# Content

This is an example of a MAPLE 7 worksheet for normalization, finding invariants and reduction of the following system:
Dx = Mx + f_2(x) + f_3(x) + higher order terms , where:

$$M := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -\alpha & 0 & 0 \\ -\alpha & -1 & 0 & 0 \end{bmatrix} \qquad f_2 := [0, 0, \mu x_1^2, 5\mu x_2 x_3] \qquad f_3 := [0, 0, 0, 0]$$

# Disclaimer

While our testing, as well as computations of examples, indicate correctness and reliability of the programs, the authors cannot guarantee the correctness of any routine.
Program developed by S. Mayer (email: mayer@mathA.rwth-aachen.de)



# Initialization

```
with(linalg):
```

The Message: "Warning, the protected names norm and trace have been redefined and unprotected" will not inflict the following calculations. It derives from using the linalg package instead of the LinearAlgebra package, for the "linalg package is useful for doing computations in abstract linear algebra" (Maple glossary). This allows parameter dependent calculations.

# Primitive operations

# Procedures

# Input

The vector field v is given in the form v(x)=Mx+f[2][x]+... where M is an nxn matrix and the f[j] are homogeneous polynomials C^n -> C^n of degree j. The following routines are capable of working with parameters, so you might want to apply them for parameter dependent vector fields. Note that maple assumes non-degeneraces of all parameters.

Enter the linearization M of the vector field. The following calculations assume that M is semisimple.

```
> M:=matrix(4,4,[0,0,1,0,0,0,0,1,-1,-alpha,0,0,-alpha,-1,0,0]);
```

$$M := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -\alpha & 0 & 0 \\ -\alpha & -1 & 0 & 0 \end{bmatrix}$$

Let targetspace be the desired degree of the normal form. targetspace then is also the degree of the taylor expansion of the vector field.

```
> targetspace:=3;
```

$$targetspace := 3$$

Just evaluate the following lines.

```
> f:=vector(targetspace,0):
> dim:=rowdim(M);
```

$$dim := 4$$

```
> for i from 1 to targetspace do
>     f[i]:=vector(dim,0);
> od:
> f[1]:=Matrix2Polynom(M);
```

$$f_1 := [x_3, x_4, -x_1 - \alpha x_2, -\alpha x_1 - x_2]$$

Enter the higher order terms of the Taylor expansion of the vector field: f[2]:=vector([...]); f[3]:=... f[targetspace]:=vector([...]);

```
> f[2]:=vector([0,0,mu*x[1]^2,mu*5*x[2]*x[3]]);f[3]:=vector([0,
  0,0,0]);
```

$$f_2 := [0, 0, \mu x_1^2, 5 \mu x_2 x_3]$$

$$f_3 := [0, 0, 0, 0]$$

Enter the maximal degree of calculated invariant polynomials.

```
> invariantsdegree:=2;
```

$$invariantsdegree := 2$$

# Automatic initialization

```
> dim:=rowdim(M):
> minimalpoly:=minpoly(M,tau);
```

$$minimalpoly := 1 - \alpha^2 + 2\tau^2 + \tau^4$$

```
> sigmaset:=symmetriceigenvalues(minimalpoly,dim):
```

# Determining annihilating polynomials

Annihilating polynomials for the action of ad M on P_r: (Remember that x -> p(-x) annihilates the action of ad M on P_0 if p annihilates M.)

```
> liebracketannihilator:=AnnihilatingPolynomials(sigmaset,subs(
  tau=-tau,minimalpoly),minimalpoly,dim,targetspace+1):
"Finished with step ", 1, "."
"Finished with step ", 2, "."
"Finished with step ", 3, "."
"Finished with step ", 4, "."
```

Now liebracketannihilator[j] annihilates the action of ad M on P_{j-1}.

```
> invpoly:=AnnihilatingPolynomials(sigmaset,minimalpoly,minimal
  poly,dim,invariantsdegree):
"Finished with step ", 1, "."
"Finished with step ", 2, "."
```

Now invpoly[j] annihilates the action of L_M on S_j.

## Output

```
> for j from 1 to targetspace+1 do
>     print("Annihilating polynomial for ad M on homogeneous
  polynomials of degree",j-1);
>     print(liebracketannihilator[j]);
> od;
```

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 0

$$(\alpha + 1 + \tau^2)(\alpha - 1 - \tau^2)$$

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 1

$$(-4\,\alpha+4+\tau^2)\,\tau\,(4\,\alpha^2+4\,\tau^2+\tau^4)\,(4\,\alpha+4+\tau^2)$$

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 2

$$(9\,\alpha+9+\tau^2)\,(25\,\alpha^2+30\,\alpha+9+6\,\tau^2\,\alpha+10\,\tau^2+\tau^4)\,(-9\,\alpha+9+\tau^2)$$
$$(25\,\alpha^2-6\,\tau^2\,\alpha-30\,\alpha+9+\tau^4+10\,\tau^2)\,(\alpha+1+\tau^2)\,(-\alpha+1+\tau^2)$$

"Annihilating polynomial for ad M on homogeneous polynomials of degree", 3

$$(-4\,\alpha+4+\tau^2)\,(-16\,\alpha+16+\tau^2)\,\tau\,(4\,\alpha^2+4\,\tau^2+\tau^4)\,(4\,\alpha+4+\tau^2)$$
$$(100\,\alpha^2-16\,\tau^2\,\alpha-160\,\alpha+\tau^4+20\,\tau^2+64)\,(16\,\alpha+16+\tau^2)$$
$$(100\,\alpha^2+16\,\tau^2\,\alpha+160\,\alpha+20\,\tau^2+\tau^4+64)\,(64\,\alpha^2+16\,\tau^2+\tau^4)$$

```
> print("Annihilating polynomial for L_M on homogeneous
  polynomials of degree",0);print(1);
  for j from 1 to invariantsdegree do
>     print("Annihilating polynomial for L_M on homogeneous
  polynomials of degree",j);
>     print(invpoly[j]);
> od;
```

"Annihilating polynomial for L_M on homogeneous polynomials of degree", 0

$$1$$

"Annihilating polynomial for L_M on homogeneous polynomials of degree", 1

$$(\alpha+1+\tau^2)\,(\alpha-1-\tau^2)$$

"Annihilating polynomial for L_M on homogeneous polynomials of degree", 2

$$(-4\,\alpha+4+\tau^2)\,\tau\,(4\,\alpha^2+4\,\tau^2+\tau^4)\,(4\,\alpha+4+\tau^2)$$

# Higher-order normalization

```
> dgl:=matrix(targetspace,targetspace): #the differential
  equation in the various transformed states
> Trafo:=vector(targetspace):Trafo[1]:=vector(dim,0):   #the
  transformation is exp(Trafo).
> for i from 1 to targetspace do
>     dgl[1,i]:=eval(f[i]);
> od:
> for i from 2 to targetspace do
      #print("starting with
  ",collect(expand(liebracketannihilator[i+1]),tau));
>
  erg:=semisimplecalculation(M,collect(expand(liebracketannihil
  ator[i+1]),tau),f[i]);
      #print("transformation and transformed part in normal
  form calculated for one degree higher");
>     for j from 1 to i-1 do
>         dgl[i,j]:=eval(dgl[i-1,j]);
>     od;
      #print("some terms are copied (degree < r)");
>     Trafo[i]:=eval(erg[1]);
>     dgl[i,i]:=eval(erg[2]);
      #print("some terms are chosen as it was calculated
  before");
>     for j from i+1 to targetspace do
```

```
>         dgl[i,j]:=eval(dgl[i-1,j]);
>         k:=1;
>         while j >= k*(i-1)+1 do
>               hlp:=scalarmul(dgl[i-1,j-k*(i-1)],1/k!);
>               for l from 1 to k do
>                   hlp:=lie(erg[1],hlp);
>               od;
>               dgl[i,j]:=matadd(dgl[i,j],hlp);
>               k:=k+1;
>         od;
          #print("transformation of higher terms up to degree
  ",j," done in this step")
>     od;
>     for j from i to targetspace do
>         f[j]:=eval(dgl[i,j]);
>     od;
      #print("transformation succesfull up to degree ",i);
> od:
```

The The previous lines compute Poincare-Dulac normal form up to degree targetspace. It is stored in the variable dgl. dgl is a field of dimension targetspace^2: dgl[i,1]+dgl[i,2]+...+dgl[i,targetspace] is the Poincare-Dulac-normal form up to degree i. The corresponding transformation is given by exp(ad(Trafo[i])) exp(ad(Trafo[i-1])) ... exp(ad(Trafo[2])). An output of the results is provided in the next steps:

***Output of the Poincare-Dulac normal form***

```
> print("Poincare-Dulac normal form up to degree",
  targetspace,":");
  for i from 1 to targetspace do
      print(dgl[targetspace,i]);
> od;
```

"Poincare-Dulac normal form up to degree", 3, ":"

$$[x_3, x_4, -x_1 - \alpha x_2, -\alpha x_1 - x_2]$$

$$[0, 0, 0, 0]$$

$$\left[\frac{1}{192} (1448 \alpha^5 x_4 x_2^2 - 1155 \alpha^5 x_4 x_3^2 + 1050 \alpha^6 x_1 x_2^2 - 69 \alpha x_4^3 - 540 x_4 x_3 x_2 \right.$$
$$+ 3330 x_4 x_1 x_2 + 900 \alpha^4 x_3^2 x_1 - 2550 \alpha^4 x_1 x_2^2 - 465 \alpha^2 x_3^2 x_1 + 1950 \alpha^2 x_1 x_2^2$$
$$- 3294 \alpha^3 x_4 x_2^2 + 2202 \alpha^3 x_4 x_3^2 - 495 \alpha x_2^3 + 900 \alpha^4 x_4^2 x_1 + 255 \alpha^4 x_3 x_4^2$$
$$+ 630 \alpha^5 x_4 x_3 x_1 - 225 \alpha x_2 x_3^2 - 5400 \alpha^4 x_2 x_4 x_3 + 90 x_3^2 x_1 - 465 \alpha^2 x_4^2 x_1$$
$$- 525 x_1 x_3^2 \alpha^6 - 165 \alpha x_1^2 x_2 + 1596 \alpha x_1^2 x_4 + 10020 \alpha^3 x_1 x_3 x_2 - 5792 \alpha^5 x_1 x_3 x_2$$
$$+ 2707 \alpha^2 x_2^2 x_3 + 930 \alpha^6 x_4 x_1 x_2 - 5794 \alpha^2 x_4 x_1 x_2 - 210 \alpha^6 x_1^3 - 780 \alpha^3 x_4 x_3 x_1$$
$$+ 1534 \alpha^4 x_4 x_1 x_2 + 3150 x_3 x_4 \alpha^6 x_2 + 1170 \alpha^3 x_2 x_3^2 - 4728 \alpha x_1 x_3 x_2$$
$$+ 2790 \alpha^2 x_2 x_4 x_3 - 390 \alpha^2 x_1^3 + 855 \alpha^3 x_1^2 x_2 - 1215 \alpha^5 x_1^2 x_2 - 3294 \alpha^3 x_1^2 x_4$$
$$+ 2707 x_1^2 x_3 \alpha^2 - 877 x_1^2 x_3 \alpha^4 - 877 \alpha^4 x_2^2 x_3 - 225 \alpha x_4^2 x_2 - 3645 \alpha^5 x_2^3$$
$$+ 1596 \alpha x_4 x_2^2 + 3676 \alpha^2 x_3^3 + 2565 \alpha^3 x_2^3 - 2211 \alpha^4 x_3^3 + 1575 \alpha^7 x_2^3 + 510 \alpha^4 x_1^3$$
$$+ 1170 \alpha^3 x_4^2 x_2 - 945 \alpha^5 x_4^2 x_2 - 945 x_3^2 \alpha^5 x_2 - 435 x_1^2 x_3 \alpha^6 + 525 \alpha^7 x_1^2 x_2$$

$$-435\,x_3\,\alpha^6\,x_2^2+1448\,\alpha^5\,x_4\,x_1^2-1395\,x_3^3+90\,x_1^3-1395\,x_2^2\,x_3+90\,x_1\,x_4^2$$

$$+1935\,x_4^2\,x_3-450\,x_1\,x_2^2-466\,\alpha^3\,x_4^3+215\,\alpha^5\,x_4^3+750\,\alpha^6\,x_4^2\,x_3+500\,x_1\,x_2\,x_3\,\alpha^7$$

$$+250\,\alpha^7\,x_4\,x_2^2+250\,\alpha^7\,x_4\,x_1^2-1980\,\alpha^2\,x_4^2\,x_3+250\,x_3^3\,\alpha^6-2007\,\alpha\,x_4\,x_3^2$$

$$-525\,\alpha^6\,x_4^2\,x_1+150\,\alpha\,x_1\,x_4\,x_3-1395\,x_1^2\,x_3)\,\mu^2\;\Big/\;((\alpha^2-1)\,(1-2\,\alpha^2+\alpha^4)$$

$$(25\,\alpha^2-9)),\;\frac{1}{192}\,(3330\,x_1\,x_2\,x_3-540\,x_4\,x_3\,x_1+750\,x_4\,x_3^2\,\alpha^6+250\,\alpha^7\,x_1^2\,x_3$$

$$+250\,\alpha^7\,x_2^2\,x_3-1980\,x_4\,x_3^2\,\alpha^2-225\,x_3^2\,\alpha\,x_1+1170\,x_3^2\,x_1\,\alpha^3-945\,\alpha^5\,x_1\,x_4^2$$

$$+3676\,\alpha^2\,x_4^3+1448\,\alpha^5\,x_1^2\,x_3-3294\,\alpha^3\,x_1^2\,x_3-1155\,x_3\,x_4^2\,\alpha^5+250\,\alpha^6\,x_4^3$$

$$+1596\,\alpha\,x_2^2\,x_3-435\,x_1^2\,x_4\,\alpha^6+630\,x_3\,x_4\,\alpha^5\,x_2-435\,x_4\,\alpha^6\,x_2^2+1170\,\alpha^3\,x_1\,x_4^2$$

$$-225\,\alpha\,x_1\,x_4^2+1935\,x_4\,x_3^2+2707\,\alpha^2\,x_1^2\,x_4+1596\,\alpha\,x_1^2\,x_3+1534\,x_1\,x_2\,x_3\,\alpha^4$$

$$-5794\,x_1\,x_2\,x_3\,\alpha^2+930\,x_1\,x_2\,x_3\,\alpha^6-465\,\alpha^2\,x_2\,x_3^2-450\,x_1^2\,x_2+1575\,\alpha^7\,x_1^3$$

$$+500\,x_4\,\alpha^7\,x_1\,x_2-3645\,\alpha^5\,x_1^3-525\,\alpha^6\,x_2\,x_3^2-780\,\alpha^3\,x_2\,x_4\,x_3+2565\,\alpha^3\,x_1^3$$

$$+900\,\alpha^4\,x_2\,x_3^2-2007\,\alpha\,x_4^2\,x_3-495\,\alpha\,x_1^3+1950\,x_1^2\,x_2\,\alpha^2-2550\,x_1^2\,x_2\,\alpha^4$$

$$-877\,\alpha^4\,x_1^2\,x_4-3294\,\alpha^3\,x_2^2\,x_3+1448\,\alpha^5\,x_2^2\,x_3+3150\,x_4\,x_3\,x_1\,\alpha^6-5792\,x_4\,\alpha^5\,x_1\,x_2$$

$$-4728\,x_4\,\alpha\,x_1\,x_2+10020\,x_4\,\alpha^3\,x_1\,x_2+90\,x_2\,x_3^2-1395\,x_1^2\,x_4-1395\,x_4\,x_2^2+90\,x_4^2\,x_2$$

$$+525\,x_1\,\alpha^7\,x_2^2-525\,\alpha^6\,x_4^2\,x_2+90\,x_2^3-1395\,x_4^3+2790\,x_3\,x_4\,\alpha^2\,x_1+150\,x_3\,x_4\,\alpha\,x_2$$

$$-165\,x_1\,\alpha\,x_2^2+855\,x_1\,\alpha^3\,x_2^2+1050\,x_1^2\,x_2\,\alpha^6-1215\,x_1\,\alpha^5\,x_2^2-69\,\alpha\,x_3^3-466\,x_3^3\,\alpha^3$$

$$+255\,x_4\,x_3^2\,\alpha^4-945\,x_3^2\,\alpha^5\,x_1-5400\,x_3\,x_4\,x_1\,\alpha^4+2707\,x_4\,\alpha^2\,x_2^2-877\,x_4\,\alpha^4\,x_2^2$$

$$-465\,x_4^2\,\alpha^2\,x_2+900\,x_4^2\,\alpha^4\,x_2-390\,\alpha^2\,x_2^3+2202\,x_3\,x_4^2\,\alpha^3-210\,\alpha^6\,x_2^3+510\,\alpha^4\,x_2^3$$

$$-2211\,\alpha^4\,x_4^3+215\,\alpha^5\,x_3^3)\,\mu^2\;\Big/\;((\alpha^2-1)\,(1-2\,\alpha^2+\alpha^4)\,(25\,\alpha^2-9)),\,-\frac{1}{192}\,($$

$$-1575\,\alpha^5\,x_4\,x_2^2+750\,\alpha^6\,x_1\,x_2^2-225\,\alpha\,x_4^3-3330\,x_4\,x_3\,x_2-540\,x_4\,x_1\,x_2$$

$$+465\,\alpha^4\,x_3^2\,x_1-3099\,\alpha^4\,x_1\,x_2^2-2212\,\alpha^2\,x_3^2\,x_1+4284\,\alpha^2\,x_1\,x_2^2+2070\,\alpha^3\,x_4\,x_2^2$$

$$+315\,\alpha^3\,x_4\,x_3^2-201\,\alpha\,x_2^3+465\,\alpha^4\,x_4^2\,x_1-2625\,\alpha^4\,x_3\,x_4^2+500\,\alpha^5\,x_4\,x_3\,x_1$$

$$+1464\,\alpha\,x_2\,x_3^2-870\,\alpha^4\,x_2\,x_4\,x_3+1395\,x_3^2\,x_1-2212\,\alpha^2\,x_4^2\,x_1+1197\,\alpha\,x_1^2\,x_2$$

$$-495\,\alpha\,x_1^2\,x_4-1380\,\alpha^3\,x_1\,x_3\,x_2+1050\,\alpha^5\,x_1\,x_3\,x_2-300\,\alpha^2\,x_2^2\,x_3+1800\,\alpha^2\,x_4\,x_1\,x_2$$

$$+250\,\alpha^6\,x_1^3+1596\,\alpha^3\,x_4\,x_3\,x_1-1260\,\alpha^4\,x_4\,x_1\,x_2-1746\,\alpha^3\,x_2\,x_3^2+330\,\alpha\,x_1\,x_3\,x_2$$

$$+4264\,\alpha^2\,x_2\,x_4\,x_3-2908\,\alpha^2\,x_1^3-2442\,\alpha^3\,x_1^2\,x_2+1245\,\alpha^5\,x_1^2\,x_2+2070\,\alpha^3\,x_1^2\,x_4$$

$$-300\,x_1^2\,x_3\,\alpha^2+210\,x_1^2\,x_3\,\alpha^4+210\,\alpha^4\,x_2^2\,x_3+1464\,\alpha\,x_4^2\,x_2-185\,\alpha^5\,x_2^3$$

$$- 495\,\alpha\,x_4\,x_2^2 - 375\,\alpha^2\,x_3^3 + 386\,\alpha^3\,x_2^3 + 525\,\alpha^4\,x_3^3 + 1263\,\alpha^4\,x_1^3 - 1746\,\alpha^3\,x_4^2\,x_2$$
$$+ 250\,\alpha^5\,x_4^2\,x_2 + 250\,x_3^2\,\alpha^5\,x_2 - 1575\,\alpha^5\,x_4\,x_1^2 + 90\,x_3^3 + 1395\,x_1^3 + 90\,x_2^2\,x_3$$
$$+ 1395\,x_1\,x_4^2 - 450\,x_4^2\,x_3 - 1935\,x_1\,x_2^2 + 945\,\alpha^3\,x_4^3 + 1875\,\alpha^2\,x_4^2\,x_3 - 75\,\alpha\,x_4\,x_3^2$$
$$- 1392\,\alpha\,x_1\,x_4\,x_3 + 90\,x_1^2\,x_3)\,\mu^2 \Big/ \left((\alpha^2-1)^2\,(25\,\alpha^2-9)\right), -\frac{1}{192}\,(-540\,x_1\,x_2\,x_3$$
$$- 3330\,x_4\,x_3\,x_1 + 1875\,x_4\,x_3^2\,\alpha^2 + 1464\,x_3^2\,\alpha\,x_1 - 1746\,x_3^2\,x_1\,\alpha^3 + 250\,\alpha^5\,x_1\,x_4^2$$
$$- 375\,\alpha^2\,x_4^3 - 1575\,\alpha^5\,x_1^2\,x_3 + 2070\,\alpha^3\,x_1^2\,x_3 - 495\,\alpha\,x_2^2\,x_3 + 500\,x_3\,x_4\,\alpha^5\,x_2$$
$$- 1746\,\alpha^3\,x_1\,x_4^2 + 1464\,\alpha\,x_1\,x_4^2 - 450\,x_4\,x_3^2 - 300\,\alpha^2\,x_1^2\,x_4 - 495\,\alpha\,x_1^2\,x_3$$
$$- 1260\,x_1\,x_2\,x_3\,\alpha^4 + 1800\,x_1\,x_2\,x_3\,\alpha^2 - 2212\,\alpha^2\,x_2\,x_3^2 - 1935\,x_1^2\,x_2 - 185\,\alpha^5\,x_1^3$$
$$+ 1596\,\alpha^3\,x_2\,x_4\,x_3 + 386\,\alpha^3\,x_1^3 + 465\,\alpha^4\,x_2\,x_3^2 - 75\,\alpha\,x_4^2\,x_3 - 201\,\alpha\,x_1^3$$
$$+ 4284\,x_1^2\,x_2\,\alpha^2 - 3099\,x_1^2\,x_2\,\alpha^4 + 210\,\alpha^4\,x_1^2\,x_4 + 2070\,\alpha^3\,x_2^2\,x_3 - 1575\,\alpha^5\,x_2^2\,x_3$$
$$+ 1050\,x_4\,\alpha^5\,x_1\,x_2 + 330\,x_4\,\alpha\,x_1\,x_2 - 1380\,x_4\,\alpha^3\,x_1\,x_2 + 1395\,x_2\,x_3^2 + 90\,x_1^2\,x_4$$
$$+ 90\,x_4\,x_2^2 + 1395\,x_4^2\,x_2 + 1395\,x_2^3 + 90\,x_4^3 + 4264\,x_3\,x_4\,\alpha^2\,x_1 - 1392\,x_3\,x_4\,\alpha\,x_2$$
$$+ 1197\,x_1\,\alpha\,x_2^2 - 2442\,x_1\,\alpha^3\,x_2^2 + 750\,x_1^2\,x_2\,\alpha^6 + 1245\,x_1\,\alpha^5\,x_2^2 - 225\,\alpha\,x_3^3$$
$$+ 945\,x_3^3\,\alpha^3 - 2625\,x_4\,x_3^2\,\alpha^4 + 250\,x_3^2\,\alpha^5\,x_1 - 870\,x_3\,x_4\,x_1\,\alpha^4 - 300\,x_4\,\alpha^2\,x_2^2$$
$$+ 210\,x_4\,\alpha^4\,x_2^2 - 2212\,x_4^2\,\alpha^2\,x_2 + 465\,x_4^2\,\alpha^4\,x_2 - 2908\,\alpha^2\,x_2^3 + 315\,x_3\,x_4^2\,\alpha^3$$
$$+ 250\,\alpha^6\,x_2^3 + 1263\,\alpha^4\,x_2^3 + 525\,\alpha^4\,x_4^3)\,\mu^2 \Big/ \left((\alpha^2-1)^2\,(25\,\alpha^2-9)\right)\Big]$$

***Output of intermediate results***

```
> for j from 1 to targetspace-1 do
  print("Poincare-Dulac normal form up to degree", j,":");
      for i from 1 to targetspace do
          print(dgl[j,i]);
      od;
  od;
```

"Poincare-Dulac normal form up to degree", 1, ":"

$$[x_3, x_4, -x_1 - \alpha\,x_2, -\alpha\,x_1 - x_2]$$

$$[0, 0, \mu\,x_1^2, 5\,\mu\,x_2\,x_3]$$

$$[0, 0, 0, 0]$$

"Poincare-Dulac normal form up to degree", 2, ":"

$$[x_3, x_4, -x_1 - \alpha\,x_2, -\alpha\,x_1 - x_2]$$

$$[0, 0, 0, 0]$$

$$\left[\frac{1}{6}\,\frac{\mu^2\,(280\,\alpha^3 - 120\,\alpha)\,x_2\,x_3^2}{25\,\alpha^6 - 59\,\alpha^4 + 43\,\alpha^2 - 9} + \frac{\frac{1}{6}\,\mu^2\,(40\,\alpha^2 - 200\,\alpha^4)\,x_2\,x_4\,x_3}{25\,\alpha^6 - 59\,\alpha^4 + 43\,\alpha^2 - 9}\right.$$

$$+\frac{\frac{1}{6}\mu^2(-250\alpha^4+250\alpha^2)x_2^{\,2}x_3}{25\alpha^6-59\alpha^4+43\alpha^2-9}+\frac{\frac{1}{6}\mu^2(250\alpha^5-550\alpha^3+300\alpha)x_1x_3x_2}{25\alpha^6-59\alpha^4+43\alpha^2-9}$$

$$+\frac{\frac{1}{6}\mu^2(-88\alpha^2+20\alpha^4+36)x_1^{\,2}x_3}{25\alpha^6-59\alpha^4+43\alpha^2-9}+\frac{\frac{1}{6}\mu^2(-24\alpha+56\alpha^3)x_1^{\,2}x_4}{25\alpha^6-59\alpha^4+43\alpha^2-9}$$

$$+\frac{\frac{1}{6}\mu^2(-75\alpha-25\alpha^5+100\alpha^3)x_1^{\,2}x_2}{25\alpha^6-59\alpha^4+43\alpha^2-9}+\frac{\frac{1}{6}\mu^2(10\alpha^4-10\alpha^2)x_1^{\,3}}{25\alpha^6-59\alpha^4+43\alpha^2-9},$$

$$-\frac{1}{6}\frac{\mu^2(200\alpha^4-40\alpha^2)x_2x_3^{\,2}}{25\alpha^6-59\alpha^4+43\alpha^2-9}+\frac{\frac{80}{3}\mu^2\alpha^3x_2x_4x_3}{25\alpha^6-59\alpha^4+43\alpha^2-9}$$

$$-\frac{1}{6}\frac{\mu^2(-200\alpha^3+75\alpha+125\alpha^5)x_2^{\,2}x_3}{25\alpha^6-59\alpha^4+43\alpha^2-9}-\frac{1}{6}\frac{\mu^2(450-1150\alpha^2+700\alpha^4)x_1x_3x_2}{25\alpha^6-59\alpha^4+43\alpha^2-9}$$

$$-\frac{1}{6}\frac{\mu^2(-56\alpha^3+24\alpha)x_1^{\,2}x_3}{25\alpha^6-59\alpha^4+43\alpha^2-9}-\frac{1}{6}\frac{\mu^2(40\alpha^4-8\alpha^2)x_1^{\,2}x_4}{25\alpha^6-59\alpha^4+43\alpha^2-9}$$

$$-\frac{1}{6}\frac{\mu^2(-55\alpha^4+100\alpha^2-45)x_1^{\,2}x_2}{25\alpha^6-59\alpha^4+43\alpha^2-9}-\frac{1}{6}\frac{\mu^2(-75\alpha+200\alpha^3-125\alpha^5)x_1^{\,3}}{25\alpha^6-59\alpha^4+43\alpha^2-9},$$

$$\frac{1}{6}\frac{(125\alpha^5-75\alpha-50\alpha^3)\mu^2x_2x_3^{\,2}}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}+\frac{\frac{1}{6}(-20\alpha^4-36+88\alpha^2)\mu^2x_1x_3^{\,2}}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$+\frac{\frac{1}{6}(-500\alpha^4+500\alpha^2)\mu^2x_4x_3x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}+\frac{\frac{1}{6}(-112\alpha^3+48\alpha)\mu^2x_1x_4x_3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$+\frac{\frac{1}{6}(-20\alpha^2+20\alpha^4)\mu^2x_2^{\,2}x_3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}+\frac{\frac{1}{6}(300\alpha-600\alpha^3+300\alpha^5)\mu^2x_1x_3x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$+\frac{\frac{1}{6}(40\alpha^4-8\alpha^2)\mu^2x_1x_4^{\,2}}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}+\frac{\frac{1}{6}(-100\alpha^2+100\alpha^4)\mu^2x_1x_4x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$+\frac{\frac{1}{6}(210\alpha^3-75\alpha^5-135\alpha)\mu^2x_1^{\,2}x_4}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}+\frac{\frac{1}{6}(-20\alpha^4+20\alpha^2)\mu^2x_1x_2^{\,2}}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$+\frac{\frac{1}{6}(-30\alpha^5-18\alpha+48\alpha^3)\mu^2x_1^{\,2}x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}+\frac{\frac{1}{6}(112\alpha^2-76\alpha^4-36)\mu^2x_1^{\,3}}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)},$$

$$-\frac{40}{3}\frac{\alpha^3\mu^2 x_3 x_4^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(-700\alpha^4-450+1150\alpha^2)\mu^2 x_1 x_4 x_3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(-250\alpha^3+250\alpha^5)\mu^2 x_4 x_3 x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(-225\alpha-375\alpha^5+600\alpha^3)\mu^2 x_1^2 x_3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(-90-230\alpha^4+320\alpha^2)\mu^2 x_1 x_3 x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(-30\alpha-200\alpha^5+230\alpha^3)\mu^2 x_2^2 x_3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(-250\alpha^4+250\alpha^2)\mu^2 x_2 x_4^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(150\alpha-400\alpha^3+250\alpha^5)\mu^2 x_1 x_4 x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(-20\alpha^2+20\alpha^4)\mu^2 x_4 x_2^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(45+85\alpha^4-130\alpha^2)\mu^2 x_1^2 x_4}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(125\alpha^6-250\alpha^4+125\alpha^2)\mu^2 x_2^3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(6\alpha-16\alpha^3+10\alpha^5)\mu^2 x_1^3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(-250\alpha^6+520\alpha^4-270\alpha^2)\mu^2 x_1^2 x_2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(75\alpha^5-150\alpha^3+75\alpha)\mu^2 x_1 x_2^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(140\alpha^3-60\alpha)\mu^2 x_3^3}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(40\alpha^2-200\alpha^4)\mu^2 x_4 x_3^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}$$

$$-\frac{1}{6}\frac{(-1200\alpha^2+750\alpha^4+450)\mu^2 x_2 x_3^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}-\frac{1}{6}\frac{(625\alpha^5+375\alpha-1000\alpha^3)\mu^2 x_1 x_3^2}{(9-34\alpha^2+25\alpha^4)(\alpha^2-1)}\Bigg]$$

```
> for j from 2 to targetspace do
>     print("Transformation from degree ",j-1," into degree
  ",j," is exp(ad(Trafo)) with Trafo=");
      print(Trafo[j]);
> od;
```

"Transformation from degree ", 1, " into degree ", 2, " is exp(ad(Trafo)) with Trafo="

$$\Bigg[\frac{1}{3}(-10 x_1\alpha^5 x_2+16 x_1\alpha^3 x_2-4\alpha^2 x_4^2+44\alpha^2 x_3^2-10\alpha^4 x_2^2-19\alpha^4 x_1^2+20\alpha^4 x_4^2$$
$$+75\alpha x_2 x_3-6 x_1\alpha x_2-9 x_1^2-56 x_3 x_4\alpha^3-50 x_4 x_1\alpha^5+25 x_3\alpha^5 x_2-10\alpha^4 x_3^2$$
$$-50 x_4\alpha^2 x_2+28\alpha^2 x_1^2+10 x_3\alpha^2 x_1+10\alpha^2 x_2^2-18 x_3^2+110 x_4\alpha^3 x_1$$
$$-100 x_3\alpha^3 x_2-10 x_3 x_1\alpha^4+50 x_4\alpha^4 x_2-60 x_4\alpha x_1+24 x_4\alpha x_3)\mu\Big/($$
$$25\alpha^6-59\alpha^4+43\alpha^2-9),-\frac{1}{3}(-10\alpha^5 x_2^2-25\alpha^5 x_1^2-25 x_4\alpha^5 x_2+125 x_3\alpha^5 x_1$$
$$-4 x_1\alpha^4 x_2-140 x_4 x_1\alpha^4+55 x_3\alpha^4 x_2-40 x_4 x_3\alpha^4+10\alpha^3 x_2^2+40 x_4\alpha^3 x_2$$
$$-200 x_3 x_1\alpha^3+28 x_3^2\alpha^3+40\alpha^3 x_1^2+16 x_4^2\alpha^3+230 x_4\alpha^2 x_1+4\alpha^2 x_1 x_2$$
$$-100 x_3\alpha^2 x_2+8 x_4 x_3\alpha^2-15 x_4\alpha x_2-12\alpha x_3^2+75 x_3\alpha x_1-15\alpha x_1^2+45 x_3 x_2$$

$$-90 x_4 x_1)\mu \Big/ (25\alpha^6 - 59\alpha^4 + 43\alpha^2 - 9), -\frac{1}{3}(25\alpha^4 x_2^2 - 50\alpha^4 x_1^2 - 10 x_3\alpha^3 x_2$$

$$+50 x_4\alpha^3 x_1 + 15 x_1\alpha^3 x_2 + 25 x_3 x_4\alpha^3 + 10\alpha^2 x_3^2 - 50\alpha^2 x_4^2 - 38 x_3\alpha^2 x_1$$

$$+50\alpha^2 x_1^2 - 25\alpha^2 x_2^2 + 4 x_4\alpha^2 x_2 - 15 x_1\alpha x_2 - 30 x_4\alpha x_1 + 15 x_4\alpha x_3 + 6\alpha x_2 x_3$$

$$+18 x_3 x_1)\mu \Big/ (9 - 34\alpha^2 + 25\alpha^4), \frac{1}{3}(100 x_1\alpha^4 x_2 + 30\alpha^3 x_2^2 - 15\alpha^3 x_1^2$$

$$+10 x_3 x_1\alpha^3 - 125 x_3^2\alpha^3 + 25 x_4^2\alpha^3 - 20 x_4\alpha^3 x_2 + 20 x_3\alpha^2 x_2 - 145\alpha^2 x_1 x_2$$

$$+85 x_4 x_3\alpha^2 - 4 x_4\alpha^2 x_1 - 6 x_3\alpha x_1 + 15\alpha x_1^2 - 30\alpha x_2^2 - 15 x_4^2\alpha + 75\alpha x_3^2$$

$$-45 x_4 x_3 + 45 x_1 x_2)\mu \Big/ (9 - 34\alpha^2 + 25\alpha^4)\Big]$$

"Transformation from degree ", 2, " into degree ", 3, " is exp(ad(Trafo)) with Trafo="

$$\Big[-\frac{1}{768}\mu^2(-28800 x_1 x_2 x_3 + 559665\alpha^7 x_4^2 x_1 - 379920\alpha^8 x_1^2 x_2 - 580175 x_1 x_3^2\alpha^7$$

$$-100800 x_4 x_3 x_1 + 972240 x_4 x_3^2\alpha^6 + 815280\alpha^7 x_1^2 x_3 + 517240\alpha^7 x_2^2 x_3$$

$$+216380 x_4 x_3^2\alpha^2 + 117360 x_3^2\alpha x_1 - 494523 x_3^2 x_1\alpha^3 - 756006\alpha^5 x_1 x_4^2$$

$$-142780\alpha^2 x_4^3 - 1096005\alpha^5 x_1^2 x_3 + 633070\alpha^3 x_1^2 x_3 - 74350 x_3 x_4^2\alpha^5$$

$$+180600\alpha^8 x_2^3 - 472880\alpha^6 x_4^3 + 26400\alpha x_2^2 x_3 + 217350 x_1^2 x_4\alpha^6$$

$$+695492 x_3 x_4\alpha^5 x_2 - 665230 x_3 x_4\alpha^7 x_2 - 528490 x_4\alpha^6 x_2^2 + 355269\alpha^3 x_1 x_4^2$$

$$-30288\alpha x_1 x_4^2 - 31680 x_4 x_3^2 + 72660\alpha^2 x_1^2 x_4 - 138720\alpha x_1^2 x_3$$

$$-431650 x_1 x_2 x_3\alpha^4 + 157640 x_1 x_2 x_3\alpha^2 + 558860 x_1 x_2 x_3\alpha^6 + 623486\alpha^2 x_2 x_3^2$$

$$-180000 x_1^2 x_2 - 63731\alpha^7 x_1^3 - 791600 x_4\alpha^7 x_1 x_2 + 188833\alpha^5 x_1^3$$

$$+1253124\alpha^6 x_2 x_3^2 - 246918\alpha^3 x_2 x_4 x_3 - 156389\alpha^3 x_1^3 - 1406025\alpha^4 x_2 x_3^2$$

$$+64800\alpha x_4^2 x_3 + 39312\alpha x_1^3 + 1024282 x_1^2 x_2\alpha^2 - 2029924 x_1^2 x_2\alpha^4$$

$$-195705\alpha^4 x_1^2 x_4 + 43430\alpha^3 x_2^2 x_3 - 352445\alpha^5 x_2^2 x_3 + 903064 x_4 x_3 x_1\alpha^6$$

$$+80000\alpha^{10} x_2^3 + 679990 x_4\alpha^5 x_1 x_2 + 230000\alpha^9 x_3 x_4 x_2 + 29760 x_4\alpha x_1 x_2$$

$$-246900 x_4\alpha^3 x_1 x_2 - 79200 x_2 x_3^2 - 2880 x_1^2 x_4 - 359650 x_4 x_3 x_1\alpha^8 + 8640 x_4 x_2^2$$

$$+99360 x_4^2 x_2 - 510721 x_1\alpha^7 x_2^2 - 256050 x_1 x_2 x_3\alpha^8 - 994612\alpha^6 x_4^2 x_2 + 99360 x_2^3$$

$$+8640 x_4^3 + 160000\alpha^9 x_3^2 x_1 - 70000\alpha^{10} x_1^2 x_2 + 453600\alpha^7 x_3^3 + 420908 x_3 x_4\alpha^2 x_1$$

$$+328750 x_4\alpha^9 x_1 x_2 + 3936 x_3 x_4\alpha x_2 - 56592 x_1\alpha x_2^2 + 59353 x_1\alpha^3 x_2^2$$

$$+1635562 x_1^2 x_2\alpha^6 + 280235 x_1\alpha^5 x_2^2 - 124320\alpha x_3^3 + 552330 x_3^3\alpha^3$$

$$-635865 x_4 x_3^2\alpha^4 + 805978 x_3^2\alpha^5 x_1 - 880802 x_3 x_4 x_1\alpha^4 - 133660 x_4\alpha^2 x_2^2$$

$$+ 423335\, x_4\, \alpha^4\, x_2^2 - 568042\, x_4^2\, \alpha^2\, x_2 + 1156079\, x_4^2\, \alpha^4\, x_2 - 610714\, \alpha^2\, x_2^3$$

$$- 128650\, x_3\, x_4^2\, \alpha^3 - 1011178\, \alpha^6\, x_2^3 + 1261932\, \alpha^4\, x_2^3 - 8025\, \alpha^9\, x_1^3 + 152375\, \alpha^8\, x_4^3$$

$$- 213625\, \alpha^9\, x_1^2\, x_3 - 234625\, \alpha^9\, x_2^2\, x_3 - 585875\, x_4\, x_3^2\, \alpha^8 + 203000\, \alpha^7\, x_4^2\, x_3$$

$$- 91425\, x_1^2\, x_4\, \alpha^8 + 230175\, x_4\, \alpha^8\, x_2^2 + 227725\, x_1\, \alpha^9\, x_2^2 - 400025\, \alpha^8\, x_2\, x_3^2$$

$$- 120000\, \alpha^9\, x_1\, x_4^2 + 298575\, \alpha^8\, x_4^2\, x_2 + 433045\, \alpha^4\, x_4^3 - 860010\, \alpha^5\, x_3^3) \Big/ (\alpha$$

$$(25\, \alpha^2 - 9)\, (\alpha^2 - 1)^3\, (25\, \alpha^2 - 16)),\ \frac{1}{768}\, \mu^2\, (-513275\, \alpha^5\, x_4\, x_2^2 + 364910\, \alpha^5\, x_4\, x_3^2$$

$$+ 2083670\, \alpha^6\, x_1\, x_2^2 - 75360\, \alpha\, x_4^3 + 31680\, x_4\, x_3\, x_2 - 17280\, x_4\, x_1\, x_2$$

$$+ 2027601\, \alpha^4\, x_3^2\, x_1 - 2461660\, \alpha^4\, x_1\, x_2^2 - 967254\, \alpha^2\, x_3^2\, x_1 + 1285350\, \alpha^2\, x_1\, x_2^2$$

$$+ 351570\, \alpha^3\, x_4\, x_2^2 - 114550\, \alpha^3\, x_4\, x_3^2 - 34320\, \alpha\, x_2^3 - 3030135\, \alpha^4\, x_4^2\, x_1$$

$$- 884135\, \alpha^4\, x_3\, x_4^2 + 2375612\, \alpha^5\, x_4\, x_3\, x_1 + 17232\, \alpha\, x_2\, x_3^2 + 657442\, \alpha^4\, x_2\, x_4\, x_3$$

$$+ 800200\, \alpha^8\, x_1^3 + 165600\, x_3^2\, x_1 + 1564226\, \alpha^2\, x_4^2\, x_1 - 1845132\, x_1\, x_3^2\, \alpha^6$$

$$- 133872\, \alpha\, x_1^2\, x_2 - 160800\, \alpha\, x_1^2\, x_4 - 579340\, \alpha^3\, x_1\, x_3\, x_2 + 934730\, \alpha^5\, x_1\, x_3\, x_2$$

$$+ 230700\, \alpha^2\, x_2^2\, x_3 - 363020\, \alpha^6\, x_4\, x_1\, x_2 + 15160\, \alpha^2\, x_4\, x_1\, x_2 - 1823702\, \alpha^6\, x_1^3$$

$$- 1414138\, \alpha^3\, x_4\, x_3\, x_1 + 166690\, \alpha^4\, x_4\, x_1\, x_2 - 273176\, x_3\, x_4\, \alpha^6\, x_2 + 120059\, \alpha^3\, x_2\, x_3^2$$

$$- 691120\, \alpha^8\, x_1\, x_2^2 + 6415\, \alpha^7\, x_3^2\, x_2 + 116160\, \alpha\, x_1\, x_3\, x_2 - 377516\, \alpha^2\, x_2\, x_4\, x_3$$

$$- 1599730\, \alpha^7\, x_4\, x_3\, x_1 - 912486\, \alpha^2\, x_1^3 + 808423\, \alpha^3\, x_1^2\, x_2 - 1721195\, \alpha^5\, x_1^2\, x_2$$

$$+ 883610\, \alpha^3\, x_1^2\, x_4 - 139620\, x_1^2\, x_3\, \alpha^2 + 387545\, x_1^2\, x_3\, \alpha^4 - 659335\, \alpha^4\, x_2^2\, x_3$$

$$- 50160\, \alpha\, x_4^2\, x_2 - 34657\, \alpha^5\, x_2^3 - 84000\, \alpha\, x_4\, x_2^2 - 132420\, \alpha^2\, x_3^3 + 85413\, \alpha^3\, x_2^3$$

$$+ 372075\, \alpha^4\, x_3^3 - 51661\, \alpha^7\, x_2^3 + 263625\, x_3^3\, \alpha^8 + 1890388\, \alpha^4\, x_1^3$$

$$+ 198450\, \alpha^8\, x_4\, x_1\, x_2 - 120000\, \alpha^{10}\, x_1^3 + 316923\, \alpha^3\, x_4^2\, x_2 - 504525\, \alpha^9\, x_1^2\, x_2$$

$$- 532378\, \alpha^5\, x_4^2\, x_2 - 21150\, \alpha^8\, x_2\, x_4\, x_3 - 232346\, x_3^2\, \alpha^5\, x_2 - 457750\, x_1^2\, x_3\, \alpha^6$$

$$+ 1551169\, \alpha^7\, x_1^2\, x_2 + 766970\, x_3\, \alpha^6\, x_2^2 + 71250\, \alpha^9\, x_1\, x_3\, x_2 + 195425\, x_1^2\, x_3\, \alpha^8$$

$$- 318175\, \alpha^8\, x_2^2\, x_3 + 330000\, \alpha^9\, x_4\, x_3\, x_1 - 1734275\, \alpha^5\, x_4\, x_1^2 + 14400\, x_3^3$$

$$- 40000\, \alpha^9\, x_4^2\, x_2 + 165600\, x_1^3 + 627825\, \alpha^8\, x_3^2\, x_1 - 20160\, x_2^2\, x_3 + 35225\, \alpha^9\, x_2^3$$

$$- 277920\, x_1\, x_4^2 - 676775\, \alpha^8\, x_4^2\, x_1 - 37440\, x_4^2\, x_3 - 246240\, x_1\, x_2^2 + 141600\, \alpha^7\, x_4^3$$

$$+ 296975\, \alpha^7\, x_4^2\, x_2 + 303990\, \alpha^3\, x_4^3 - 391830\, \alpha^5\, x_4^3 - 262125\, \alpha^8\, x_4^2\, x_3$$

$$+ 933040\, \alpha^6\, x_4^2\, x_3 - 542800\, x_1\, x_2\, x_3\, \alpha^7 + 248080\, \alpha^7\, x_4\, x_2^2 + 1504840\, \alpha^7\, x_4\, x_1^2$$

$$- 2375\,\alpha^{9}\,x_4\,x_2^{2} + 315460\,\alpha^{2}\,x_4^{2}\,x_3 - 496080\,x_3^{3}\,\alpha^{6} - 493375\,\alpha^{9}\,x_1^{2}\,x_4$$
$$- 331000\,\alpha^{7}\,x_4\,x_3^{2} + 15840\,\alpha\,x_4\,x_3^{2} + 2429244\,\alpha^{6}\,x_4^{2}\,x_1 + 80000\,\alpha^{9}\,x_3^{2}\,x_2$$
$$+ 290976\,\alpha\,x_1\,x_4\,x_3 + 14400\,x_1^{2}\,x_3 + 30000\,\alpha^{10}\,x_1\,x_2^{2}) \Big/ (\alpha\,(25\,\alpha^{2} - 9)\,(\alpha^{2} - 1)^{3}$$
$$(25\,\alpha^{2} - 16)),\ -\frac{1}{768}\,\mu^{2}\,(-100800\,x_1\,x_2\,x_3 + 300440\,\alpha^{7}\,x_4^{2}\,x_1 + 930095\,\alpha^{8}\,x_1^{2}\,x_2$$
$$- 826080\,x_1\,x_3^{2}\,\alpha^{7} + 28800\,x_4\,x_3\,x_1 + 1862068\,x_4\,x_3^{2}\,\alpha^{6} - 260607\,\alpha^{7}\,x_1^{2}\,x_3$$
$$- 179879\,\alpha^{7}\,x_2^{2}\,x_3 + 915946\,x_4\,x_3^{2}\,\alpha^{2} + 132960\,x_3^{2}\,\alpha\,x_1 - 645990\,x_3^{2}\,x_1\,\alpha^{3}$$
$$- 440695\,\alpha^{5}\,x_1\,x_4^{2} - 572458\,\alpha^{2}\,x_4^{3} + 126501\,\alpha^{5}\,x_1^{2}\,x_3 + 42567\,\alpha^{3}\,x_1^{2}\,x_3$$
$$+ 153806\,x_3\,x_4^{2}\,\alpha^{5} - 282115\,\alpha^{8}\,x_2^{3} - 934252\,\alpha^{6}\,x_4^{3} + 8592\,\alpha\,x_2^{2}\,x_3$$
$$+ 2295894\,x_1^{2}\,x_4\,\alpha^{6} + 151410\,x_3\,x_4\,\alpha^{5}\,x_2 - 188400\,x_3\,x_4\,\alpha^{7}\,x_2 - 1317730\,x_4\,\alpha^{6}\,x_2^{2}$$
$$+ 271330\,\alpha^{3}\,x_1\,x_4^{2} - 55200\,\alpha\,x_1\,x_4^{2} - 180000\,x_4\,x_3^{2} + 716942\,\alpha^{2}\,x_1^{2}\,x_4$$
$$- 31536\,\alpha\,x_1^{2}\,x_3 - 717752\,x_1\,x_2\,x_3\,\alpha^{4} + 453836\,x_1\,x_2\,x_3\,\alpha^{2} + 540076\,x_1\,x_2\,x_3\,\alpha^{6}$$
$$+ 54060\,\alpha^{2}\,x_2\,x_3^{2} + 31680\,x_1^{2}\,x_2 - 1181550\,\alpha^{7}\,x_1^{3} - 483262\,x_4\,\alpha^{7}\,x_1\,x_2$$
$$+ 1436310\,\alpha^{5}\,x_1^{3} + 135730\,\alpha^{6}\,x_2\,x_3^{2} - 21020\,\alpha^{3}\,x_2\,x_4\,x_3 - 760170\,\alpha^{3}\,x_1^{3}$$
$$- 152195\,\alpha^{4}\,x_2\,x_3^{2} + 97488\,\alpha\,x_4^{2}\,x_3 + 147360\,\alpha\,x_1^{3} - 285500\,x_1^{2}\,x_2\,\alpha^{2}$$
$$+ 910805\,x_1^{2}\,x_2\,\alpha^{4} - 1989436\,\alpha^{4}\,x_1^{2}\,x_4 - 115953\,\alpha^{3}\,x_2^{2}\,x_3 + 245965\,\alpha^{5}\,x_2^{2}\,x_3$$
$$- 683900\,x_4\,x_3\,x_1\,\alpha^{6} + 77125\,\alpha^{10}\,x_2^{3} + 623210\,x_4\,\alpha^{5}\,x_1\,x_2 + 76250\,\alpha^{9}\,x_3\,x_4\,x_2$$
$$+ 58656\,x_4\,\alpha\,x_1\,x_2 - 333554\,x_4\,\alpha^{3}\,x_1\,x_2 + 2880\,x_2\,x_3^{2} - 79200\,x_1^{2}\,x_4$$
$$+ 263850\,x_4\,x_3\,x_1\,\alpha^{8} + 99360\,x_4\,x_2^{2} - 8640\,x_4^{2}\,x_2 + 566190\,x_1\,\alpha^{7}\,x_2^{2}$$
$$- 235360\,x_1\,x_2\,x_3\,\alpha^{8} + 258530\,\alpha^{6}\,x_4^{2}\,x_2 - 8640\,x_2^{3} + 99360\,x_4^{3} + 211125\,\alpha^{9}\,x_3^{2}\,x_1$$
$$- 247625\,\alpha^{10}\,x_1^{2}\,x_2 - 343075\,\alpha^{7}\,x_3^{3} - 249800\,x_3\,x_4\,\alpha^{2}\,x_1 + 134950\,x_4\,\alpha^{9}\,x_1\,x_2$$
$$- 18240\,x_3\,x_4\,\alpha\,x_2 + 60000\,x_1\,x_2\,x_3\,\alpha^{10} - 64800\,x_1\,\alpha\,x_2^{2} + 349290\,x_1\,\alpha^{3}\,x_2^{2}$$
$$- 1339455\,x_1^{2}\,x_2\,\alpha^{6} - 677430\,x_1\,\alpha^{5}\,x_2^{2} + 28080\,\alpha\,x_3^{3} - 119759\,x_3^{3}\,\alpha^{3}$$
$$- 1945199\,x_4\,x_3^{2}\,\alpha^{4} + 1127985\,x_3^{2}\,\alpha^{5}\,x_1 + 641050\,x_3\,x_4\,x_1\,\alpha^{4} - 597850\,x_4\,\alpha^{2}\,x_2^{2}$$
$$+ 1350140\,x_4\,\alpha^{4}\,x_2^{2} + 81820\,x_4^{2}\,\alpha^{2}\,x_2 - 226435\,x_4^{2}\,\alpha^{4}\,x_2 + 75580\,\alpha^{2}\,x_2^{3}$$
$$- 211869\,x_3\,x_4^{2}\,\alpha^{3} + 391155\,\alpha^{6}\,x_2^{3} - 253105\,\alpha^{4}\,x_2^{3} + 358050\,\alpha^{9}\,x_1^{3} + 277075\,\alpha^{8}\,x_4^{3}$$
$$+ 123075\,\alpha^{9}\,x_1^{2}\,x_3 + 41275\,\alpha^{9}\,x_2^{2}\,x_3 - 644175\,x_4\,x_3^{2}\,\alpha^{8} + 135000\,\alpha^{9}\,x_3^{3}$$
$$+ 145000\,x_1^{2}\,x_4\,\alpha^{10} - 15000\,x_4\,\alpha^{10}\,x_2^{2} + 35000\,\alpha^{9}\,x_4^{2}\,x_3 - 83065\,\alpha^{7}\,x_4^{2}\,x_3$$

$$- 1089200\, x_1^{2} x_4 \alpha^{8} + 481080\, x_4 \alpha^{8} x_2^{2} - 173250\, x_1 \alpha^{9} x_2^{2} - 40475\, \alpha^{8} x_2 x_3^{2}$$

$$- 75875\, \alpha^{9} x_1 x_4^{2} - 105275\, \alpha^{8} x_4^{2} x_2 + 1133155\, \alpha^{4} x_4^{3} + 296874\, \alpha^{5} x_3^{3}) \Big/ (\alpha$$

$$(9 - 34\, \alpha^{2} + 25\, \alpha^{4})\, (\alpha^{2} - 1)^{2}\, (25\, \alpha^{2} - 16)),\ \frac{1}{768}\, \mu^{2}\, (630171\, \alpha^{5} x_4 x_2^{2}$$

$$+ 1928946\, \alpha^{5} x_4 x_3^{2} - 662145\, \alpha^{6} x_1 x_2^{2} + 39120\, \alpha x_4^{3} + 17280\, x_4 x_3 x_2 + 31680\, x_4 x_1 x_2$$

$$- 323965\, \alpha^{4} x_3^{2} x_1 + 649195\, \alpha^{4} x_1 x_2^{2} + 116580\, \alpha^{2} x_3^{2} x_1 - 267460\, \alpha^{2} x_1 x_2^{2}$$

$$- 312839\, \alpha^{3} x_4 x_2^{2} - 973859\, \alpha^{3} x_4 x_3^{2} + 98400\, \alpha x_2^{3} + 41475\, \alpha^{4} x_4^{2} x_1$$

$$- 2714321\, \alpha^{4} x_3 x_4^{2} - 870770\, \alpha^{5} x_4 x_3 x_1 + 51360\, \alpha x_2 x_3^{2} + 582630\, \alpha^{4} x_2 x_4 x_3$$

$$- 869565\, \alpha^{8} x_1^{3} - 14400\, x_3^{2} x_1 - 59820\, \alpha^{2} x_4^{2} x_1 + 385310\, x_1 x_3^{2} \alpha^{6} - 15840\, \alpha x_1^{2} x_2$$

$$+ 220272\, \alpha x_1^{2} x_4 + 1089906\, \alpha^{3} x_1 x_3 x_2 - 1834730\, \alpha^{5} x_1 x_3 x_2 + 1679858\, \alpha^{2} x_2^{2} x_3$$

$$- 356140\, \alpha^{6} x_4 x_1 x_2 - 68300\, \alpha^{2} x_4 x_1 x_2 + 1136205\, \alpha^{6} x_1^{3} + 553500\, \alpha^{3} x_4 x_3 x_1$$

$$+ 145720\, \alpha^{4} x_4 x_1 x_2 - 703620\, x_3 x_4 \alpha^{6} x_2 - 110690\, \alpha^{3} x_2 x_3^{2} + 259345\, \alpha^{8} x_1 x_2^{2}$$

$$+ 222440\, \alpha^{7} x_3^{2} x_2 - 247584\, \alpha x_1 x_3 x_2 - 195640\, \alpha^{2} x_2 x_4 x_3 + 521200\, \alpha^{7} x_4 x_3 x_1$$

$$+ 178500\, \alpha^{2} x_1^{3} + 91350\, \alpha^{3} x_1^{2} x_2 - 177930\, \alpha^{5} x_1^{2} x_2 - 1235023\, \alpha^{3} x_1^{2} x_4$$

$$- 1146918\, x_1^{2} x_3 \alpha^{2} + 3159876\, x_1^{2} x_3 \alpha^{4} - 3792004\, \alpha^{4} x_2^{2} x_3 + 89760\, \alpha x_4^{2} x_2$$

$$+ 1100970\, \alpha^{5} x_2^{3} + 54960\, \alpha x_4 x_2^{2} - 962838\, \alpha^{2} x_3^{3} - 548310\, \alpha^{3} x_2^{3} + 2050525\, \alpha^{4} x_3^{3}$$

$$- 950610\, \alpha^{7} x_2^{3} + 649325\, x_3^{3} \alpha^{8} - 681615\, \alpha^{4} x_1^{3} + 387040\, \alpha^{8} x_4 x_1 x_2$$

$$+ 250875\, \alpha^{10} x_1^{3} - 485530\, \alpha^{3} x_4^{2} x_2 - 42750\, \alpha^{9} x_1^{2} x_2 + 928975\, \alpha^{5} x_4^{2} x_2$$

$$+ 299350\, \alpha^{8} x_2 x_4 x_3 - 665000\, x_1^{2} x_3 \alpha^{10} + 295000\, \alpha^{10} x_2^{2} x_3 + 485000\, \alpha^{9} x_4 x_3^{2}$$

$$- 15000\, \alpha^{9} x_4^{3} - 30985\, x_3^{2} \alpha^{5} x_2 - 4207838\, x_1^{2} x_3 \alpha^{6} + 145170\, \alpha^{7} x_1^{2} x_2$$

$$+ 3945066\, x_3 \alpha^{6} x_2^{2} - 400550\, \alpha^{9} x_1 x_3 x_2 + 2694280\, x_1^{2} x_3 \alpha^{8} - 1850000\, \alpha^{8} x_2^{2} x_3$$

$$- 76250\, \alpha^{9} x_4 x_3 x_1 - 140000\, \alpha^{10} x_4 x_1 x_2 + 2414195\, \alpha^{5} x_4 x_1^{2} + 165600\, x_3^{3}$$

$$+ 204875\, \alpha^{9} x_4^{2} x_2 - 14400\, x_1^{3} - 163525\, \alpha^{8} x_3^{2} x_1 - 277920\, x_2^{2} x_3 + 299550\, \alpha^{9} x_2^{3}$$

$$+ 20160\, x_1 x_4^{2} - 58725\, \alpha^{8} x_4^{2} x_1 - 246240\, x_4^{2} x_3 + 37440\, x_1 x_2^{2} + 59875\, \alpha^{7} x_4^{3}$$

$$- 738080\, \alpha^{7} x_4^{2} x_2 - 57841\, \alpha^{3} x_4^{3} - 23274\, \alpha^{5} x_4^{3} - 813425\, \alpha^{8} x_4^{2} x_3$$

$$+ 2450188\, \alpha^{6} x_4^{2} x_3 + 1392958\, x_1 x_2 x_3 \alpha^{7} - 535617\, \alpha^{7} x_4 x_2^{2} - 2002969\, \alpha^{7} x_4 x_1^{2}$$

$$+ 163325\, \alpha^{9} x_4 x_2^{2} + 1315158\, \alpha^{2} x_4^{2} x_3 - 1905492\, x_3^{3} \alpha^{6} + 603525\, \alpha^{9} x_1^{2} x_4$$

$$- 1615815\, \alpha^{7} x_4 x_3^{2} + 184368\, \alpha x_4 x_3^{2} + 56910\, \alpha^{6} x_4^{2} x_1 - 132125\, \alpha^{9} x_3^{2} x_2$$

$$-127680\,\alpha\,x_1\,x_4\,x_3+165600\,x_1^{\,2}\,x_3-16375\,\alpha^{10}\,x_1\,x_2^{\,2}\Big)\Big/\Big(\alpha\,(9-34\,\alpha^2+25\,\alpha^4)\,(\alpha^2-1)^2\,(25\,\alpha^2-16)\Big)\Big]$$

# Invariants at fixed degree

```
> eval(invariantsdegree);
```

$$2$$

Choose a degree invdegree <= invariantsdegree;

```
> invdegree:=invariantsdegree;
```

$$invdegree := 2$$

## *Invariant providing map*

A map from the space of homogeneous polynomials of invdegree onto the invariant polynomials of invdegree is calculated.

```
> eval(invpoly);
```

$$[(\alpha+1+\tau^2)\,(\alpha-1-\tau^2),\ (-4\,\alpha+4+\tau^2)\,\tau\,(4\,\alpha^2+4\,\tau^2+\tau^4)\,(4\,\alpha+4+\tau^2)]$$

Note that the command InvariantsProvidingPolynomial(invpoly[deg],M); calculates the invariant providing map for degree deg, if deg<invariantsdegree.

```
> invpolymap:=InvariantsProvidingPolynomial(invpoly[invdegree],M);
```

$$invpolymap := -64\,\alpha^4+64\,\alpha^2+\tau^2\,(-32\,\alpha^2+64)+\tau^4\,(48-12\,\alpha^2)+12\,\tau^6+\tau^8$$

## *Finding the invariants*

A basis of invariants of degree 'invariantsdegree' is calculated using 'invpolymap'. If you already know the dimension of the space of invariants from some theoretical argument, set 'number' to some positive value (in order to save calculation time) and thus only do 'number' tries for invariants. Another call of invariants(...,eval(thistry),...) will then try more invariants...

```
> Invarianten:={}: InvZahl:=0: firsttry:=vector(dim,0):
> firsttry[dim]:=invdegree:number:=-1:eval(firsttry);
```

$$[0, 0, 0, 2]$$

```
> thistry,Invarianten,InvZahl:=invariants(invpolymap,eval(Invarianten),InvZahl,eval(firsttry),number,invdegree,dim,M);
```

$$thistry,\ Invarianten,\ InvZahl := [3, 0, 0, 0],\ \{-32\,x_4\,x_3\,\alpha^4+32\,x_4\,x_3\,\alpha^2+32\,\alpha^2\,x_1\,x_2-32\,x_1\,\alpha^4\,x_2+16\,\alpha^3\,x_1^{\,2}-16\,\alpha^5\,x_1^{\,2}+16\,\alpha^3\,x_2^{\,2}-16\,\alpha^5\,x_2^{\,2},\ -16\,\alpha^4\,x_4^{\,2}+16\,\alpha^2\,x_4^{\,2}+32\,x_1\,\alpha^3\,x_2-32\,x_1\,\alpha^5\,x_2+16\,\alpha^2\,x_1^{\,2}-16\,\alpha^4\,x_1^{\,2}+16\,\alpha^2\,x_2^{\,2}-16\,\alpha^4\,x_2^{\,2}+16\,\alpha^2\,x_3^{\,2}-16\,\alpha^4\,x_3^{\,2}\},\ 2$$

If g is an invariant, and c a constant, c*g is also an invariant. So it is 'legal' to try to simplify further calculations by eliminating parameters.

```
> Inva:=vector(InvZahl):
> for i from 1 to InvZahl do

  Inva[i]:=factor(polynomialsimplify(Invarianten[i],dim));
  od;
```

$$Inva_1 := \alpha\,x_1^{\,2}+\alpha\,x_2^{\,2}+2\,x_1\,x_2+2\,x_4\,x_3$$

$$Inva_2 := x_1\,\alpha\,x_2 + \frac{1}{2}x_2^{\,2} + \frac{1}{2}x_1^{\,2} + \frac{1}{2}x_4^{\,2} + \frac{1}{2}x_3^{\,2}$$

# Reduced vector field

## Matrix form of the invariants and positive semi definit invariants

```
> A:=matrixform(4,Inva[1]);
```

$$A := \begin{bmatrix} \alpha & 1 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> B:=matrixform(4,Inva[2]);
```

$$B := \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\alpha & 0 & 0 \\ \frac{1}{2}\alpha & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

```
> if A[3,4]=0 then A1:=A; else
    A1:=simplify(matadd(B,scalarmul(A,-B[3,4]/A[3,4])));
    C:=eval(A):A:=eval(B);B:=eval(C);
    fi;
```

$$A1 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\alpha & 0 & 0 \\ \frac{1}{2}\alpha & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$C := \begin{bmatrix} \alpha & 1 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A := \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\alpha & 0 & 0 \\ \frac{1}{2}\alpha & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$B := \begin{bmatrix} \alpha & 1 & 0 & 0 \\ 1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> A1:=matrixsimplify(A1); #this produces A1new=c*A1old with
  c a real (komplex) polynomial in alpha[i]
```

$$A1 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\alpha & 0 & 0 \\ \frac{1}{2}\alpha & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

```
> if A1[4,4]<0 then A1:= scalarmul(A1,-1);fi;
> unassign('delta');delta:=[solve((B[3,3]+delta*A1[3,3])*(B[
  4,4]+delta*A1[4,4])-(B[3,4]*B[4,3])=0,delta)];
```

$$\delta := [2, -2]$$

```
> A2:=simplify(matadd(B,scalarmul(A1,delta[1])));B2:=simplif
  y(matadd(B,scalarmul(A1,delta[2])));
```

$$A2 := \begin{bmatrix} \alpha+1 & \alpha+1 & 0 & 0 \\ \alpha+1 & \alpha+1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$B2 := \begin{bmatrix} \alpha-1 & 1-\alpha & 0 & 0 \\ 1-\alpha & \alpha-1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

```
> A3:=vecor(2):B3:=vector(2):
```

```
for k from 0 to 1 do
    A3[k+1]:=matrix(2,2): B3[k+1]:=matrix(2,2):
    for i from 1 to 2 do
        for j from 1 to 2 do
            A3[k+1][i,j]:=simplify(A2[i+2*k,j+2*k]);

B3[k+1][i,j]:=simplify(factor(B2[i+2*k,j+2*k]));
            od;
        od;
    od:
```

Check for semi-simple definite matrices

```
> factor(simplify(det(A3[1])));factor(simplify(det(A3[2])));
factor(simplify(det(B3[1])));factor(simplify(det(B3[2])));
```

$$0$$
$$0$$
$$0$$
$$0$$

So all submatrices in question are singular.

```
> factor(expand((trace(A3[1]))));factor(expand((trace(A3[2])
)));
```

$$2\,\alpha+2$$
$$2$$

```
> factor(expand((trace(B3[1]))));factor(expand((trace(B3[2])
)));
```

$$2\,\alpha-2$$
$$-2$$

Now we know that A2 and -B2 are positive semi definit. (The theory for this case guarantees two independent positive semi definit invariants.)

```
> Inva1:=multiply(transpose(x),A2,x);
```

$$Inva1 := (x_1\,(\alpha+1)+x_2\,(\alpha+1))\,x_1+(x_1\,(\alpha+1)+x_2\,(\alpha+1))\,x_2+(x_3+x_4)\,x_3+(x_3+x_4)\,x_4$$

```
> Inva2:=multiply(transpose(x),-B2,x);
```

$$Inva2 := (x_1\,(1-\alpha)+x_2\,(\alpha-1))\,x_1+(x_1\,(\alpha-1)+x_2\,(1-\alpha))\,x_2+(x_3-x_4)\,x_3+(-x_3+x_4)\,x_4$$

```
> Inva3:=simplify(lieableitung(f[3],Inva1)):
> Inva4:=simplify(lieableitung(f[3],Inva2)):
> pbasis:=vector([eval(Inva1^2),eval(Inva1*Inva2),eval(Inva2^2)
]):
> LK3:=LinearCoefficients(pbasis,Inva3,dim);
```

$$LK3 := \left[\frac{5}{32}\,\frac{(7\,\alpha^2+7\,\alpha+2)\,\mu^2}{(\alpha^2-1)\,(\alpha+1)\,(5\,\alpha+3)},\,-\frac{5}{16}\,\frac{(7\,\alpha^2-7\,\alpha+2)\,\mu^2}{(5\,\alpha-3)\,(\alpha^3-\alpha^2-\alpha+1)},\,0\right]$$

```
> LK4:=LinearCoefficients(pbasis,Inva4,dim);
```

$$LK4 := \left[0,\,-\frac{5}{16}\,\frac{(7\,\alpha^2+7\,\alpha+2)\,\mu^2}{(\alpha^2-1)\,(\alpha+1)\,(5\,\alpha+3)},\,\frac{5}{32}\,\frac{(7\,\alpha^2-7\,\alpha+2)\,\mu^2}{(5\,\alpha-3)\,(\alpha^3-\alpha^2-\alpha+1)}\right]$$

```
>
```

Dx1=LK3[1]*x1^2+LK3[2]*x1*x2 + 0
Dx2=0 + LK4[2]*x1*x2+LK4[3]*x2^2