{13}, -1+\frac{5}{13}\sqrt{13}\right\}$	$\{\}$
13	$\{\}$	$\left\{-\frac{3}{2}+\frac{1}{2}\sqrt{13}\right\}$	$\left\{-2+\frac{9}{17}\sqrt{17}, 2+\frac{9}{17}\sqrt{17}\right\}$
16	$\left\{\frac{4}{5}\sqrt{5}\right\}$	$\left\{\frac{4}{13}\sqrt{13}\right\}$	$\left\{\begin{matrix}\frac{4}{17}\sqrt{17}, 3+\frac{13}{17}\sqrt{17}, \\ \frac{1}{2}+\frac{9}{34}\sqrt{17}, -\frac{1}{2}+\frac{9}{34}\sqrt{17}, \\ -3+\frac{13}{17}\sqrt{17}\end{matrix}\right\}$
17	$\{\}$	$\left\{\frac{1}{2}+\frac{9}{26}\sqrt{13}, -\frac{1}{2}+\frac{9}{26}\sqrt{13}\right\}$	$\left\{\sqrt{17}-4\right\}$
18	$\{\}$	$\{\}$	$\left\{\frac{3}{2}+\frac{15}{34}\sqrt{17}, -\frac{3}{2}+\frac{15}{34}\sqrt{17}\right\}$
19	$\left\{\begin{matrix}\frac{1}{2}+\frac{9}{10}\sqrt{5}, \\ -\frac{1}{2}+\frac{9}{10}\sqrt{5}\end{matrix}\right\}$	$\{\}$	$\left\{-1+\frac{6}{17}\sqrt{17}, 1+\frac{6}{17}\sqrt{17}\right\}$
20	$\left\{-1+\sqrt{5}\right\}$	$\{\}$	$\{\}$
23	$\{\}$	$\left\{1+\frac{6}{13}\sqrt{13}, -1+\frac{6}{13}\sqrt{13}\right\}$	$\{\}$
24	$\{\}$	$\{\}$	$\{\}$
25	$\left\{\sqrt{5}\right\}$	$\left\{\frac{5}{13}\sqrt{13}\right\}$	$\left\{\frac{5}{17}\sqrt{17}\right\}$

Table A.6: $R(W,-n)$: For $p \in \{5,13,17\}$ and $n \in \{6,7,14,15,21,22\}$ the set $R(W,-n)$ is empty.

Ψ_k	weight	Fourier expansion	divisor
μ	other	Fourier expansion on the diagonal (if not 0)	diagonal
Ψ_1	5	$g(h-\frac{1}{h})-10g^2(h^2-\frac{1}{h^2})-g^2(h^4-\frac{1}{h^4})+O(g^3)$	F_1
1	$\overline{\Psi_1}=-\Psi_1$		
$\frac{\Psi_4}{\Psi_1}$	10	$g\left(h+\frac{1}{h}\right)+g^2\left(454+228\left(h^2+\frac{1}{h^2}\right)+\left(h^4+\frac{1}{h^4}\right)\right)$	F_4
1	$\overline{\left(\frac{\Psi_4}{\Psi_1}\right)}=\frac{\Psi_4}{\Psi_1}$	$2g+912g^2+101304g^3-632704g^4+O(g^5)$	$2E_4^2\cdot\Delta$
Ψ_4	15	$g^2(h^2-\frac{1}{h^2})+216g^3(h+h^3-\frac{1}{h}-\frac{1}{h^3})+O(g^4)$	F_1+F_4
1	$\overline{\Psi_4}=-\Psi_4$		
Ψ_5	15	$g^2-275g^3(h+\frac{1}{h})-g^3(h^5+\frac{1}{h^5})+O(g^4)$	F_5
1	$\overline{\Psi_5}=\Psi_5$	$g^2-552g^3+8640g^4+116000g^5+O(g^6)$	$E_6\cdot\Delta^2$
Ψ_6	10	$1-264g(h+\frac{1}{h})+O(g^2)$	F_6
1	$\overline{\Psi_6}=\Psi_6$	$1-528g-201168g^2+61114944g^3+O(g^4)$	$E_4^2E_6^2$
Ψ_9	35	$g^3(h^3-\frac{1}{h^3})+3555g^4(h^2+h^4-\frac{1}{h^2}-\frac{1}{h^4})+O(g^5)$	F_1+F_9
1	$\overline{\Psi_9}=-\Psi_9$		
Ψ_{10}	10	$1-3400g(h+\frac{1}{h})+O(g^2)$	F_{10}
1	$\overline{\Psi_{10}}=\Psi_{10}$	$1-6800g-3061200g^2-256574400g^3+O(g^4)$	$\frac{5^2}{3^3}E_4^2E_6^2-\frac{2\cdot7^2}{3^3}E_4^5$
Ψ_{11}	60	$-g^6+3256g^7(h+\frac{1}{h})+g^7(h^7+\frac{1}{h^7})+O(g^8)$	F_{11}
1	$\overline{\Psi_{11}}=\Psi_{11}$	$-g^6+6514g^7+O(g^8)$	
Ψ_{14}	30	$1+25704g(h+\frac{1}{h})+O(g^2)$	F_{14}
1	$\overline{\Psi_{14}}=\Psi_{14}$	$1+51408g+146187664g^2+O(g^3)$	
Ψ_{15}	20	$1-22425f(h+\frac{1}{h})+O(g^2)$	F_{15}
1	$\overline{\Psi_{15}}=\Psi_{15}$	$1-44850g-428741775g^2+O(g^3)$	

Table A.7: Borcherds products in case $p=5$ for the Weyl chamber $W(-i\overline{\varepsilon_0},i\varepsilon_0)$

	weight	μ	diagonal
Fourier expansion			
Fourier expansion on the diagonal			
E_2^H	2	1	$E_2^H(\tau,\tau) = E_4(\tau)$
$1 + 120g\left(h + \frac{1}{h}\right) + g^2\left(720 + 600\left(h^2 + \frac{1}{h^2}\right) + 120\left(h^4 + \frac{1}{h^4}\right)\right) + O\left(g^3\right)$			
$1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + O\left(q^6\right)$			
E_4^H	4	1	$E_4^H(\tau,\tau) = \left(E_4(\tau)\right)^2 = E_8(\tau)$
$1 + 240g\left(h + \frac{1}{h}\right) + g^2\left(30240 + 15600\left(h^2 + \frac{1}{h^2}\right) + 240\left(h^4 + \frac{1}{h^4}\right)\right) + O\left(g^3\right)$			
$1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + 37500480q^5 + O\left(q^6\right)$			
E_6^H	6	1	$E_6^H(\tau,\tau) = \frac{42}{67}\left(E_4(\tau)\right)^3 + \frac{25}{67}\left(E_6(\tau)^2\right.$
$1 + \frac{2520}{67}g\left(h + \frac{1}{h}\right) + g^2\left(\frac{7877520}{67} + \frac{2583000}{67}\left(h^2 + \frac{1}{h^2}\right) + \frac{2520}{67}\left(h^4 + \frac{1}{h^4}\right)\right) + O\left(g^3\right)$			
$1 + \frac{5040}{67}q + \frac{13048560}{67}q^2 + \frac{1125069120}{67}q^3 + \frac{26660859120}{67}q^4 + \frac{310192878240}{67}q^6 + O\left(q^5\right)$			

Table A.8: Eisenstein series in case $p = 5$

<table>
<tr><td>Ψ_k</td><td>weight</td><td>divisor</td><td>ρ_W</td><td>μ</td><td>other</td><td>diagonal</td></tr>
<tr><td colspan="7">Fourier expansion</td></tr>
<tr><td colspan="7">Fourier expansion on the diagonal (if not 0)</td></tr>
<tr><td>Ψ_1</td><td>1</td><td>F_1</td><td>$\frac{1}{6}+\frac{\sqrt{13}}{26}$</td><td>$\mu_{1,2}$</td><td>$\overline{\Psi_1}=-\Psi_1$</td><td>$\Psi_1(\tau,\tau)\equiv 0$</td></tr>
<tr><td colspan="7">$g^{1/3}\left(h-\frac{1}{h}\right)-g^{4/3}\left(2\left(h^2-\frac{1}{h^2}\right)+\left(h^4-\frac{1}{h^4}\right)\right)+O\left(g^{7/3}\right)$</td></tr>
<tr><td>Ψ_3</td><td>4</td><td>F_3</td><td>$\frac{5}{6}+5\frac{\sqrt{13}}{26}$</td><td>$\mu_{2,1}$</td><td></td><td>$\Psi(\tau,\tau)=(\eta(\tau))^{16}$</td></tr>
<tr><td colspan="7">$-g^{2/3}+g^{5/3}\left(-2\left(h+\frac{1}{h}\right)+9\left(h^3+\frac{1}{h^3}\right)+\left(h^5+\frac{1}{h^5}\right)\right)+g^{8/3}\left(16+O(h^2+\frac{1}{h^2})\right)+O\left(g^{11/3}\right)$</td></tr>
<tr><td colspan="7">$-q^{2/3}+16q^{5/3}-104q^{8/3}+O(q^{11/3})$</td></tr>
<tr><td>Ψ_4</td><td>3</td><td>F_4+F_1</td><td>$\frac{1}{3}+\frac{\sqrt{13}}{13}$</td><td>$\mu_{2,1}$</td><td>$\overline{\Psi_4}=-\Psi_4$</td><td>$\Psi_4(\tau,\tau)\equiv 0$</td></tr>
<tr><td colspan="7">$g^{2/3}\left(h^2-\frac{1}{h^2}\right)+g^{5/3}\left(-24\left(h-\frac{1}{h}\right)-16\left(h^3-\frac{1}{h^3}\right)+8\left(h^5-\frac{1}{h^5}\right)\right)+O\left(g^{8/3}\right)$</td></tr>
<tr><td>$\frac{\Psi_4}{2\Psi_1}$</td><td>2</td><td>F_4</td><td>$\frac{1}{6}+\frac{\sqrt{13}}{26}$</td><td>$\mu_{1,2}$</td><td>symmetric</td><td>$\frac{\Psi_4}{2\Psi_1}(\tau,\tau)=\eta^8(\tau)$</td></tr>
<tr><td colspan="7">$\frac{1}{2}g^{1/3}\left(h+\frac{1}{h}\right)+g^{4/3}\left(-26-4\left(h^2+\frac{1}{h^2}\right)+9\left(h^4+\frac{1}{h^4}\right)\right)+O\left(g^{7/3}\right)$</td></tr>
<tr><td colspan="7">$-q^{1/3}\left(-1+8q-20q^2+70q^4\right)+O\left(q^{16/3}\right)$</td></tr>
<tr><td>Ψ_{10}</td><td>4</td><td>F_{10}</td><td>0</td><td>1</td><td>$\overline{\Psi_{10}}=\Psi_{10}$</td><td>$\Psi_{10}(\tau,\tau)=(E_4(\tau))^2$</td></tr>
<tr><td colspan="7">$1+g\left(200\left(h+\frac{1}{h}\right)+40\left(h^3+\frac{1}{h^3}\right)\right)+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1+480q+61920q^2+1050240q^3+7926240q^4+O\left(q^5\right)$</td></tr>
<tr><td>Ψ_{13}</td><td>7</td><td>F_{13}</td><td>$\frac{1}{3}$</td><td>$\mu_{2,1}$</td><td>symmetric</td><td>$\Psi_{13}(\tau,\tau)=\eta^{16}(\tau)\cdot E_6(\tau)$</td></tr>
<tr><td colspan="7">$g^{2/3}+g^{5/3}\left(-221\left(h+\frac{1}{h}\right)-39\left(h^3+\frac{1}{h^3}\right)\right)$</td></tr>
<tr><td colspan="7">$q^{2/3}-520q^{5/3}-8464q^{8/3}+O\left(q^{11/3}\right)$</td></tr>
<tr><td>Ψ_{14}</td><td>6</td><td>F_{14}</td><td>0</td><td>1</td><td>$\overline{\Psi_{14}}=\Psi_{14}$</td><td>$\Psi_{14}(\tau,\tau)=E_6^2(\tau)$</td></tr>
<tr><td colspan="7">$1-504g\left(h+\frac{1}{h}\right)+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1-1008q+220752q^2+16519104q^3+399517776q^4+O\left(q^5\right)$</td></tr>
<tr><td colspan="7">The restriction of Ψ_{14} to the diagonal has trivial character.</td></tr>
<tr><td>Ψ_{26}</td><td>6</td><td>F_{26}</td><td>0</td><td>1</td><td>$\overline{\Psi_{26}}=\Psi_{26}$</td><td>$\Psi_{26}(\tau,\tau)=\frac{125}{27}\left(E_6(\tau)\right)^2-\frac{98}{27}\left(E_4(\tau)\right)^3$</td></tr>
<tr><td colspan="7">$1-g\left(3432\left(h+\frac{1}{h}\right)+208\left(h^3+\frac{1}{h^3}\right)\right)+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1-7280q+371280q^2+14938560q^3+408750160q^4+O\left(q^5\right)$</td></tr>
</table>

Table A.9: Borcherds products in case $p=13$ for the Weyl chamber $W(-i\overline{\varepsilon_0},i\varepsilon_0)$

<table>
<tr><td></td><td>weight</td><td>μ</td><td>diagonal</td></tr>
<tr><td colspan="4">Fourier expansion</td></tr>
<tr><td colspan="4">Fourier expansion on the diagonal</td></tr>
<tr><td>E_2^H</td><td>2</td><td>1</td><td>$E_2^H(\tau,\tau) = E_4(\tau)$</td></tr>
<tr><td colspan="4">$1 + g\left(96\left(h + \frac{1}{h}\right) + 24\left(h^3 + \frac{1}{h^3}\right)\right) + O\left(g^2\right)$</td></tr>
<tr><td colspan="4">$1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + 82560q^7 + O\left(q^8\right)$</td></tr>
<tr><td colspan="4">$4\Psi_1^4 \cdot E_2^H = \Psi_4^2 + 4\Psi_1^2\Psi_3$</td></tr>
<tr><td>E_4^H</td><td>4</td><td>1</td><td>$E_4^H(\tau,\tau) = \left(E_4(\tau)\right)^2$</td></tr>
<tr><td colspan="4">$1 + g\left(\frac{6720}{29}\left(h + \frac{1}{h}\right) + \frac{240}{29}\left(h^3 + \frac{1}{h^3}\right)\right) + O\left(g^2\right)$</td></tr>
<tr><td colspan="4">$1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + 37500480q^5 + 135480960q^6 + O\left(q^7\right)$</td></tr>
<tr><td>E_6^H</td><td>6</td><td>1</td><td>$E_6^H(\tau,\tau) = \frac{21378}{33463}E_4^3(\tau) + \frac{12085}{33463}E_6^2(\tau)$</td></tr>
<tr><td colspan="4">$1 + g\left(\frac{1598688}{33463}\left(h + \frac{1}{h}\right) + \frac{6552}{33463}g\left(h^3 + \frac{1}{h^3}\right)\right)$</td></tr>
<tr><td colspan="4">$1 + \frac{3210480}{33463}q + \frac{6500435760}{33463}q^2 + \frac{562087955520}{33463}q^3 + \frac{13314685915440}{33463}q^4 + \frac{154928487036960}{33463}q^5 + O\left(q^6\right)$</td></tr>
</table>

Table A.10: Eisenstein series in case $p = 13$

<table>
<tr><td>Ψ_k</td><td>weight</td><td>divisor</td><td>ρ_W</td><td>μ</td><td>other</td><td>diagonal</td></tr>
<tr><td colspan="7">Fourier expansion</td></tr>
<tr><td colspan="7">Fourier expansion on the diagonal (if not 0)</td></tr>
<tr><td>Ψ_1</td><td>$\frac{1}{2}$</td><td>F_1</td><td>$\frac{1}{8}+\frac{\sqrt{17}}{34}$</td><td>$\mu_{3,4}$</td><td>$\overline{\Psi_1}=-\Psi_1$</td><td>0</td></tr>
<tr><td colspan="7">$g^{1/4}\left(h-h^{-1}\right)-g^{5/4}\left(\left(h^2-\frac{1}{h^2}\right)+\left(h^4-\frac{1}{h^4}\right)\right)+g^{9/4}\left(h^9-\frac{1}{h^9}\right)+O\left(g^{13/4}\right)$</td></tr>
<tr><td>Ψ_2</td><td>$\frac{3}{2}$</td><td>F_2</td><td>$\frac{5}{8}+\frac{5\sqrt{17}}{34}$</td><td>$\mu_{3,3}=\mu_{3,4}^5$</td><td>$\overline{\Psi_2}=\Psi_2$</td><td>$\Psi_2(\tau,\tau)=-\left(\eta(\tau)\right)^6$</td></tr>
<tr><td colspan="7">$-g^{1/4}+g^{5/4}\left(-\left(h+\frac{1}{h}\right)+3\left(h^3+\frac{1}{h^3}\right)+\left(h^5+\frac{1}{h^5}\right)\right)+O\left(g^{9/4}\right)$</td></tr>
<tr><td colspan="7">$-q^{1/4}+6q^{5/4}-9q^{9/4}-10q^{13/4}+30q^{17/4}+O\left(q^{21/4}\right)$</td></tr>
<tr><td>Ψ_4</td><td>$\frac{7}{2}$</td><td>F_1+F_4</td><td>$\frac{15}{8}+\frac{15\sqrt{17}}{34}$</td><td>$\mu_{3,5}=\mu_{3,4}^7$</td><td>$\overline{\Psi_4}=-\Psi_4$</td><td>0</td></tr>
<tr><td colspan="7">$-g^{3/4}\left(h^2-\frac{1}{h^2}\right)+g^{7/4}\left(13\left(h+\frac{1}{h}\right)+11\left(h^3-\frac{1}{h^3}\right)-2\left(h^5-\frac{1}{h^5}\right)\right)+O\left(g^{11/4}\right)$</td></tr>
<tr><td>$\frac{\Psi_4}{\Psi_1}$</td><td>3</td><td>F_4</td><td>$\frac{7}{4}+\frac{7\sqrt{17}}{17}$</td><td>$\mu_{2,3}=\mu_{3,4}^6$</td><td>$\overline{\frac{\Psi_4}{\Psi_1}}=\frac{\Psi_4}{\Psi_1}$</td><td>$\frac{\Psi_4}{\Psi_1}=2\left(\eta(\tau)\right)^{12}$</td></tr>
<tr><td>Ψ_8</td><td>$\frac{15}{2}$</td><td>F_2+F_8</td><td>$\frac{17}{8}+\frac{\sqrt{17}}{2}$</td><td>$\mu_{3,4}$</td><td>$\overline{\Psi_8}=\Psi_8$</td><td>$\Psi_8(\tau,\tau)=\left(\eta(\tau)\right)^{30}=\Delta(\tau)\cdot\left(\eta(\tau)\right)^6$</td></tr>
<tr><td colspan="7">$g^{5/4}+g^{9/4}\left(10\left(h+\frac{1}{h}\right)-24\left(h^3+\frac{1}{h^3}\right)-\left(h^7+\frac{1}{h^7}\right)\right)+O\left(g^{13/4}\right)$</td></tr>
<tr><td>Ψ_9</td><td>$\frac{7}{2}$</td><td>F_1+F_9</td><td>$\frac{3}{8}+\frac{3\sqrt{17}}{34}$</td><td>$\mu_{3,6}=\mu_{3,4}^3$</td><td>$\overline{\Psi_9}=-\Psi_9$</td><td>0</td></tr>
<tr><td colspan="7">$g^{3/4}\left(h^3-h^{-3}\right)+g^{7/4}\left(-36\left(h^2-\frac{1}{h^2}\right)-36\left(h^4-\frac{1}{h^4}\right)+27\left(h^6-\frac{1}{h^6}\right)\right)+O\left(g^{7/4}\right)$</td></tr>
<tr><td>$\frac{\Psi_9}{\Psi_1}$</td><td>3</td><td>F_9</td><td>$\frac{1}{4}+\frac{\sqrt{17}}{17}$</td><td>$\mu_{2,2}=\mu_{3,4}^2$</td><td>$\overline{\frac{\Psi_9}{\Psi_1}}=\frac{\Psi_9}{\Psi_1}$</td><td>$\frac{\Psi_9}{\Psi_1}(\tau,\tau)=3\cdot\left(\eta(\tau)\right)^{12}$</td></tr>
<tr><td colspan="7">$g^{1/2}\left(h^2+1+\frac{1}{h^2}\right)g^{3/2}\left(-40\left(h+\frac{1}{h}\right)-6\left(h^3+\frac{1}{h^3}\right)+28\left(h^5+\frac{1}{h^5}\right)\right)+O\left(g^{5/2}\right)$</td></tr>
<tr><td>Ψ_{13}</td><td>7</td><td>F_{13}</td><td>$\frac{9}{4}+\frac{9\sqrt{17}}{17}$</td><td>$\mu_{2,2}=\mu_{3,4}^2$</td><td>$\overline{\Psi_{13}}=\Psi_{13}$</td><td>$\Psi_{13}(\tau,\tau)=-E_4(\tau)^2\cdot\left(\eta(\tau)\right)^{12}$</td></tr>
<tr><td colspan="7"></td></tr>
<tr><td>Ψ_{15}</td><td>4</td><td>F_{15}</td><td>0</td><td>1</td><td>$\overline{\Psi_{15}}=\Psi_{15}$</td><td>$\Psi_{15}(\tau,\tau)=E_4^2(\tau)=E_8(\tau)$</td></tr>
<tr><td colspan="7">$1+240g\left(h+\frac{1}{h}\right)+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1+480q+61920q^2+1050240q^3+7926240q^4+O\left(q^5\right)$</td></tr>
<tr><td>Ψ_{17}</td><td>$\frac{9}{2}$</td><td>F_{17}</td><td>$\frac{1}{8}$</td><td>$\mu_{3,3}=\mu_{3,4}^5$</td><td>$\overline{\Psi_{17}}=\Psi_{17}$</td><td>$\Psi_{17}(\tau,\tau)=\left(\eta(\tau)\right)^6\cdot E_6(\tau)$</td></tr>
<tr><td colspan="7">$g^{1/4}-g^{5/4}\left(204\left(h+\frac{1}{h}\right)+51\left(h^3+\frac{1}{h^3}\right)\right)+O\left(g^{9/4}\right)$</td></tr>
<tr><td colspan="7">$q^{1/4}-510q^{5/4}-13599q^{9/4}-27710q^{13/4}+50370q^{17/4}+360194q^{21/4}-19479432q^{25/4}$</td></tr>
<tr><td>Ψ_{21}</td><td>6</td><td>F_{21}</td><td>0</td><td>1</td><td>$\overline{\Psi_{21}}=\Psi_{21}$</td><td>$\Psi_{21}=E_6^2(\tau)$</td></tr>
<tr><td colspan="7">$1-630g(h+h^{-1})+126g(h^3+h^{-3})+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1-1008q+220752q^2+16519104q^3+399517776q^4+O\left(q^5\right)$</td></tr>
</table>

Table A.11: Borcherds products in case $p=17$ for the Weyl chamber $W(-i\overline{\varepsilon_0},i\varepsilon_0)$

	weight	μ	diagonal
Fourier expansion			
Fourier expansion on the diagonal			
E_2^H	2	1	$E_2^H(\tau,\tau) = E_4(\tau)$
$1 + g\left(84\left(h + \frac{1}{h}\right) + 36\left(h^3 + \frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 66312q^6 + 82560q^7 + O\left(q^8\right)$			
$3\Psi_1^3 \cdot E_2^H = 3\Psi_1 \cdot \Psi_2^2 - \Psi_9$			
E_4^H	4	1	$E_4^H(\tau,\tau) = \left(E_4(\tau)\right)^2 = E_8(\tau)$
$1 + g\left(\frac{8760}{41}\left(h + \frac{1}{h}\right) + \frac{1080}{41}\left(h^3 + \frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + O\left(q^5\right)$			
E_6^H	6	1	$E_6^H(\tau,\tau) = \frac{3696}{5791}E_4^3(\tau) + \frac{2095}{5791}E_6^2(\tau)$
$1 + g\left(\frac{266364}{5791}\left(h + \frac{1}{h}\right) + \frac{8316}{5791}\left(h^3 + \frac{1}{h^3}\right)\right)$			
$1 + \frac{549360}{5791}q + \frac{1125094320}{5791}q^2 + \frac{97271576640}{5791}q^3 + \frac{2304206236080}{5791}q^4 + O\left(q^5\right)$			
$\frac{5791}{2095}\left(E_6^H - \frac{3696}{5791}\left(E_2^H\right)^3\right)$	6	1	$\frac{5791}{2095}\left(E_6^H - \frac{3696}{5791}\left(E_2^H\right)^3\right)(\tau,\tau) = E_6^2(\tau)$
$1 - g\left(\frac{665028}{2095}\left(h + \frac{1}{h}\right) + \frac{390852}{2095}\left(h^3 + \frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 - 1008q + 220752q^2 + 16519104q^3 + 399517776q^4 + 4624512480q^5 + O\left(q^6\right)$			
$\frac{41}{2^4 \cdot 3^2}\left(E_4^H - \left(E_2^H\right)^2\right)$	4	1	0
$13g\left(h + \frac{1}{h} - h^3 - \frac{1}{h^3}\right) + g^2\left(-784 + 349\left(h^2 + \frac{1}{h^2}\right) + 14\left(h^4 + \frac{1}{h^4}\right) + O\left(h^6 + \frac{1}{h^6}\right)\right)$			

Table A.12: Eisenstein series in case $p = 17$

Bibliography

[Ao06] Aoki, H., *Estimate of the dimensions of Hilbert modular forms by means of differential operators*, Automorphic forms and zeta functions, 20–28, World Sci. Publ., Hackensack, NJ, (2006).

[AI05] Aoki, H., Ibukiyama, T., *Simple graded rings of Siegel modular forms, differential operators and Borcherds products*, Int. J. Math. **16-3**, 249-279 (2005).

[Ap90] Apostol, T. M., *Modular Functions and Dirichlet series in number theory*, 2. Ed., Springer, Berlin Heidelberg, New York (1990).

[Be58] Behnke, H., *Otto Blumenthal zum Gedächtnis*, Mathematische Annalen **136**, 387-392 (1958).

[Bl03] Blumenthal, O., *Über Modulfunktionen von mehreren Veränderlichen (Erste Hälfte)*, Math. Ann. **56**, 509-548 (1903).

[Bl04a] Blumenthal, O., *Über Modulfunktionen von mehreren Veränderlichen (Zweite Hälfte)*, Math. Ann. **58**, 497-527 (1904).

[Bl04b] Blumenthal, O., *Über Thetafunktionen und Modulfunktionen mehrerer Veränderlicher*, Jahresber. Deutsche Math. Ver. **13**, 120-132 (1904).

[Bl18] Blumenthal, O., *Berechnung eines einstieligen Doppeldeckers mit Berücksichtigung der Kabelverspannungen*, Technische Berichte herausgegeben von der Flugzeugmeisterei der Inspektion der Luftschiffertruppen 3, 152-169 (1918).

[Bl22] Blumenthal, O., *David Hilbert*, Naturwissenschaften **10**, 67-72 (1922).

[Bo98] Borcherds, R. E., *Automorphic Forms with singularities on Grassmannians*, Invent. Math. **132**, 491-562 (1998).

[Bo99] Borcherds, R. E., *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97**, 219-233 (1999).

[Br98] Bruinier, J. H., *Borcherdsprodukte und Chernsche Klassen von Hirzebruch-Zagier-zykeln*, Dissertation, Heidelberg (1998).
`http://www.mi.uni-koeln.de/~bruinier/diss.ps`

[Br02] Bruinier, J. H., *Borcherds products on O(2,l) and Chern classes of Heegner divisors*, Springer Lecture Notes in Mathematics **1780**, Springer-Verlag (2002), revised Version of his Habilitationsschrift (Universität Heidelberg, May 2000).
`http://www.mi.uni-koeln.de/~bruinier/chern.ps`

[Br04] Bruinier, J. H., *Infinite products in number theory and geometry*, Jahresber. Dtsch. Math. Ver. **106**, Heft 4, 151-184 (2004).

[BB03] Bruinier, J. H., Bundschuh, M., *On Borcherds products associated with lattices of prime discriminant*, Ramanujan J., **7**, 49-61, (2003).
`http://www.mi.uni-koeln.de/~bruinier/prim.ps`

[BK01] Bruinier, J. H., Kuss, M., *Eisenstein series attached to lattices and modular forms on orthogonal groups*, Manuscr. Math. **106**, 443-459 (2001).

[Bu97] Bump, D., *Automorphic Forms and Representation*, Cambridge Studies in Advanced Mathematics **55**, Cambridge university press (1997).

[BV06] Butzer, P., Volkmann, L., *Otto Blumenthal (1876-1944) in retrospect*, J. Approximation Theory **138**, No. 1, 1-36 (2006).

[BDJ] Buzzard, K., Diamond, F., Jarvis, A. F., *On Serre's conjecture for mod l Galois representations over totally real fields*, preprint.
`http://www.unet.brandeis.edu/~fdiamond/bdj12.pdf`

[Ch85] 近岡 宣吉, ある 実２次体上 の Hilbert modular 形式 の なす graded ring について, 東北 大学 修士 論文, 67 ページ, 1985 年.
Chikaoka, *On certain graded rings of Hilbert modular forms over real quadratic fields*, Thesis at Touhoku University, p. 67 (1985).

[Da00] Darmon, H., *Rigid local systems, Hilbert modular forms, and Fermat's last theorem*, Duke Math. J. **102**, no. 3, 413-449 (2000).

[De06] Dembele, L., Computing Hilbert modular forms over real quadratic fi elds, draft (2006).
`http://modular.math.washington.edu/msri06/lassina/MSRI_Notes.pdf`

[El06] Elstrodt, J., *A very simple proof of the eta transformation formula*, manuscripta math. **121**, 457-459 (2006).

[Fe03] Felsch, V., *Der Aachener Mathematikprofessor Otto Blumenthal*, talk at the Volkshochschule Aachen (2003).
`http://www.math.rwth-aachen.de/~Blumenthal/Vortrag/vortrag.pdf`

[Fra78] Franke, H.-G., *Kurven in Hilbertschen Modulflächen und Humbertschen Flächen im Siegelraum*, Bonner Math. Schriften **104** (1978).

[Fr90] Freitag, E., *Hilbert Modular Forms*, Springer, Berlin, Heidelberg (1990).

[Fr01] Freitag, E., *Modular embeddings of Hilbert modular surfaces* (2001). `http://www.rzuser.uni-heidelberg.de/~t91/skripten/preprints/minimal.ps`

[Ge78] van der Geer, G., *Hilbert Modular forms for the field* $\mathbb{Q}(\sqrt{6})$, Math. Ann. **233**, 163-179 (1978).

[Ge88] van der Geer, G., *Hilbert Modular Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 16, Springer, Berlin, Heidelberg, New York (1988).

[GZ77] van der Geer, G., Zagier, D., *The Hilbert Modular group for the field* $\mathbb{Q}(\sqrt{13})$, Invent. Math. **42**, 93-133 (1977).

[GN98] Gritsenko, V., Nikulin, V., *Automorphic Forms and Lorentzian Kac-Moody Algebras II*, Internat. J. Math. **9**, 201-275 (1998).

[Gu63] Gundlach, K.-B., *Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers* $\mathbb{Q}(\sqrt{5})$, Math. Ann. **152**, 226-256 (1963).

[Gu81] Gundlach, K.-B., *Nullstellen Hilbertscher Modulformen*, Nachr. Akad. Wiss. Gött., II. Math.-Phys. Kl. 1981, 1-38 (1981).

[Gu85] Gundlach, K.-B., *Multiplier Systems for Hilbert's and Siegel's modular groups*, Glasg. Math. J. **27**, 57-80 (1985).

[Gu88] Gundlach, K.-B., *Die Berechnung von Multiplikatorsystemen zu Hilbertschen Modulgruppen*, Nachr. Akad. Wiss. Gött., II. Math.-Phys. Kl. 1988, **7, 1-28** (1988).

[Ha66a] Hammond, W. *The modular groups of Hilbert and Siegel*, Amer. J. Math. **88**, 497-516 (1966)

[Ha66b] Hammond, W. *The modular groups of Hilbert and Siegel*, Proc. Sympos. Pure Math. **9**, 358-360 (1966)

[Ha80] Hausmann, *Kurven auf Hilbertschen Modulflächen*, Bonner Math. Schriften **123**, (1980).

[He40] Hecke, E., *Analytische Arithmetik der positiv definiten quadratischen Formen*, Kgl. Danske Vid. Selskab. Math. fys. Med. XIII **12** (1940). Hecke, E., *Mathematische Werke*, Vandenhoeck & Ruprecht, 789-918 (1959).

[He81] Hermann, C.F. *Symmetrische Hilbertsche Modulformen und Modulfunktionen zu* $\mathbb{Q}(\sqrt{17})$, Math. Ann. **256**, 191-197 (1981).

[He83] Hermann, C.F. *Thetafunktionen und symmetrische Hilbertsche Modulformen zu* $\mathbb{Q}(\sqrt{65})$, J. Reine Angew. Math. **339**, 147-162 (1983).

[Hi76] Hirzebruch, F., *Hilbert's modular group of the field* $\mathbb{Q}(\sqrt{5})$ *and the cubic diagonal surface of Clebsch and Klein*, Russ. Math. Surveys **31**, 96-110 (1976).

[KK98] Koecher, M., Krieg, A., *Elliptische Funktionen und Modulformen*, Springer Verlag Berlin Heidelberg (1998).

[Le96] Leutbecher, A. *Zahlentheorie*, Springer, Berlin, Heidelberg, New York (1996).

[Le78] Leutbecher, A. *Euklidischer Algorithmus und die Gruppe* GL_2, Math. Ann. **231**, 269-285 (1978).

[Li81] Liehl, B. *On the group* SL_2 *over orders of arithmetic type*, J. Reine Angew. Math. **323**, 153-171 (1981).

[Li01] Livné, R. *Communication networks and Hilbert modular forms*, in: Ciliberto, C. (ed.) et al., Applications of algebraic geometry to coding theory, physics and computation, Proc. of the NATO advanced research workshop, Eilat, Israel (2001).
NATO Sci. Ser II, Math. Phys. Chem. **36**, 255-270 (2001).

[MWS] Maple 7.0. Waterloo Maple Inc., Waterloo, Ontario.

[Mi89] Miyake, T., *Modular Forms*, Springer, Berlin Heidelberg (1989).

[Mü83] Müller, R. *Hilbertsche Modulformen und Modulfunktionen zu* $\mathbb{Q}(\sqrt{8})$, Math. Ann. **266**, 83-103 (1983).

[Mü85] Müller, R. *Hilbertsche Modulformen und Modulfunktionen zu* $\mathbb{Q}(\sqrt{5})$, Arch. Math. **45**, 239-251 (1985).

[Na82] Nagaoka, S. *On Hilbert modular forms. II.* Proc. Japan Acad., Ser. **A 58**, 44-46 (1982).

[Na83a] Nagaoka, S. *On Hilbert modular forms. III.* Proc. Japan Acad., Ser. **A 59**, 346-348 (1983).

[Na83b] Nagaoka, S. *On the ring of Hilbert modular forms over* $\mathbb{Z}$ J. Math. Soc. Japan., **35**, no. 4, 589-608 (1983).

[Na86] Nagaoka, S. *On Hilbert modular forms with integral Fourier coefficients* Abh. Math. Sem. Univ. Hamburg **56**, 157-168 (1986).

[Og73] Ogg, A., *Survey on modular functions of one variable*, in: Modular functions of one variable I, Lecture notes in Mathematics, Springer- Verlag **320**, 1-36 (1973).

[Ra29] Rademacher, H., *Über die Erzeugenden von Kongruenzuntergruppen der Modulgruppe*, Abh. Math. Sem. Hamb. Univ. **7**, 134-146 (1929).

[Re56] Remmert, R., *Meromorphe Funktionen in kompakten komplexen Räumen*, Math. Ann. **132**, 277-288 (1956).

[Re74] Resnikoff, H.L. *On the graded ring of Hilbert modular forms associated with* $\mathbb{Q}(\sqrt{5})$, Math. Ann. **208**, 161-170 (1974).

[Sh71] Shimura, G. *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten, Tokyo and Priceton University Press, Princeton (1971).

[Si69] Siegel, C.L., *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen, **II**. Math.-Phys. Kl., 87-102 (1969).

[Ste04] Stein, W., *The Modular Forms Database*, `http://modular.ucsd.edu/Tables` (2004).

[Ta89] Taylor, R., *On Galois representations associated to Hilbert modular forms*, Invent. Math. **98(2)**, 265-280 (1989).

[Th06] Thieler, K., *„[...] des Tragens eines deutschen akademischen Grades unwürdig.“ Die Entziehung von Doktortiteln an der Georg-August-Universität Göttingen im „Dritten Reich“*, second edition, Göttinger Biliotheksschriften **32**, (2006).

[Th54] Thimm, W., *Meromorphe Abbildungen von Riemannschen Bereichen*, Math. Zeitschr. **60**, 435-457 (1954).

[TV83] Thomas, E., Vasquez, A.T., *Rings of Hilbert modular forms*, Compos. Math. **48**, 139-165 (1983).

[Va72] Vaserstein, L.N., *On the group* SL_2 *over Dedekind rings of arithmetic type*, Math. USSR Sbornik **18**, no. 2, 321-332 (1972) [from: Mat. Sborn. **89 (131)**, No. 2, 313-330 (1972)].

[Za81] Zagier, D. B., *Zetafunktionen und quadratische Körper*, Springer, Berlin, Heidelberg, New York (1981).

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