

Content

This is an example of a MAPLE 7 worksheet for normalization, finding invariants and reduction of the following system:

$Dx = Mx + f_2(x) + f_3(x) + \text{higher order terms}$, where:

$$M := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & 0 & 0 \\ -\alpha_2 & -\alpha_3 & 0 & 0 \end{bmatrix} \quad f_2 := [0, 0, \beta x_1^2, 5 \beta x_2 x_3] \quad f_3 := [0, 0, 0, 0]$$

Disclaimer

While our testing, as well as computations of examples, indicate correctness and reliability of the programs, the authors cannot guarantee the correctness of any routine.

Program developed by S. Mayer (email: mayer@mathA.rwth-aachen.de)

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The programs may be used for any non-commercial purpose by individuals and scientific organizations.

Initialization

```
with(linalg):  
Warning, the protected names norm and trace have been redefined and  
unprotected
```

The Message: "Warning, the protected names norm and trace have been redefined and unprotected" will not inflict the following calculations. It derives from using the linalg package instead of the LinearAlgebra package, for the "linalg package is useful for doing computations in abstract linear algebra" (Maple glossary). This allows parameter dependent calculations.

Primitive operations

Procedures

Input

The vector field v is given in the form $v(x)=Mx+f_2[x]+...$ where M is an $n \times n$ matrix and the f_j are homogeneous polynomials $C^n \rightarrow C^n$ of degree j . The following routines are capable of working with parameters, so you might want to apply them for parameter dependent vector fields. Note that maple assumes non-degeneracies of all parameters.

Enter the linearization M of the vector field. The following calculations assume that M is semisimple.

```
> M:=matrix(4,4,[0,0,1,0,0,0,0,1,-alpha[1],-alpha[2],0,0,-alpha  
[2],-alpha[3],0,0]);
```

$$M := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & 0 & 0 \\ -\alpha_2 & -\alpha_3 & 0 & 0 \end{bmatrix}$$

Let targetspace be the desired degree of the normal form. targetspace then is also the degree of the Taylor expansion of the vector field.

```
> targetspace:=3;
```

```
targetspace := 3
```

Just evaluate the following lines.

```
> f:=vector(targetspace,0);
```

```
> dim:=rowdim(M);
```

```
dim := 4
```

```
> for i from 1 to targetspace do
```

```
>   f[i]:=vector(dim,0);
```

```
> od:
```

```
> f[1]:=Matrix2Polynom(M);
```

$$f_1 := [x_3, x_4, -\alpha_1 x_1 - \alpha_2 x_2, -\alpha_2 x_1 - \alpha_3 x_2]$$

Enter the higher order terms of the Taylor expansion of the vector field: f[2]:=vector(...);

```
f[3]:=... f[targetspace]:=vector(...);
```

```
> f[2]:=vector([0,0,beta*x[1]^2,beta*5*x[2]*x[3]]);f[3]:=vector([0,0,0,0]);
```

$$f_2 := [0, 0, \beta x_1^2, 5 \beta x_2 x_3]$$

$$f_3 := [0, 0, 0, 0]$$

Enter the maximal degree of calculated invariant polynomials.

```
> invariantsdegree:=2;
```

```
invariantsdegree := 2
```

Automatic initialization

```
> dim:=rowdim(M);
```

```
> minimalpoly:=minpoly(M,tau);
```

$$\text{minimalpoly} := -\alpha_2^2 + \alpha_1 \alpha_3 + (\alpha_1 + \alpha_3) \tau^2 + \tau^4$$

```
> sigmaset:=symmetriceigenvalues(minimalpoly,dim);
```

Determining annihilating polynomials

Annihilating polynomials for the action of ad M on P_r: (Remember that x -> p(-x) annihilates the action of ad M on P_0 if p annihilates M.)

```
> liebracketannihilator:=AnnihilatingPolynomials(sigmaset,subs(
tau=-tau,minimalpoly),minimalpoly,dim,targetspace+1):
```

```
"Finished with step ", 1, "."
```

```
"Finished with step ", 2, "."
```

```
"Finished with step ", 3, "."
```

```
"Finished with step ", 4, "."
```

Now liebracketannihilator[j] annihilates the action of ad M on P_{j-1}.

```
> invpoly:=AnnihilatingPolynomials(sigmaset,minimalpoly,minimal
poly,dim,invariantsdegree):
```

```
"Finished with step ", 1, "."
```

```
"Finished with step ", 2, "."
```

[Now invpoly[j] annihilates the action of L_M on S_j.

Output

```

> for j from 1 to targetspace+1 do
>   print("Annihilating polynomial for ad M on homogeneous
polynomials of degree",j-1);
>   print(liebracketannihilator[j]);
> od;
"Annihilating polynomial for ad M on homogeneous polynomials of degree", 0

$$-\alpha_2^2 + \alpha_1 \alpha_3 + \tau^2 \alpha_1 + \tau^2 \alpha_3 + \tau^4$$

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 1

$$\tau(-16 \alpha_2^2 + 4 \tau^2 \alpha_1 + \tau^4 + 16 \alpha_1 \alpha_3 + 4 \tau^2 \alpha_3)$$


$$(\alpha_1^2 + 2 \tau^2 \alpha_1 + \tau^4 - 2 \alpha_1 \alpha_3 + 2 \tau^2 \alpha_3 + \alpha_3^2 + 4 \alpha_2^2)$$

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 2

$$(-\alpha_2^2 + \alpha_1 \alpha_3 + \tau^2 \alpha_1 + \tau^2 \alpha_3 + \tau^4)(16 \alpha_1^4 + 40 \tau^2 \alpha_1^3 - 136 \alpha_1^3 \alpha_3 + 321 \alpha_1^2 \alpha_3^2$$


$$+ 50 \alpha_1^2 \alpha_3 \tau^2 + 33 \tau^4 \alpha_1^2 + 200 \alpha_1^2 \alpha_2^2 - 136 \alpha_1 \alpha_3^3 + 50 \tau^2 \alpha_1 \alpha_3^2 - 850 \alpha_2^2 \alpha_1 \alpha_3$$


$$+ 52 \tau^4 \alpha_1 \alpha_3 + 70 \tau^2 \alpha_1 \alpha_2^2 + 10 \tau^6 \alpha_1 + 16 \alpha_3^4 + 40 \alpha_3^3 \tau^2 + 33 \tau^4 \alpha_3^2 + 200 \alpha_3^2 \alpha_2^2$$


$$+ 70 \alpha_2^2 \tau^2 \alpha_3 + 10 \tau^6 \alpha_3 + 14 \tau^4 \alpha_2^2 + \tau^8 + 625 \alpha_2^4)$$


$$(81 \alpha_1 \alpha_3 + 9 \tau^2 \alpha_1 - 81 \alpha_2^2 + 9 \tau^2 \alpha_3 + \tau^4)$$

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 3

$$(\tau^8 + 118 \tau^4 \alpha_3^2 - 56 \tau^4 \alpha_2^2 + 292 \tau^4 \alpha_1 \alpha_3 + 1800 \alpha_3^2 \alpha_2^2 + 1800 \alpha_1^2 \alpha_2^2$$


$$+ 1100 \tau^2 \alpha_1 \alpha_3^2 - 560 \tau^2 \alpha_1 \alpha_2^2 - 560 \alpha_2^2 \tau^2 \alpha_3 + 1100 \alpha_1^2 \alpha_3 \tau^2 + 20 \tau^6 \alpha_1$$


$$+ 6886 \alpha_1^2 \alpha_3^2 - 1476 \alpha_1 \alpha_3^3 - 1476 \alpha_1^3 \alpha_3 + 180 \tau^2 \alpha_1^3 - 16400 \alpha_2^2 \alpha_1 \alpha_3$$


$$+ 118 \tau^4 \alpha_1^2 + 180 \alpha_3^3 \tau^2 + 20 \tau^6 \alpha_3 + 81 \alpha_3^4 + 81 \alpha_1^4 + 10000 \alpha_2^4)$$


$$(256 \alpha_1 \alpha_3 + 16 \tau^2 \alpha_1 - 256 \alpha_2^2 + \tau^4 + 16 \tau^2 \alpha_3)$$


$$(16 \alpha_1^2 + 8 \tau^2 \alpha_1 + \tau^4 - 32 \alpha_1 \alpha_3 + 8 \tau^2 \alpha_3 + 16 \alpha_3^2 + 64 \alpha_2^2) \tau$$


$$(-16 \alpha_2^2 + 4 \tau^2 \alpha_1 + \tau^4 + 16 \alpha_1 \alpha_3 + 4 \tau^2 \alpha_3)$$


$$(\alpha_1^2 + 2 \tau^2 \alpha_1 + \tau^4 - 2 \alpha_1 \alpha_3 + 2 \tau^2 \alpha_3 + \alpha_3^2 + 4 \alpha_2^2)$$

> print("Annihilating polynomial for L_M on homogeneous
polynomials of degree",0);print(1);
for j from 1 to invariantsdegree do
>   print("Annihilating polynomial for L_M on homogeneous
polynomials of degree",j);
>   print(invpoly[j]);
> od;
"Annihilating polynomial for L_M on homogeneous polynomials of degree", 0
1
"Annihilating polynomial for L_M on homogeneous polynomials of degree", 1

```

$$-\alpha_2^2 + \alpha_1 \alpha_3 + \tau^2 \alpha_1 + \tau^2 \alpha_3 + \tau^4$$

"Annihilating polynomial for L_M on homogeneous polynomials of degree", 2

$$\tau(-16 \alpha_2^2 + 4 \tau^2 \alpha_1 + \tau^4 + 16 \alpha_1 \alpha_3 + 4 \tau^2 \alpha_3)$$

$$(\alpha_1^2 + 2 \tau^2 \alpha_1 + \tau^4 - 2 \alpha_1 \alpha_3 + 2 \tau^2 \alpha_3 + \alpha_3^2 + 4 \alpha_2^2)$$

Higher-order normalization

```
[ > dgl:=matrix(targetspace,targetspace): #the differential
  equation in the various transformed states
[ > Trafo:=vector(targetspace):Trafo[1]:=vector(dim,0): #the
  transformation is exp(Trafo).
[ > for i from 1 to targetspace do
  >   dgl[1,i]:=eval(f[i]);
[ > od:
[ > for i from 2 to targetspace do
  #print("starting with
  ",collect(expand(liebracketannihilator[i+1]),tau));
  >
  erg:=semisimplecalculation(M,collect(expand(liebracketannihilator[i+1]),tau),f[i]);
  #print("transformation and transformed part in normal
  form calculated for one degree higher");
  >   for j from 1 to i-1 do
  >     dgl[i,j]:=eval(dgl[i-1,j]);
  >   od;
  #print("some terms are copied (degree < r)");
  >   Trafo[i]:=eval(erg[1]);
  >   dgl[i,i]:=eval(erg[2]);
  #print("some terms are chosen as it was calculated
  before");
  >   for j from i+1 to targetspace do
  >     dgl[i,j]:=eval(dgl[i-1,j]);
  >     k:=1;
  >     while j >= k*(i-1)+1 do
  >       hlp:=scalarmul(dgl[i-1,j-k*(i-1)],1/k!);
  >       for l from 1 to k do
  >         hlp:=lie(erg[l],hlp);
  >       od;
  >       dgl[i,j]:=matadd(dgl[i,j],hlp);
  >       k:=k+1;
  >     od;
  #print("transformation of higher terms up to degree
  ",j," done in this step")
  >   od;
  >   for j from i to targetspace do
  >     f[j]:=eval(dgl[i,j]);
  >   od;
  #print("transformation succesfull up to degree ",i);
[ > od:
```

The The previous lines compute Poincare-Dulac normal form up to degree targetspace. It is stored in the variable dgl. dgl is a field of dimension targetspace^2: dgl[i,1]+dgl[i,2]+...+dgl[i,targetspace] is the Poincare-Dulac-normal form up to degree i. The

corresponding transformation is given by $\exp(\text{ad}(\text{Trafo}[i])) \exp(\text{ad}(\text{Trafo}[i-1])) \dots \exp(\text{ad}(\text{Trafo}[2]))$. An output of the results is provided in the next steps:

+ *Output of the Poincare-Dulac normal form*

+ *Output of intermediate results*

Invariants at fixed degree

```
> eval(invariantsdegree);
2
[ Choose a degree invdegree <= invariantsdegree;
> invdegree:=invariantsdegree;
invdegree := 2
```

Invariant providing map

A map from the space of homogeneous polynomials of invdegree onto the invariant polynomials of invdegree is calculated.

```
> eval(invpoly);
[-α22 + α1 α3 + τ2 α1 + τ2 α3 + τ4, τ(-16 α22 + 4 τ2 α1 + τ4 + 16 α1 α3 + 4 τ2 α3)
(α12 + 2 τ2 α1 + τ4 - 2 α1 α3 + 2 τ2 α3 + α32 + 4 α22)]
[ Note that the command InvariantsProvidingPolynomial(invpoly[deg],M); calculates the
invariant providing map for degree deg, if deg<invariantsdegree.
> invpolymap:=InvariantsProvidingPolynomial(invpoly[invdegree],M);
invpolymap := -16 α12 α22 + 96 α22 α1 α3 - 64 α24 - 16 α32 α22 + 16 α13 α3
- 32 α12 α32 + 16 α1 α33
+ τ2 (-16 α22 α3 - 16 α1 α22 + 28 α32 α1 + 28 α12 α3 + 4 α13 + 4 α33)
+ τ4 (-12 α22 + 30 α1 α3 + 9 α12 + 9 α32) + τ6 (6 α1 + 6 α3) + τ8
```

Finding the invariants

A basis of invariants of degree 'invariantsdegree' is calculated using 'invpolymap'. If you already know the dimension of the space of invariants from some theoretical argument, set 'number' to some positive value (in order to save calculation time) and thus only do 'number' tries for invariants. Another call of invariants(...,eval(thistry),...) will then try more invariants...

```
> Invarianten:={}: InvZahl:=0: firsttry:=vector(dim,0):
> firsttry[dim]:=invdegree:number:=-1:eval(firsttry);
[0,0,0,2]
> thistry, Invarianten, InvZahl:=invariants(invpolymap,eval(Invarianten),InvZahl,eval(firsttry),number,invdegree,dim,M);
thistry, Invarianten, InvZahl := [3,0,0,0], {8 x22 α1 α24 - 16 x42 α12 α32 + 8 x42 α1 α33
- 8 x42 α12 α22 + 8 x42 α13 α3 - 16 x22 α12 α33 - 8 x22 α33 α22 + 8 x22 α13 α32
+ 8 x22 α1 α34 - 16 x22 α12 α22 α3 + 16 x32 α22 α1 α3 + 16 α23 x4 x3 α1
- 16 α2 x4 x3 α12 α3 + 16 α2 x4 x3 α1 α32 - 16 α23 x4 x3 α3 - 8 α22 x42 α32
- 24 α24 α3 x22 - 32 α25 x1 x2 - 16 α24 x42 + 32 α22 x42 α1 α3 - 16 x32 α24
```

$$\begin{aligned}
& -8x_1^2\alpha_2^4\alpha_1 - 8x_1^2\alpha_2^4\alpha_3 - 16\alpha_2^3x_1x_2\alpha_3^2 + 8x_1^2\alpha_2^2\alpha_1^2\alpha_3 + 8x_1^2\alpha_2^2\alpha_1\alpha_3^2 \\
& - 16\alpha_2x_1x_2\alpha_1^2\alpha_3^2 + 16\alpha_2x_1x_2\alpha_1\alpha_3^3 + 48\alpha_2^3x_1x_2\alpha_1\alpha_3 + 40\alpha_2^2\alpha_3^2x_2^2\alpha_1, \\
& -16\alpha_2^5x_2^2 + 8\alpha_2^3x_4^2\alpha_1 + 32x_3\alpha_2^2x_4\alpha_1\alpha_3 - 8\alpha_2^3x_3^2\alpha_1 + 8\alpha_2x_3^2\alpha_1^2\alpha_3 \\
& - 8\alpha_2x_3^2\alpha_1\alpha_3^2 - 16\alpha_2^5x_1^2 + 16\alpha_2^2x_1x_2\alpha_1\alpha_3^2 + 16\alpha_2^2x_1x_2\alpha_1^2\alpha_3 - 8\alpha_2^3x_1^2\alpha_1^2\alpha_3^2 \\
& - 16\alpha_2^4x_1x_2\alpha_1 + 24\alpha_2^3x_1^2\alpha_1\alpha_3 + 8\alpha_2x_1^2\alpha_1^3\alpha_3 - 8\alpha_2x_1^2\alpha_1^2\alpha_3^2 \\
& + 24\alpha_2^3x_2^2\alpha_1\alpha_3 - 32x_3\alpha_2^4x_4 - 8x_2^2\alpha_1^2\alpha_3^2\alpha_2 + 8x_2^2\alpha_1\alpha_3^3\alpha_2 - 8x_2^2\alpha_3^2\alpha_2^3 \\
& - 8x_4^2\alpha_1^2\alpha_2\alpha_3 + 8x_4^2\alpha_1\alpha_2\alpha_3^2 + 8\alpha_2^3x_3^2\alpha_3 - 8\alpha_2^3x_4^2\alpha_3 - 16\alpha_2^4x_1\alpha_3x_2\}, 2
\end{aligned}$$

If g is an invariant, and c a constant, c*g is also an invariant. So it is 'legal' to try to simplify further calculations by eliminating parameters.

```
> Inva:=vector(InvZahl):
```

```
> for i from 1 to InvZahl do
```

```
  Inva[i]:=factor(polynomialsimplify(Invarianten[i],dim));
od;
```

$$\begin{aligned}
Inva_1 := & 4x_1x_2\alpha_2^3 + 2\alpha_2^2x_3^2 + 3\alpha_2^2x_2^2\alpha_3 + \alpha_1\alpha_2^2x_1^2 + 2\alpha_2^2x_4^2 - \alpha_2^2x_2^2\alpha_1 \\
& + \alpha_2^2x_1^2\alpha_3 - 2\alpha_2x_4x_3\alpha_1 + 2\alpha_2x_4x_3\alpha_3 + 2\alpha_2x_1x_2\alpha_3^2 - 2\alpha_2x_1x_2\alpha_1\alpha_3 \\
& + x_2^2\alpha_1^2\alpha_3 + x_4^2\alpha_1^2 + x_2^2\alpha_3^3 - 2\alpha_3^2x_2^2\alpha_1 + x_4^2\alpha_3^2 - 2x_4^2\alpha_1\alpha_3
\end{aligned}$$

$$\begin{aligned}
Inva_2 := & 2\alpha_2^2x_2^2 + 2\alpha_2^2x_1^2 + 4\alpha_2x_3x_4 + 2\alpha_2x_1x_2\alpha_3 + 2\alpha_2x_1x_2\alpha_1 + \alpha_1x_3^2 - x_4^2\alpha_1 \\
& - x_1^2\alpha_1\alpha_3 + \alpha_1^2x_1^2 - x_2^2\alpha_1\alpha_3 - x_3^2\alpha_3 + x_4^2\alpha_3 + x_2^2\alpha_3^2
\end{aligned}$$

– Reduced vector field

– Matrix from of the invariants and positive semi definit invariants

```
> A:=matrixform(4,Inva[1]);
```

$$A := \begin{bmatrix} \alpha_1\alpha_2^2 + \alpha_2^2\alpha_3, & 2\alpha_2^3 + \alpha_2\alpha_3^2 - \alpha_2\alpha_1\alpha_3, & 0, & 0 \\ 2\alpha_2^3 + \alpha_2\alpha_3^2 - \alpha_2\alpha_1\alpha_3, & 3\alpha_2^2\alpha_3 - \alpha_1\alpha_2^2 + \alpha_1^2\alpha_3 + \alpha_3^3 - 2\alpha_3^2\alpha_1, & 0, & 0 \\ 0, & 0, & 2\alpha_2^2, & -\alpha_1\alpha_2 + \alpha_2\alpha_3 \\ 0, & 0, & -\alpha_1\alpha_2 + \alpha_2\alpha_3, & 2\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3 \end{bmatrix}$$

```
> B:=matrixform(4,Inva[2]);
```

$$B := \begin{bmatrix} 2\alpha_2^2 - \alpha_1\alpha_3 + \alpha_1^2 & \alpha_2\alpha_3 + \alpha_1\alpha_2 & 0 & 0 \\ \alpha_2\alpha_3 + \alpha_1\alpha_2 & 2\alpha_2^2 - \alpha_1\alpha_3 + \alpha_3^2 & 0 & 0 \\ 0 & 0 & \alpha_1 - \alpha_3 & 2\alpha_2 \\ 0 & 0 & 2\alpha_2 & -\alpha_1 + \alpha_3 \end{bmatrix}$$

```
> if A[3,4]=0 then A1:=A; else
  A1:=simplify(matadd(B,scalarmul(A,-B[3,4]/A[3,4])));
  C:=eval(A):A:=eval(B);B:=eval(C);
fi;
```

$$A1 := \begin{bmatrix} \frac{\alpha_1(-2\alpha_1\alpha_3 + \alpha_3^2 + \alpha_1^2 + 4\alpha_2^2)}{\alpha_1 - \alpha_3}, & \frac{\alpha_2(-2\alpha_1\alpha_3 + \alpha_3^2 + \alpha_1^2 + 4\alpha_2^2)}{\alpha_1 - \alpha_3}, & 0, & 0 \\ \frac{\alpha_2(-2\alpha_1\alpha_3 + \alpha_3^2 + \alpha_1^2 + 4\alpha_2^2)}{\alpha_1 - \alpha_3}, & \frac{\alpha_3(-2\alpha_1\alpha_3 + \alpha_3^2 + \alpha_1^2 + 4\alpha_2^2)}{\alpha_1 - \alpha_3}, & 0, & 0 \\ 0, & 0, & \frac{-2\alpha_1\alpha_3 + \alpha_3^2 + \alpha_1^2 + 4\alpha_2^2}{\alpha_1 - \alpha_3}, & 0 \\ 0, & 0, & 0, & \frac{-2\alpha_1\alpha_3 + \alpha_3^2 + \alpha_1^2 + 4\alpha_2^2}{\alpha_1 - \alpha_3} \end{bmatrix}$$

$$C := \begin{bmatrix} \alpha_1\alpha_2^2 + \alpha_2^2\alpha_3, & 2\alpha_2^3 + \alpha_2\alpha_3^2 - \alpha_2\alpha_1\alpha_3, & 0, & 0 \\ 2\alpha_2^3 + \alpha_2\alpha_3^2 - \alpha_2\alpha_1\alpha_3, & 3\alpha_2^2\alpha_3 - \alpha_1\alpha_2^2 + \alpha_1^2\alpha_3 + \alpha_3^3 - 2\alpha_3^2\alpha_1, & 0, & 0 \\ 0, & 0, & 2\alpha_2^2, & -\alpha_1\alpha_2 + \alpha_2\alpha_3 \\ 0, & 0, & -\alpha_1\alpha_2 + \alpha_2\alpha_3, & 2\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3 \end{bmatrix}$$

$$A := \begin{bmatrix} 2\alpha_2^2 - \alpha_1\alpha_3 + \alpha_1^2 & \alpha_2\alpha_3 + \alpha_1\alpha_2 & 0 & 0 \\ \alpha_2\alpha_3 + \alpha_1\alpha_2 & 2\alpha_2^2 - \alpha_1\alpha_3 + \alpha_3^2 & 0 & 0 \\ 0 & 0 & \alpha_1 - \alpha_3 & 2\alpha_2 \\ 0 & 0 & 2\alpha_2 & -\alpha_1 + \alpha_3 \end{bmatrix}$$

$$B := \begin{bmatrix} \alpha_1 \alpha_2^2 + \alpha_2^2 \alpha_3, & 2 \alpha_2^3 + \alpha_2 \alpha_3^2 - \alpha_2 \alpha_1 \alpha_3, & 0, & 0 \\ 2 \alpha_2^3 + \alpha_2 \alpha_3^2 - \alpha_2 \alpha_1 \alpha_3, & 3 \alpha_2^2 \alpha_3 - \alpha_1 \alpha_2^2 + \alpha_1^2 \alpha_3 + \alpha_3^3 - 2 \alpha_3^2 \alpha_1, & 0, & 0 \\ 0, & 0, & 2 \alpha_2^2, & -\alpha_1 \alpha_2 + \alpha_2 \alpha_3 \\ 0, & 0, & -\alpha_1 \alpha_2 + \alpha_2 \alpha_3, & 2 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3 \end{bmatrix}$$

> A1:=matrix(simplify(A1)); #this produces A1new=c*A1old with c a real (komplex) polynomial in alpha[i]

$$A1 := \begin{bmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \alpha_2 & \alpha_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[> if A1[4,4]<0 then A1:= scalarmul(A1,-1);fi;

> unassign('delta');delta:=[solve((B[3,3]+delta*A1[3,3])*(B[4,4]+delta*A1[4,4])-(B[3,4]*B[4,3])=0,delta)];

$$\delta := \left[-2 \alpha_2^2 - \frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_3^2 + \alpha_1 \alpha_3 \right. \\ \left. + \frac{1}{2} \sqrt{\alpha_1^4 - 4 \alpha_1^3 \alpha_3 + 4 \alpha_1^2 \alpha_2^2 + 6 \alpha_1^2 \alpha_3^2 - 4 \alpha_1 \alpha_3^3 - 8 \alpha_2^2 \alpha_1 \alpha_3 + 4 \alpha_3^2 \alpha_2^2 + \alpha_3^4}, \right. \\ \left. -2 \alpha_2^2 - \frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_3^2 + \alpha_1 \alpha_3 \right. \\ \left. - \frac{1}{2} \sqrt{\alpha_1^4 - 4 \alpha_1^3 \alpha_3 + 4 \alpha_1^2 \alpha_2^2 + 6 \alpha_1^2 \alpha_3^2 - 4 \alpha_1 \alpha_3^3 - 8 \alpha_2^2 \alpha_1 \alpha_3 + 4 \alpha_3^2 \alpha_2^2 + \alpha_3^4} \right]$$

> A2:=simplify(matadd(B,scalarmul(A1,delta[1])));B2:=simplify(matadd(B,scalarmul(A1,delta[2])));

A2:=

$$\begin{bmatrix} -\alpha_1 \alpha_2^2 + \alpha_2^2 \alpha_3 - \frac{1}{2} \alpha_1^3 - \frac{1}{2} \alpha_3^2 \alpha_1 + \alpha_1^2 \alpha_3 \\ + \frac{1}{2} \alpha_1 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, \\ \frac{1}{2} \alpha_2 \alpha_3^2 - \frac{1}{2} \alpha_1^2 \alpha_2 + \frac{1}{2} \alpha_2 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, 0, 0 \\ \left[\frac{1}{2} \alpha_2 \alpha_3^2 - \frac{1}{2} \alpha_1^2 \alpha_2 + \frac{1}{2} \alpha_2 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, \alpha_2^2 \alpha_3 - \alpha_1 \alpha_2^2 \right. \\ \left. + \frac{1}{2} \alpha_1^2 \alpha_3 + \frac{1}{2} \alpha_3^3 - \alpha_3^2 \alpha_1 + \frac{1}{2} \alpha_3 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, 0, 0 \right] \\ \left[0, 0, -\frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_3^2 + \alpha_1 \alpha_3 + \frac{1}{2} \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, \right. \end{bmatrix}$$


```


$$\begin{bmatrix} -\alpha_1 \alpha_2 + \alpha_2 \alpha_3 \\ 0, 0, -\alpha_1 \alpha_2 + \alpha_2 \alpha_3, \\ \frac{1}{2} \alpha_1^2 + \frac{1}{2} \alpha_3^2 - \alpha_1 \alpha_3 + \frac{1}{2} \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \end{bmatrix}$$

B2 :=

$$\begin{bmatrix} -\alpha_1 \alpha_2^2 + \alpha_2^2 \alpha_3 - \frac{1}{2} \alpha_1^3 - \frac{1}{2} \alpha_3^2 \alpha_1 + \alpha_1^2 \alpha_3 \\ -\frac{1}{2} \alpha_1 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, \\ \frac{1}{2} \alpha_2 \alpha_3^2 - \frac{1}{2} \alpha_1^2 \alpha_2 - \frac{1}{2} \alpha_2 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, 0, 0 \end{bmatrix}$$


$$\begin{bmatrix} \frac{1}{2} \alpha_2 \alpha_3^2 - \frac{1}{2} \alpha_1^2 \alpha_2 - \frac{1}{2} \alpha_2 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, \alpha_2^2 \alpha_3 - \alpha_1 \alpha_2^2 \\ + \frac{1}{2} \alpha_1^2 \alpha_3 + \frac{1}{2} \alpha_3^3 - \alpha_3^2 \alpha_1 - \frac{1}{2} \alpha_3 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, 0, 0 \end{bmatrix}$$


$$\begin{bmatrix} 0, 0, -\frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_3^2 + \alpha_1 \alpha_3 - \frac{1}{2} \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2}, \\ -\alpha_1 \alpha_2 + \alpha_2 \alpha_3 \end{bmatrix}$$


$$\begin{bmatrix} 0, 0, -\alpha_1 \alpha_2 + \alpha_2 \alpha_3, \\ \frac{1}{2} \alpha_1^2 + \frac{1}{2} \alpha_3^2 - \alpha_1 \alpha_3 - \frac{1}{2} \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \end{bmatrix}$$

```

```

> A3:=vecor(2):B3:=vector(2):
for k from 0 to 1 do
  A3[k+1]:=matrix(2,2): B3[k+1]:=matrix(2,2):
  for i from 1 to 2 do
    for j from 1 to 2 do
      A3[k+1][i,j]:=simplify(A2[i+2*k,j+2*k]);
      B3[k+1][i,j]:=simplify(factor(B2[i+2*k,j+2*k]));
    od;
  od;
od:
[ Check for semi-simple definite matrices
> factor(simplify(det(A3[1])));factor(simplify(det(A3[2])));
factor(simplify(det(B3[1])));factor(simplify(det(B3[2])));

0
0
0
0
[ So all submatrices in question are singular.

```

```
> factor(expand((trace(A3[1]))));factor(expand((trace(A3[2]))));
```

$$\begin{aligned}
& -2\alpha_1\alpha_2^2 + 2\alpha_2^2\alpha_3 - \frac{1}{2}\alpha_1^3 - \frac{3}{2}\alpha_3^2\alpha_1 + \frac{3}{2}\alpha_1^2\alpha_3 \\
& + \frac{1}{2}\alpha_1\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} + \frac{1}{2}\alpha_3^3 \\
& + \frac{1}{2}\alpha_3\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \\
& \qquad \qquad \qquad \sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2}
\end{aligned}$$

```
> factor(expand((trace(B3[1]))));factor(expand((trace(B3[2]))));
```

$$\begin{aligned}
& -2\alpha_1\alpha_2^2 + 2\alpha_2^2\alpha_3 - \frac{1}{2}\alpha_1^3 - \frac{3}{2}\alpha_3^2\alpha_1 + \frac{3}{2}\alpha_1^2\alpha_3 \\
& - \frac{1}{2}\alpha_1\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} + \frac{1}{2}\alpha_3^3 \\
& - \frac{1}{2}\alpha_3\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \\
& \qquad \qquad \qquad -\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2}
\end{aligned}$$

Now we know that A2 and -B2 are positive semi definit. (The theory for this case guarantees two independent positive semi definit invariants.)

```
>
```

```
> Inval:=multiply(transpose(x),A2,x);
```

$$\begin{aligned}
Inval := & \left(x_1 \left(-\alpha_1\alpha_2^2 + \alpha_2^2\alpha_3 - \frac{1}{2}\alpha_1^3 - \frac{1}{2}\alpha_3^2\alpha_1 + \alpha_1^2\alpha_3 \right. \right. \\
& \left. \left. + \frac{1}{2}\alpha_1\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \right) \right) \\
& + x_2 \left(\frac{1}{2}\alpha_2\alpha_3^2 - \frac{1}{2}\alpha_1^2\alpha_2 + \frac{1}{2}\alpha_2\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \right) x_1 + \left(\right. \\
& \left. x_1 \left(\frac{1}{2}\alpha_2\alpha_3^2 - \frac{1}{2}\alpha_1^2\alpha_2 + \frac{1}{2}\alpha_2\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \right) + x_2 \left(\right. \right. \\
& \left. \left. \alpha_2^2\alpha_3 - \alpha_1\alpha_2^2 + \frac{1}{2}\alpha_1^2\alpha_3 + \frac{1}{2}\alpha_3^3 - \alpha_3^2\alpha_1 + \frac{1}{2}\alpha_3\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \right) \right) \\
& \left. \right) x_2 + \left(x_3 \left(-\frac{1}{2}\alpha_1^2 - \frac{1}{2}\alpha_3^2 + \alpha_1\alpha_3 + \frac{1}{2}\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \right) \right) \\
& + x_4 \left(-\alpha_1\alpha_2 + \alpha_2\alpha_3 \right) x_3 + \left(x_3 \left(-\alpha_1\alpha_2 + \alpha_2\alpha_3 \right) \right. \\
& \left. + x_4 \left(\frac{1}{2}\alpha_1^2 + \frac{1}{2}\alpha_3^2 - \alpha_1\alpha_3 + \frac{1}{2}\sqrt{(4\alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2\alpha_1\alpha_3)(\alpha_1 - \alpha_3)^2} \right) \right) x_4
\end{aligned}$$

```
> Inva2:=multiply(transpose(x),-B2,x);
```

$$Inva2 := \left(x_1 \left(\right. \right.$$

$$\begin{aligned}
& \alpha_1 \alpha_2^2 - \alpha_2^2 \alpha_3 + \frac{1}{2} \alpha_1^3 + \frac{1}{2} \alpha_3^2 \alpha_1 - \alpha_1^2 \alpha_3 + \frac{1}{2} \alpha_1 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \\
& \left. \right) + x_2 \left(-\frac{1}{2} \alpha_2 \alpha_3^2 + \frac{1}{2} \alpha_1^2 \alpha_2 + \frac{1}{2} \alpha_2 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \right) x_1 + \left(\right. \\
& x_1 \left(-\frac{1}{2} \alpha_2 \alpha_3^2 + \frac{1}{2} \alpha_1^2 \alpha_2 + \frac{1}{2} \alpha_2 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \right) + x_2 \left(-\alpha_2^2 \alpha_3 \right. \\
& \left. + \alpha_1 \alpha_2^2 - \frac{1}{2} \alpha_1^2 \alpha_3 - \frac{1}{2} \alpha_3^3 + \alpha_3^2 \alpha_1 + \frac{1}{2} \alpha_3 \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \right) \left. \right) x_2 \\
& + \left(\right. \\
& x_3 \left(\frac{1}{2} \alpha_1^2 + \frac{1}{2} \alpha_3^2 - \alpha_1 \alpha_3 + \frac{1}{2} \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \right) + x_4 (\alpha_1 \alpha_2 - \alpha_2 \alpha_3) \\
& \left. \right) x_3 + \left(x_3 (\alpha_1 \alpha_2 - \alpha_2 \alpha_3) \right. \\
& \left. + x_4 \left(-\frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_3^2 + \alpha_1 \alpha_3 + \frac{1}{2} \sqrt{(4 \alpha_2^2 + \alpha_1^2 + \alpha_3^2 - 2 \alpha_1 \alpha_3)(\alpha_1 - \alpha_3)^2} \right) \right) x_4
\end{aligned}$$

[> Inva3:=simplify(lieableitung(f[3],Inva1)):

[> Inva4:=simplify(lieableitung(f[3],Inva2)):

[> pbasis:=vector([eval(Inva1^2),eval(Inva1*Inva2),eval(Inva2^2)]):

[> #LK3:=LinearCoefficients(pbasis,Inva3,dim); #The results are quite long and can be found in nofos-4d.mws!

[> #LK4:=LinearCoefficients(pbasis,Inva4,dim); #The results are quite long and can be found in nofos-4d.mws!

[Dx1=LK3[1]*x1^2+LK3[2]*x1*x2 + 0

[Dx2=0 + LK4[2]*x1*x2+LK4[3]*x2^2