Normal form computations: A brief synopsis

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1 Introduction

The authors recently proposed a strategy to compute the Poncaré-Dulac normal form of a vector field f at the critical point 0, as well as a reduced vector field, even if the linearization $B = Df(0)$ is not in any particular canonical form (e.g. Jordan form), see [2] and [1]. This is of some relevance for practical computations, since normal forms depend in a subtle manner on the eigenvalues of the linearization, whence numerical approximations are of limited value. Moreover, the vector field may depend on parameters.

In [2] it was shown how a normal form up to degree m can be computed without explicit knowledge of the eigenvalues, provided that, for each degree $r \leq m$, a polynomial is given which annihilates the adjoint action of B (or of the semisimple part B_s) on each space $\mathcal{P}_r$ of homogeneous vector polynomials of degree r.

In [1] we outline a general systematic approach to computing the Poincaré-Dulac normal form, and determining a reduced vector field via invariants. Only rational operations are required for all computations, and only the minimum polynomial of B_s acting on the underlying vector space is needed. Only standard computer algebra software (MAPLE) was used for computations.

The present synopsis is intended for those readers who are mainly interested in using the programs contained in this web site. Only a rough outline of the essential steps is given.

2 Normal forms and invariants at a fixed degree

Let $\mathbf{K}$ denote the field of real or complex numbers. Consider the vector field

$$f(x) = Bx + f_2(x) + f_3(x) + \ldots$$

over $\mathbf{K}$ (with B linear and each f_j homogeneous of degree j, thus $f(0) = 0$) with f sufficiently differentiable. Moreover let $B = B_s + B_n$ be the decomposition into semisimple and nilpotent part.

Let us briefly recall the setting and the central result of [2] with regard to the computation of normal forms. If a vector field is already normalized up to degree $r - 1$ then for further normalization the equation

$$[B, h_r] = f_r - g_r$$

has to be solved in $\mathcal{P}_r$. Here h_r is the first nontrivial term in a near-identity normalizing transformation, and g_r is the remaining degree r term in a normalized vector field. (The Lie bracket is given, as usual, by $[p,q]\,(x) = Dq(x)p(x) - Dp(x)q(x)$.) In [2] it was shown that the essential problem lies in solving

$$[B_s, h_r] = f_r - g_r \tag{$*$}$$

for the semisimple part B_s.

Proposition 1. *Let*

$$p(\tau) = \tau^m + \sum_{j=0}^{m} \alpha_i \tau^{m-i}$$

be a polynomial that annihilates the linear map $\operatorname{ad} B_s$ *on* $\mathcal{P}_r$*, with the additional property that* $\alpha_{m-1} \neq 0$ *if* $\alpha_m = 0$*. (Such a polynomial exists due to the semisimplicity of* $\operatorname{ad} B_s$*.)*
In case $\alpha_m \neq 0$ *equation* $(*)$ *is solved by* $g_r = 0$ *and*

$$h_r = -\frac{1}{\alpha_m}\left((\operatorname{ad} B_s)^{m-1} f_r + \alpha_1 (\operatorname{ad} B_s)^{m-2} f_r + \ldots + \alpha_{m-1} f_r\right).$$

In case $\alpha_m = 0$ *a solution is given by*

$$h_r = -\frac{1}{\alpha_{m-1}}\left((\operatorname{ad} B_s)^{m-2} f_r + \alpha_1 (\operatorname{ad} B_s)^{m-3} f_r + \ldots + \alpha_{m-2} f_r\right),$$

$$g_r = f_r - [B_s,\, h_r]\,.$$

We refer to [2], Prop. 6.1, for the proof. The fundamental ingredient from linear algebra is a decomposition of the identity map into projections onto kernel and image of $\operatorname{ad} B_s$.

The same basic approach is used in [1] for the computation of polynomial invariants of B_s. Denote by S_r the space of homogeneous scalar-valued polynomials of degree r. Then B_s acts on S_r via the Lie derivative

$$L_{B_s}(\phi)(x) = D\phi(x) B_s x,$$

and the homogeneous invariants of degree r are just the functions annihilated by L_{B_s}. For ease of notation we will assume $B = B_s$ in the following.

For fixed r let

$$q(\tau) = \tau^\ell + \sum_{i=1}^{\ell} \beta_i \tau^{\ell-i}$$

be a polynomial annihilating L_B on S_r. Since B is semisimple, the action of L_B on S_r is semisimple, whence we can assume that $\beta_{\ell-1} \neq 0$ in case $\beta_\ell = 0$. The following result allows a systematic computation of the invariants of B in S_r. Nontrivial invariants exist only in case $\beta_\ell = 0$.

Proposition 2. *Let $\beta_\ell = 0$, and $q(\tau) = \tau \cdot q^*(\tau)$. Moreover, define*

$$\begin{aligned} s_1(\tau) &= \left(-\frac{1}{\beta_{\ell-1}}\right)\left(\tau^{\ell-2} + \sum_{i=1}^{\ell-2} \beta_i \tau^{\ell-i-2}\right) \\ s_2(\tau) &= \frac{1}{\beta_{\ell-1}} \end{aligned}$$

such that

$$s_1(\tau) \cdot \tau + s_2(\tau) \cdot q^*(\tau) = 1.$$

Then

$$s_2(L_B) q^*(L_B) : \ S_r \to S_r$$

is a projection onto the subspace of invariants in S_r. Thus, all homogeneous invariants of degree r can be obtained by applying this map to a set of generators of the vector space S_r.

3 Further ingredients

• After normalizing the vector field at degree r, one also has to transform the terms of degree $> r$, up to the desired truncation order. The easiest way to do this is to apply $\exp(\mathrm{ad}\, h_r)$ to f; see [2].

• Obtaining an annihilating polynomial for $\mathrm{ad}(B_s)$ on $\mathcal{P}_r$, respectively, of L_{B_s} on S_r is a matter of (relatively straightforward) computer algebra, if the minimum polynomial of B_s is given; see [1].

• The truncated vector field in normal form admits nontrivial symmetries, and thus allows a canonical reduction procedure via invariants. The problem of finding invariants is taken care of by Proposition 2.

• Computing a reduced vector field, on the other hand, seems to be a tricky matter, which has to be addressed on a case-by-case basis. See the examples in [1] and on this site.

References

[1] S. Mayer, J. Scheurle, S. Walcher: *Practical normal form computations for vector fields.* Preprint, 11 pp. (2002).

[2] J. Scheurle, S. Walcher: *On normal form computations.* In: *Geometry, Dynamics and Mechanics* (P.K. Newton, P. Holmes, A. Weinstein, Eds.), pp. 309 - 325. Springer-Verlag, New York (2002).