Calculating local densities

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In this article we perform explicit calculations for Siegel type representation numbers of positive definite quadratic forms. The material is quite technical and is based on calculations with Gaussian sums. The formulas given here can be applied e.g. to figure out the exact values of the Fourier coefficients of lattice indexed Jacobi-Eisenstein series.

1 Gaussian sums and representation numbers

Let $n \in \mathbb{N}$ and consider the module $\mathbb{Z}^n$. We denote by $b_1, \ldots, b_n$ some basis of $\mathbb{Z}^n$. For $x \in \mathbb{Z}^n$ we write $x = \sum_i x_i b_i$. Let

$$f : \mathbb{Z}^n \to \mathbb{Z} \quad \text{such that for all } x \in \mathbb{Z}^n \quad f(x) \in \mathbb{Z}[x_1, \ldots, x_n]\,.$$

This requirement on f does obviously not depend on the chioce of the basis. Let $a \in \mathbb{Z}$ be an integer. Then we have

$$f(x) \equiv f(y) \pmod a \quad \text{whenever} \quad x, y \in \mathbb{Z}^n \text{ and } x \equiv y \pmod a\,.$$

We put

$$\mathrm{N}_a(f) = \sharp \{x \in \mathbb{Z}^n/a\mathbb{Z}^n\,;\, f(x) \equiv 0 \pmod a\}\ .$$

The first lemma says that $\mathrm{N}(f)$ is multiplicative.

Lemma 1.1 *Let $a, b \in \mathbb{Z}$ and $(a, b) = 1$. Then we have $\mathrm{N}_{ab}(f) = \mathrm{N}_a(f) \cdot \mathrm{N}_b(f)$.*

Proof
For $c \in \mathbb{Z}$ we define the set

$$\mathrm{M}_c(f) = \{x \in \mathbb{Z}^n/c\mathbb{Z}^n\,;\, f(x) \equiv 0 \pmod c\}\ .$$

We consider the map

$$\Phi : \mathrm{M}_{ab}(f) \to \mathrm{M}_a(f) \times \mathrm{M}_b(f) \quad , \quad x + ab\mathbb{Z}^n \mapsto (x + a\mathbb{Z}^n, x + b\mathbb{Z}^n)$$

which is obviously well-defined. If $\Phi(x) = \Phi(y)$ we have $x \equiv y \pmod a$ and $x \equiv y \pmod b$ and therefore $x \equiv y \pmod{ab}$ since a, b are coprime. Thus Φ is injective. Now let $(y + a\mathbb{Z}^n, z + b\mathbb{Z}^n) \in \mathrm{M}_a(f) \times \mathrm{M}_b(f)$. By the chinese remainder theorem there exists a unique $x \pmod{ab}$ such that $(x + a\mathbb{Z}^n, x + b\mathbb{Z}^n) = (y + a\mathbb{Z}^n, z + b\mathbb{Z}^n)$ and $f(x) \equiv 0 \pmod a$ and $f(x) \equiv 0 \pmod b$. This shows by again using that a and b are coprime $x \in \mathrm{M}_{ab}(f)$. So Φ is bijective and the claim follows. $\square$

The next lemma derives a sum expansion of $\mathrm{N}_a(f)$.

Lemma 1.2 *For every $a \in \mathbb{Z}$ we have*

$$a\,\mathrm{N}_a(f) = \sum_{\mu \in \mathbb{Z}/a\mathbb{Z}} \sum_{x \in \mathbb{Z}^n/a\mathbb{Z}^n} e^{\frac{2\pi \mathrm{i}}{a}\mu f(x)}\,.$$

Proof
Let $c \in \mathbb{Z}$. We first evaluate the expression $\sum_{\mu \in \mathbb{Z}/a\mathbb{Z}} e^{\frac{2\pi \mathrm{i}}{a}\mu c}$. One observes by substituting $\mu \mapsto \mu + 1$

$$\sum_{\mu \in \mathbb{Z}/a\mathbb{Z}} e^{\frac{2\pi \mathrm{i}}{a}\mu c} = e^{\frac{2\pi \mathrm{i}}{a}c} \cdot \sum_{\mu \in \mathbb{Z}/a\mathbb{Z}} e^{\frac{2\pi \mathrm{i}}{a}\mu c} = \begin{cases} a & \text{if } c = 0 \pmod a \\ 0 & \text{if } c \neq 0 \pmod a \end{cases}\,.$$

This yields

$$\sum_{\mu\in\mathbb{Z}/a\mathbb{Z}}\ \sum_{x\in\mathbb{Z}^n/a\mathbb{Z}^n} e^{\frac{2\pi i}{a}\mu f(x)} = \sum_{x\in\mathbb{Z}^n/a\mathbb{Z}^n}\ \sum_{\mu\in\mathbb{Z}/a\mathbb{Z}} e^{\frac{2\pi i}{a}\mu f(x)} = \sum_{x\in\mathbb{Z}^n/a\mathbb{Z}^n} a \sum_{f(x)\equiv 0 \bmod a} 1 = a\,\mathrm{N}_a(f) \qquad \square$$

Definition 1.3 Let $S \in \mathrm{Sym}(n,\mathbb{Z})$,i.e. S is symmetric and has entries in $\mathbb{Z}$. We call S *even* if $x'Sx \in 2\mathbb{Z}$ for all $x \in \mathbb{Z}^n$. A matrix $R \in \mathrm{Sym}(n,\mathbb{Q})$ is called *semi-integral* if $2R$ is even.

We often write $S[x]$ for the scalarproduct $x'Sx$. We now want to investigate $\mathrm{N}_a(f)$ closer. For this purpose we concentrate on the case where f is of the type

$$f(x) = R[x] - \Delta \quad \text{where } R \text{ is semi-integral and } \Delta \in \mathbb{Z}\,.$$

The corresponding solution number $\mathrm{N}_a(f)$ will be denoted by $\mathrm{N}_a(R,\Delta)$. Since these numbers are mutiplicative in a it suffices to evaluate them at prime powers. The following Proposition is needed to treat the odd primes and is due to Minkowski.

Proposition 1.4 *Let p be an odd prime, $l \in \mathbb{N}$ and $S \in \mathrm{Sym}(n,\mathbb{Z})$. Then there exists an invertible $U \in \mathrm{GL}(n,\mathbb{Z})$ and a diagonal matrix D such that*

$$S[U] \equiv D \pmod{p^l}\,.$$

The matrix D can always be chosen even.

Proof
Suppose $n > 1$ and $S = (s_{i,j}) \neq 0$. Let d denote the gcd of the entries of S. By replacing S by $\frac{1}{d}S$ we can assume that $d = 1$. Now suppose that there exists an diagonal element of S not divisible by p. Without loss of generality we can assume that it is the first. Now choose $u_2,\ldots,u_n \in \mathbb{Z}$ such that $s_{1,1}u_j + s_{1,j} \equiv 0 \pmod{p^l}$ and put

$$U = \begin{pmatrix} 1 & u_2 & \cdots & u_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{such that} \quad S[U] \equiv \begin{pmatrix} s_{1,1} & 0 \\ 0 & R \end{pmatrix} \pmod{p^l} \quad \text{where} \quad R \in \mathrm{Sym}(n-1,\mathbb{Z})\,.$$

If p divides all diagonal entries then there exists $i, j, i \neq j$ such that $p \nmid s_{i,j}$ in view of $d = 1$. Choose $V \in \mathrm{GL}(n,\mathbb{Z})$ having $e_i + e_j$ as its first column where e_k denotes the k-th standard basis vector of $\mathbb{Z}^n$. Then the first diagonal element of $S[V]$ equals $s_{i,i} + s_{j,j} + 2s_{i,j}$ which is not divisible by p. An induction finishes the proof. $\square$

We introduce the notion of a Gaussian sum. We first remind to the classical Gaussian sums. For integers $a, r \in \mathbb{Z}$ we define

$$g_a(r) = \sum_{k=0}^{a-1} e^{2\pi i r k^2/a}\,.$$

We recall some properties of Gaussian sums.

Lemma 1.5 *Let $a, r, t \in \mathbb{Z}$ and $t > 0$.*

(i) $g_{ta}(tr) = t \cdot g_a(r)$.

(ii) If p is an odd prime not dividing r we have

$$g_p(r) = \left(\frac{r}{p}\right) g_p(1)$$

where $\left(\frac{r}{p}\right)$ denotes the Legendre symbol.

(iii) If p is an odd prime, $t \geq 2$ and p is coprime to r we have

$$g_{p^t}(r) = \begin{cases} p^{t/2} & \text{if } t = 0 \pmod 2 \\ p^{(t-1)/2} g_p(r) & \text{if } t \neq 0 \pmod 2 \end{cases}$$

(iv) We have

$$g_a(1) = \begin{cases} (1+i)\sqrt{a} & \text{if } a \equiv 0 \pmod 4 \\ \sqrt{a} & \text{if } a \equiv 1 \pmod 4 \\ 0 & \text{if } a \equiv 2 \pmod 4 \\ i\sqrt{a} & \text{if } a \equiv 3 \pmod 4 \end{cases} .$$

(v) If p is coprime to r we have

$$\sum_{k=0}^{p-1} \left(\frac{k}{p}\right) e^{2\pi \mathrm{i} rk/p} = \left(\frac{r}{p}\right) g_p(1) .$$

The next definition generalizes Gaussian sums to semi-integral R.

Definition 1.6 Let $R \in \mathrm{Sym}(n, \mathbb{Q})$ be semi-integral and $a \in \mathbb{Z}$. The *generalized Gaussian sum* is defined as

$$G_a(R) = \sum_{\substack{x \in \mathbb{Z}^n \\ x \bmod a}} e^{2\pi \mathrm{i} R[x]/a} .$$

Since R is semi-integral and $p > 2$ we can always find an integral matrix A such that $R \equiv A \pmod p$. In this context the congruence relation can be extended to semi-integral matrices and odd modules. We state some elementary properties of G.

Proposition 1.7 *Let $R \in \mathrm{Sym}(n, \mathbb{Q})$ be semi-integral and $a \in \mathbb{Z}$.*

(a) If $n = 1$ we have $G_a(R) = g_a(R)$.

(b) We have $G_a(R[U]) = G_a(R)$, whenever $U \in \mathrm{GL}(n, \mathbb{Z})$.

(c) If $R \equiv \mathrm{diag}(r_1, \ldots, r_n) \pmod a$ we have $G_a(R) = G_a(r_1) \cdots G_a(r_n)$.

Lemma 1.8 *Let $R \in \mathrm{Sym}(n, \mathbb{Q})$ be semi-integral, $a \in \mathbb{Z}, l \in \mathbb{N}$ such that $\mathrm{ord}_p(a) < l$ and p be an odd prime satisfying $p \nmid \det(2R)$. Choose $d \in \mathbb{Z}$ such that $\det(2R) \equiv 2^n d \pmod{p^l}$. Then*

$$G_{p^l}(aR) = \begin{cases} p^{\frac{n}{2}(l+\mathrm{ord}_p(a))} & \text{if } l - \mathrm{ord}_p(a) \equiv 0 \pmod 2 \\ \varepsilon_{n,p} \left(\frac{a^n p^{-n \,\mathrm{ord}_p(a)}}{p}\right) \left(\frac{d}{p}\right) p^{\frac{n}{2}(l+\mathrm{ord}_p(a))} & \text{if } l - \mathrm{ord}_p(a) \equiv 1 \pmod 2 \end{cases}$$

where

$$\varepsilon_{n,p} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ \mathrm{i}^n & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

for $l \geq 2$ and

$$G_p(aR) = \varepsilon_{n,p} \left(\frac{a^n}{p}\right) \left(\frac{d}{p}\right) p^{\frac{n}{2}} .$$

Proof
According to Proposition 1.4 there exists an $U \in \mathrm{GL}(n, \mathbb{Z})$ and an integral diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ satisfying

$$2R[U] \equiv 2D \pmod{p^l} .$$

The choice of d implies $d \equiv d_1 \cdots d_n \pmod{p^l}$. The last proposition gives

$$G_{p^l}(aR) = G_{p^l}(ad_1) \cdots G_{p^l}(ad_n) .$$

By Lemma 1.5 (i),(iii) we have for $l \geq 2$

$$G_{p^l}(ad_j) = \begin{cases} p^{\frac{1}{2}(l+\mathrm{ord}_p(a))} & \text{if } l - \mathrm{ord}_p(a) \equiv 0 \pmod 2 \\ p^{\frac{1}{2}(l+\mathrm{ord}_p(a)-1)} \cdot G_p(ap^{-\mathrm{ord}_p(a)} d_j) & \text{if } l - \mathrm{ord}_p(a) \equiv 1 \pmod 2 \end{cases}$$

Now 1.5 (ii),(iv) say that

$$G_p(ap^{-\mathrm{ord}_p(a)} d_j) = \begin{cases} \left(\frac{ap^{-\mathrm{ord}_p(a)} d_j}{p}\right) \sqrt{p} & \text{if } p \equiv 1 \pmod 4 \\ \left(\frac{ap^{-\mathrm{ord}_p(a)} d_j}{p}\right) \mathrm{i}\sqrt{p} & \text{if } p \equiv 3 \pmod 4 \end{cases} .$$

The claim now follows by the multiplicativity of the Legendre symbol. □

Remark 1.9 If S is an even matrix we have under the same assumtions as in the lemma above

$$G_{p^l}(aS) = \begin{cases} p^{\frac{n}{2}(l+\mathrm{ord}_p(a))} & \text{if } l-\mathrm{ord}_p(a) \equiv 0 \pmod 2 \\ \varepsilon_{n,p} \left(\frac{a^n p^{-n\,\mathrm{ord}_p(a)}}{p}\right) \left(\frac{\det S}{p}\right) p^{\frac{n}{2}(l+\mathrm{ord}_p(a))} & \text{if } l-\mathrm{ord}_p(a) \equiv 1 \pmod 2 \end{cases}$$

where

$$\varepsilon_{n,p} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ \mathrm{i}^n & \text{if } p \equiv 3 \pmod 4 \end{cases}.$$

We can use the generalized Gaussian sums to describe the representation numbers. According to Lemma 1.2 we have for $f(x) = R[x] - \Delta$

$$p^l \, \mathrm{N}_{p^l}(R,\Delta) = \sum_{\mu \in \mathbb{Z}/p^l\mathbb{Z}} \; \sum_{x \in \mathbb{Z}^n/p^l\mathbb{Z}^n} e^{\frac{2\pi \mathrm{i}}{p^l}\mu f(x)} = \sum_{\mu=1}^{p^l} G_{p^l}(\mu R) e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta}. \tag{1}$$

2 Even unimodular matrices

Theorem 2.1 *Let $S \in \mathrm{Sym}(n,\mathbb{Z})$ be an even unimodular matrix, i.e, $\det S = 1$. Put $R = \frac{1}{2}S$. Let p be an odd prime and $\Delta \in \mathbb{Z}$ and let $l \in \mathbb{N}$ be a positive integer. Then we have the following formula for the representation number*

$$p^{l(1-n)} \, \mathrm{N}_{p^l}(R,\Delta) = \begin{cases} (1-p^{-n/2}) \frac{1-p^{(1+\mathrm{ord}_p(\Delta))(1-\frac{n}{2})}}{1-p^{1-\frac{n}{2}}} & \textit{if } l > \mathrm{ord}_p(\Delta) \\ p^{l(1-\frac{n}{2})} + (1-p^{-n/2}) \frac{1-p^{l(1-\frac{n}{2})}}{1-p^{1-\frac{n}{2}}} & \textit{if } l \le \mathrm{ord}_p(\Delta) \\ 1-p^{-\frac{n}{2}} & \textit{if } \Delta \textit{ and } p \textit{ are coprime} \end{cases}$$

Proof
We first note that n is necessarily divisible by 8. According to (1) and Lemma 1.8 we have

$$\begin{aligned} p^l \, \mathrm{N}_{p^l}(R,\Delta) = \quad & p^{nl} + \sum_{\mu=1}^{p^l-1} G_{p^l}(\mu R) e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta} \quad = \quad p^{nl} + \sum_{\mu=1}^{p^l-1} p^{\frac{n}{2}(l+\mathrm{ord}_p(\mu))} e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta} \\ & = \quad p^{nl} + S_{\mathcal{T}} + S_{\mathcal{T}^{\mathrm{C}}} \end{aligned}$$

where

$$\mathcal{T} = \left\{ d \in [1, p^l-1] \cap \mathbb{Z}\, ;\, p \mid d \right\} \subseteq [1,p^l-1] \cap \mathbb{Z}.$$

If we put

$$\mathcal{T}_\mu = \{ t \in \mathcal{T}\, ;\, \mathrm{ord}_p(t) = \mu \} = \left\{ \sum_{j=\mu}^{l-1} \alpha_j p^j \, ;\, \alpha_j \in \{0,\dots,p-1\}\, ,\, \alpha_\mu \neq 0 \right\}$$

we receive

$$\begin{aligned} S_{\mathcal{T}} = \sum_{\mu \in \mathcal{T}} p^{\frac{n}{2}(l+\mathrm{ord}_p(\mu))} e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta} &= \sum_{\mu=1}^{l-1} p^{\frac{n}{2}(l+\mu)} \sum_{t \in \mathcal{T}_\mu} e^{-\frac{2\pi\mathrm{i}}{p^l}t\Delta} \\ &= \sum_{\mu=1}^{l-1} p^{\frac{n}{2}(l+\mu)} \sum_{\substack{t=1 \\ p \nmid t}}^{p^{l-\mu}-1} e^{-\frac{2\pi\mathrm{i}}{p^{l-\mu}}t\Delta}. \end{aligned}$$

The complementary sum reads

$$S_{\mathcal{T}^{\mathrm{C}}} = \sum_{\mu \in \mathcal{T}^{\mathrm{C}}} p^{\frac{nl}{2}} e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta}.$$

We now distinguish between several cases. For convenience put $\alpha_\mu(t) = e^{-\frac{2\pi\mathrm{i}}{p^{l-\mu}}t\Delta}$, $\beta(t) = e^{-\frac{2\pi\mathrm{i}}{p^l}t\Delta}$ and write $\Delta = \bar{\Delta}\, p^{\mathrm{ord}_p(\Delta)}$.

(i) First consider the case where Δ and p are coprime. We note that

$$\sum_{\substack{t=1\\ p\nmid t}}^{p^{l-\mu}-1} \alpha_\mu(t) = \sum_{t=1}^{p^{l-\mu}-1} \alpha_\mu(t) - \sum_{\substack{t=1\\ p\mid t}}^{p^{l-\mu}-1} \alpha_\mu(t) = \sum_{t=1}^{p^{l-\mu}-1} \alpha_\mu(t) - \sum_{t=1}^{p^{l-\mu-1}-1} \alpha_\mu(tp)\,. \tag{2}$$

Now μ ranges over all natural numbers between 1 and $l-1$ and we have $\alpha_\mu(1)\neq 1$ for all μ and $\alpha_\mu(p)\neq 1$ for all $\mu\in[0,l-2]\cap\mathbb{N}$ and $\alpha_{l-1}(p)=1$. So the formula of geometric sums yields

$$\sum_{\substack{t=1\\ p\nmid t}}^{p^{l-\mu}-1} \alpha_\mu(t) = \begin{cases} 0 & \text{if } \mu\in[0,l-2]\cap\mathbb{N} \text{ and } l>1 \\ -1 & \text{if } \mu=l-1 \qquad\qquad \text{and } l>1 \end{cases} \quad\text{and thus}\quad S_{\mathcal{T}} = \begin{cases} -p^{nl-\frac{n}{2}} & \text{if } l>1 \\ 0 & \text{if } l=1 \end{cases}$$

On the other hand by again using (2) we obtain

$$S_{\mathcal{T}^{\mathrm{C}}} = p^{\frac{nl}{2}} \sum_{\substack{t=1\\ p\nmid t}}^{p^l-1} \beta(t) = p^{\frac{nl}{2}} \left[\sum_{t=1}^{p^l-1} \beta(t) - \sum_{t=1}^{p^{l-1}-1} \beta(tp) \right] = \begin{cases} 0 & \text{if } l>1 \\ -p^{\frac{n}{2}} & \text{if } l=1 \end{cases}$$

Summing up we have shown

$$p^{l(1-n)}\,\mathrm{N}_{p^l}(R,\Delta) = 1-p^{-\frac{n}{2}}\,.$$

(ii) Now assume that $\mathrm{ord}_p(\Delta)>0$ and $l\le \mathrm{ord}_p(\Delta)$. In this case we have the formula

$$S_{\mathcal{T}} = \sum_{\mu=1}^{l-1} p^{\frac{n}{2}(l+\mu)}\,\varphi(p^{l-\mu}) = \left(1-\frac{1}{p}\right) p^{\left(\frac{n}{2}+1\right)l} \sum_{\mu=1}^{l-1} p^{\left(\frac{n}{2}-1\right)\mu} = \left(1-\frac{1}{p}\right) p^{l\left(\frac{n}{2}+1\right)} \cdot \left\{ \frac{1-p^{l\left(\frac{n}{2}-1\right)}}{1-p^{\frac{n}{2}-1}} - 1 \right\}$$

where we have used Euler's totient function φ and the identity

$$\sharp\mathcal{T}_\mu = \varphi(p^{l-\mu}) = p^{l-\mu}-p^{l-\mu-1}\,.$$

Moreover we have $\mathcal{T}^{\mathrm{C}}=\varphi(p^l)$ and $\beta(t)=1$ which implies

$$S_{\mathcal{T}^{\mathrm{C}}} = \sum_{t\in\mathcal{T}^{\mathrm{C}}} p^{\frac{nl}{2}}\beta(t) = p^{\frac{nl}{2}}\cdot\varphi(p^l) = \left(1-\frac{1}{p}\right)\cdot p^{l\left(\frac{n}{2}+1\right)}\,.$$

Summarizing these calculations we finally obtain

$$p^{l(1-n)}\,\mathrm{N}_{p^l}(R,\Delta) = 1+\left(1-\frac{1}{p}\right) p^{l\left(1-\frac{n}{2}\right)}\cdot \frac{1-p^{l\left(\frac{n}{2}-1\right)}}{1-p^{\frac{n}{2}-1}}\,.$$

(iii) It remains to consider the case where $\mathrm{ord}_p(\Delta)>0$ and $l>\mathrm{ord}_p(\Delta)$. We rewrite the quantity α as

$$\alpha_\mu(t) = \exp\left(2\pi\mathrm{i}t\,\bar{\Delta}\,p^{\mathrm{ord}_p(\Delta)-l\,+\mu}\right)\,.$$

Using (2) we establish the identity

$$\sum_{\substack{t=1\\ p\nmid t}}^{p^{l-\mu}-1} \alpha_\mu(t) = \sum_{t=1}^{p^{l-\mu}-1} \alpha_\mu(t) - \sum_{t=1}^{p^{l-\mu-1}-1} \alpha_\mu(tp) = \begin{cases} 0 & \text{if } \mathrm{ord}_p(\Delta)-l\,+\mu<-1 \\ -p^{l-\mu-1} & \text{if } \mathrm{ord}_p(\Delta)-l\,+\mu=-1 \\ \varphi(p^{l-\mu}) & \text{if } \mathrm{ord}_p(\Delta)-l\,+\mu>-1 \end{cases} \tag{3}$$

where φ again denotes Euler's totient function. Next we write the index set $\mathbb{N}_{\le l-1}$ as the union of the two (possibly empty) sets

$$(\,[1,l-\mathrm{ord}_p(\Delta)-1)\cap\mathbb{N}\,)\cup(\,[l-\mathrm{ord}_p(\Delta)-1,l-1]\cap\mathbb{N}\,)\,.$$

The contribution of (3) to $S_{\mathcal{T}}$ for the index μ being strictly smaller than $l-\mathrm{ord}_p(\Delta)-1$ is always 0. If $l-\mathrm{ord}_p(\Delta)>1$ we also have to take account of the summand $p^{l-\mu-1}$. This means

$$S_{\mathcal{T}} = \sum_{\mu=1}^{l-1} p^{\frac{n}{2}(l+\mu)} \sum_{\substack{t=1\\ p\nmid t}}^{p^{l-\mu}-1} \alpha_\mu(t)$$

$$= \begin{cases} p^{(\frac{n}{2}+1)l}\left(1-\frac{1}{p}\right)\sum_{\mu=1}^{l-1} p^{(\frac{n}{2}-1)\mu} & \text{if } l-\mathrm{ord}_p(\Delta)=1\\ p^{(\frac{n}{2}+1)l}\left(1-\frac{1}{p}\right)\sum_{\mu=l-\mathrm{ord}_p(\Delta)}^{l-1} p^{(\frac{n}{2}-1)\mu} - p^{\frac{n}{2}(2l-\mathrm{ord}_p(\Delta)-1)}\,p^{\mathrm{ord}_p(\Delta)} & \text{if } l-\mathrm{ord}_p(\Delta)>1 \end{cases}$$

$$= \begin{cases} p^{(\frac{n}{2}+1)l}\left(1-\frac{1}{p}\right)\left(\frac{1-p^{l(\frac{n}{2}-1)}}{1-p^{\frac{n}{2}-1}}-1\right) & \text{if } l-\mathrm{ord}_p(\Delta)=1\\ p^{(\frac{n}{2}+1)l}\left(1-\frac{1}{p}\right)\left(\frac{p^{(l-\mathrm{ord}_p(\Delta))(\frac{n}{2}-1)}-p^{l(\frac{n}{2}-1)}}{1-p^{\frac{n}{2}-1}}\right) - p^{\frac{n}{2}(2l-\mathrm{ord}_p(\Delta)-1)}\,p^{\mathrm{ord}_p(\Delta)} & \text{if } l-\mathrm{ord}_p(\Delta)>1 \end{cases}$$

We need to determine the remaining terms in $S_{\mathcal{T}^{\mathrm{C}}}$. To this end we first write

$$\beta(t)=\exp\left(2\pi\mathrm{i}t\,\bar{\Delta}\,p^{\mathrm{ord}_p(\Delta)-l}\right)$$

and note that

$$S_{\mathcal{T}^{\mathrm{C}}} = p^{\frac{nl}{2}}\sum_{\substack{t=1\\ p\nmid t}}^{p^l-1}\beta(t) = p^{\frac{nl}{2}}\left[\sum_{t=1}^{p^l-1}\beta(t)-\sum_{t=1}^{p^{l-1}-1}\beta(tp)\right] = \begin{cases} -p^{l(\frac{n}{2}+1)-1} & \text{if } l-\mathrm{ord}_p(\Delta)=1\\ 0 & \text{if } l-\mathrm{ord}_p(\Delta)>1\end{cases}$$

Finally we combine all these results and obtain

$$p^l\,\mathrm{N}_{p^l}(R,\Delta) = \begin{cases} p^{nl}+p^{(\frac{n}{2}+1)l}\left(1-\frac{1}{p}\right)\left(\frac{1-p^{l(\frac{n}{2}-1)}}{1-p^{\frac{n}{2}-1}}-1\right)-p^{l(\frac{n}{2}+1)-1} & ,\ l-\mathrm{ord}_p(\Delta)=1\\ p^{nl}-p^{\frac{n}{2}(2l-\mathrm{ord}_p(\Delta)-1)}\,p^{\mathrm{ord}_p(\Delta)}+p^{(\frac{n}{2}+1)l}\left(1-\frac{1}{p}\right)\left(\frac{p^{(l-\mathrm{ord}_p(\Delta))(\frac{n}{2}-1)}-p^{l(\frac{n}{2}-1)}}{1-p^{\frac{n}{2}-1}}\right) & ,\ l-\mathrm{ord}_p(\Delta)>1\end{cases}$$

or equivalently

$$p^{l(1-n)}\,\mathrm{N}_{p^l}(R,\Delta) = \begin{cases} 1-p^{l(1-\frac{n}{2})-1}+p^{(1-\frac{n}{2})l}\left(1-\frac{1}{p}\right)\left(\frac{1-p^{l(\frac{n}{2}-1)}}{1-p^{\frac{n}{2}-1}}-1\right) & ,\ l-\mathrm{ord}_p(\Delta)=1\\ 1-p^{\mathrm{ord}_p(\Delta)(1-\frac{n}{2})-\frac{n}{2}}-\left(1-\frac{1}{p}\right)\left(\frac{1-p^{\mathrm{ord}_p(\Delta)(1-\frac{n}{2})}}{1-p^{\frac{n}{2}-1}}\right) & ,\ l-\mathrm{ord}_p(\Delta)>1\end{cases}$$

If we finally use the identity

$$(1-p^{-n/2})\frac{1-p^{m(1-\frac{n}{2})}}{1-p^{1-\frac{n}{2}}} = 1-p^{m(1-\frac{n}{2})}-\left(1-\frac{1}{p}\right)\left(\frac{1-p^{m(1-\frac{n}{2})}}{1-p^{\frac{n}{2}-1}}\right)\quad \text{for all } m\in\mathbb{N}\,.$$ □

we obtain the desired identities.

It remains to calculate the representation numbers for the prime $p=2$. To this end we cite a theorem from the arithmetic theory of quadratic forms.

Theorem 2.2 *Let S_1, $S_2\in\mathrm{Sym}(n,\mathbb{Z})$ be even unimodular matrices and $q\in\mathbb{N}$. Then there exists a $U=U(q)\in\mathrm{GL}(n,\mathbb{Z})$ such that*

$$S_1[U]\equiv S_2 \bmod q\,.$$

Theorem 2.3 *Let $S\in\mathrm{Sym}(n,\mathbb{Z})$ be an even unimodular matrix, i.e, $\det S=1$. Let $\Delta\in\mathbb{Z}$ and let $l\in\mathbb{N}$ be a positive integer. Then we have the following formula for the representation number*

$$2^{l(1-n)}\,\mathrm{N}_{2^l}(S,\Delta) = \begin{cases} 2(1-2^{-n/2})\frac{1-2^{\mathrm{ord}_2(\Delta)(1-\frac{n}{2})}}{1-2^{1-\frac{n}{2}}} & \textit{if } l>\mathrm{ord}_2(\Delta)\\ 1+2^{\frac{n}{2}}(2^{l(1-\frac{n}{2})}-1)+2^{\frac{n}{2}}(1-2^{-n/2})\frac{1-2^{l(1-\frac{n}{2})}}{1-2^{1-\frac{n}{2}}} & \textit{if } l\le\mathrm{ord}_2(\Delta)\\ 0 & \textit{if } \Delta \textit{ and } p \textit{ are coprime}\end{cases}$$

and for $R = \frac{1}{2}S$

$$2^{l(1-n)}\,\mathrm{N}_{2^l}(R,\Delta) = \begin{cases} (1-2^{-n/2})\frac{1-2^{(\mathrm{ord}_2(\Delta)+1)(1-\frac{n}{2})}}{1-2^{1-\frac{n}{2}}} & \textit{if } l > \mathrm{ord}_2(\Delta) \\ 2^{l(1-\frac{n}{2})} + (1-2^{-n/2})\frac{1-2^{l(1-\frac{n}{2})}}{1-2^{1-\frac{n}{2}}} & \textit{if } l \le \mathrm{ord}_2(\Delta) \\ 1-2^{-\frac{n}{2}} & \textit{if } \Delta \textit{ and } p \textit{ are coprime} \end{cases}$$

Proof
We apply the previous theorem to $q = 2^l$, $S_1 = S$ and $S_2 = \mathrm{diag}(H,\ldots,H)$ where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that there is a $U \in \mathrm{GL}(n,\mathbb{Z})$ such that

$$S[U] \equiv \begin{pmatrix} H & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H \end{pmatrix} \bmod 2^l .$$

The Gaussian sum thus equals

$$G_{2^l}(aS) = \left(\sum_{x\in\mathbb{Z}/2^l\mathbb{Z}} \sum_{y\in\mathbb{Z}/2^l\mathbb{Z}} e^{\frac{4\pi\mathrm{i}\,axy}{2^l}} \right)^{\frac{n}{2}} .$$

We have

$$\sum_{x\in\mathbb{Z}/2^l\mathbb{Z}} \sum_{y\in\mathbb{Z}/2^l\mathbb{Z}} e^{\frac{2\pi\mathrm{i}\,axy}{2^{l-1}}} = \sum_{\mu=0}^{2^l-1}\sum_{\nu=0}^{2^l-1} e^{\frac{2\pi\mathrm{i}\,a\mu\nu}{2^l-1}} = \sum_{\mu=0}^{2^l-1}\sum_{\nu=0}^{2^l-1} e^{2\pi\mathrm{i}\,\bar{a}\,2^{\mathrm{ord}_2(a)+1-l}\mu\nu}$$

Now write $\mu = q\,2^{l-\mathrm{ord}_2(a)-1} + r$ by division with remainder. Then we obtain

$$\sum_{\mu=0}^{2^l-1}\sum_{\nu=0}^{2^l-1} e^{2\pi\mathrm{i}\,\bar{a}\,2^{\mathrm{ord}_2(a)+1-l}\mu\nu} = 2^{1+\mathrm{ord}_2(a)} \sum_{\mu=0}^{2^{l-\mathrm{ord}_2(a)-1}-1}\sum_{\nu=0}^{2^l-1} e^{2\pi\mathrm{i}\,\bar{a}\,2^{\mathrm{ord}_2(a)+1-l}\mu\nu} = 2^{1+l+\mathrm{ord}_2(a)}$$

hence

$$G_{2^l}(aS) = 2^{\frac{n}{2}(1+l+\mathrm{ord}_2(a))} .$$

With the notation from the proof of the last theorem we have

$$2^l\,\mathrm{N}_{2^l}(S,\Delta) = 2^{nl} + S_{\mathcal{T}} + S_{\mathcal{T}^{\mathrm{C}}}$$

where

$$S_{\mathcal{T}} = \sum_{\mu=1}^{l-1} 2^{\frac{n}{2}(1+l+\mu)} \sum_{\substack{t=1\\ 2\nmid t}}^{2^{l-\mu}-1} e^{-\frac{2\pi\mathrm{i}}{2^{l-\mu}}t\Delta}$$

and

$$S_{\mathcal{T}^{\mathrm{C}}} = \sum_{\mu\in\mathcal{T}^{\mathrm{C}}} 2^{\frac{n}{2}(1+l)} e^{-\frac{2\pi\mathrm{i}}{2^l}\mu\Delta} .$$

(i) If 2 and Δ are coprime we have

$$S_{\mathcal{T}} = \begin{cases} -2^{nl} & \text{if } l > 1 \\ 0 & \text{if } l = 1 \end{cases} \quad \text{and} \quad S_{\mathcal{T}^{\mathrm{C}}} = \begin{cases} 0 & \text{if } l > 1 \\ -2^n & \text{if } l = 1 \end{cases}$$

which implies

$$2^{l(1-n)}\,\mathrm{N}_{2^l}(S,\Delta) = 0 \quad \text{respectively} \quad 2^{l(1-n)}\,\mathrm{N}_{2^l}(R,\Delta) = 1 - 2^{-\frac{n}{2}} .$$

(ii) If $\mathrm{ord}_2(\Delta) > 0$ and $l \le \mathrm{ord}_2(\Delta)$ the corresponding formulas are

$$S_{\mathcal{T}} = \left(1 - \frac{1}{2}\right) 2^{l(\frac{n}{2}+1)+\frac{n}{2}} \cdot \left\{ \frac{1-2^{l(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}} - 1 \right\}$$

and

$$S_{\mathcal{T}^{\mathrm{C}}} = \left(1-\frac{1}{2}\right)\cdot 2^{l(\frac{n}{2}+1)+\frac{n}{2}}\,.$$

which implies

$$2^{l(1-n)}\,\mathrm{N}_{2^l}(S,\Delta) = 1+\left(1-\frac{1}{2}\right)2^{l(1-\frac{n}{2})+\frac{n}{2}}\cdot\frac{1-2^{l(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}$$

(iii) It remains th case where $\mathrm{ord}_2(\Delta)>0$ and $l>\mathrm{ord}_2(\Delta)$. In this case we have

$$2^l\,\mathrm{N}_{2^l}(S,\Delta) = \begin{cases} 2^{nl}+2^{(\frac{n}{2}+1)l+\frac{n}{2}}\left(1-\frac{1}{2}\right)\left(\frac{1-2^{l(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}-1\right)-2^{l(\frac{n}{2}+1)+(\frac{n}{2}-1)} & ,\ l-\mathrm{ord}_2(\Delta)=1 \\ 2^{nl}-2^{\frac{n}{2}(2l-\mathrm{ord}_2(\Delta))}\,2^{\mathrm{ord}_2(\Delta)}+2^{(\frac{n}{2}+1)l+\frac{n}{2}}\left(1-\frac{1}{2}\right)\left(\frac{2^{(l-\mathrm{ord}_2(\Delta))(\frac{n}{2}-1)}-2^{l(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}\right) & ,\ l-\mathrm{ord}_2(\Delta)>1 \end{cases}$$

or equiavalently

$$2^{l(1-n)}\,\mathrm{N}_{2^l}(S,\Delta) = \begin{cases} 1+2^{(1-\frac{n}{2})l+\frac{n}{2}}\left(1-\frac{1}{2}\right)\left(\frac{1-2^{l(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}-1\right)-2^{(l-1)(1-\frac{n}{2})} & ,\ l-\mathrm{ord}_2(\Delta)=1 \\ 1-2^{-\frac{n}{2}\,\mathrm{ord}_2(\Delta)}\,2^{\mathrm{ord}_2(\Delta)}+2^{(1-\frac{n}{2})l+\frac{n}{2}}\left(1-\frac{1}{2}\right)\left(\frac{2^{(l-\mathrm{ord}_2(\Delta))(\frac{n}{2}-1)}-2^{l(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}\right) & ,\ l-\mathrm{ord}_2(\Delta)>1 \end{cases}$$

□

3 Arbitrary matrices

The first theorem describes the case where $n=\mathrm{rank}(S)$ is even and p is an odd prime. In this case we note that

$$\mathrm{N}_{p^l}(R,\Delta)=\mathrm{N}_{p^l}(S,2\Delta)\quad,\quad R=\frac{1}{2}S\,.$$

Theorem 3.1 *Let $S\in\mathrm{Sym}(n,\mathbb{Z})$ where n is even. Let p be an odd prime, $\Delta\in\mathbb{Z}$ and let $l\in\mathbb{N}$ be a positive integer. Then we have the following formula for the representation number*

$$p^{l(1-n)}\,\mathrm{N}_{p^l}(S,\Delta) = \begin{cases} (1-\varepsilon(p,S)\,p^{-n/2})\frac{1-\varepsilon(p,S)^{1+\mathrm{ord}_p(\Delta)}\,p^{(1+\mathrm{ord}_p(\Delta))(1-\frac{n}{2})}}{1-\varepsilon(p,S)\,p^{1-\frac{n}{2}}} & \text{if } l>\mathrm{ord}_p(\Delta) \\ \varepsilon(p,S)^l\,p^{l(1-\frac{n}{2})}+(1-\varepsilon(p,S)\,p^{-n/2})\frac{1-\varepsilon(p,S)^l\,p^{l(1-\frac{n}{2})}}{1-\varepsilon(p,S)\,p^{1-\frac{n}{2}}} & \text{if } l\le\mathrm{ord}_p(\Delta) \\ 1-\varepsilon(p,S)\,p^{-\frac{n}{2}} & \text{if } \Delta \text{ and } p \text{ are coprime} \end{cases}$$

where we have used the abbreviation

$$\varepsilon(p,S)=\left(\frac{(-1)^{n/2}\det S}{p}\right)\,.$$

Proof
Considering (1) leads to the expression

$$p^l\,\mathrm{N}_{p^l}(R,\Delta)=p^{nl}+\sum_{\mu=1}^{p^l-1}G_{p^l}(\mu R)e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta}=p^{nl}+S_{\mathcal{T}}+S_{\mathcal{T}^{\mathrm{C}}}$$

where

$$\mathcal{T}=\left\{d\in[1,p^l-1]\cap\mathbb{Z}\,;\,p\mid d\right\}\subseteq[1,p^l-1]\cap\mathbb{Z}\,.$$

If we put

$$\mathcal{T}_\mu=\{t\in\mathcal{T}\,;\,\mathrm{ord}_p(t)=\mu\}=\left\{\sum_{j=\mu}^{l-1}\alpha_j p^j\,;\,\alpha_j\in\{0,\ldots,p-1\}\,,\,\alpha_\mu\neq 0\right\}$$

we receive by using remark 1.9 and the fact that n is even

$$\begin{aligned} S_{\mathcal{T}}=\sum_{\mu\in\mathcal{T}}G_{p^l}(\mu R)e^{-\frac{2\pi\mathrm{i}}{p^l}\mu\Delta} &= \sum_{\mu=1}^{l-1}\varepsilon(p,S)^{l+\mu}\,p^{\frac{n}{2}(l+\mu)}\sum_{t\in\mathcal{T}_\mu}e^{-\frac{2\pi\mathrm{i}}{p^l}t\Delta} \\ &= \sum_{\mu=1}^{l-1}\varepsilon(p,S)^{l+\mu}\,p^{\frac{n}{2}(l+\mu)}\sum_{\substack{t=1\\ p\nmid t}}^{p^{l-\mu}-1}e^{-\frac{2\pi\mathrm{i}}{p^{l-\mu}}t\Delta}\,. \end{aligned}$$

The complementary sum reads

$$S_{\mathcal{T}^{\mathrm{C}}} = \sum_{\mu\in\mathcal{T}^{\mathrm{C}}} \varepsilon(p,S)^l\, p^{\frac{nl}{2}}\, e^{-\frac{2\pi \mathrm{i}}{p^l}\mu\Delta}\,.$$

Now an analogue discussion as in the proof of Theorem 2.1 yields the formulae stated above. □

Again the case $p = 2$ has to be treated separetely. For this purpose we consider another generalization of Gaussian sums where we also allow the exponent to be half-integral.

Lemma 3.2 *Let $l \in \mathbb{N}$ and $0 \neq a \in \mathbb{Z}$ such that $\mathrm{ord}_2(a) < l$ and $\lambda \in \mathbb{Z}$. Write $a = 2^{\mathrm{ord}_2(a)} a_{\bar{2}}$ Let $\lambda \in \mathbb{Z}$ be an even integer. Then we have*

$$\sum_{x=0}^{2^l-1} e^{\frac{2\pi \mathrm{i} a(x^2-\lambda x)}{2^l}} = e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}} \left(\frac{2^{l-\mathrm{ord}_2(a)}}{a_{\bar{2}}}\right) (1+\mathrm{i}^{\,a_{\bar{2}}})\, 2^{\frac{l+\mathrm{ord}_2(a)}{2}} \cdot \begin{cases} \mathrm{sign}(l-\mathrm{ord}_2(a)-1) & \text{if } \lambda \equiv 0 \bmod 2 \\ 1-\mathrm{sign}(l-\mathrm{ord}_2(a)-1) & \text{if } \lambda \equiv 1 \bmod 2 \end{cases}$$

Proof
We rewrite our sum as

$$\sum_{x=0}^{2^l-1} e^{\frac{2\pi \mathrm{i} a(x^2-\lambda x)}{2^l}} = e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}} \cdot \sum_{x=0}^{2^l-1} e^{\frac{2\pi \mathrm{i} a(x-\frac{\lambda}{2})^2}{2^l}} =: e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}} \cdot g(\lambda/2; 2^l; a)$$

Actually we have

$$e^{\frac{2\pi \mathrm{i} a(x-\frac{\lambda}{2})^2}{2^l}} = e^{\frac{2\pi \mathrm{i} a(x-\frac{\lambda}{2}+2^l)^2}{2^l}}$$

and thus

$$\sum_{x=0}^{2^l-1} e^{\frac{2\pi \mathrm{i} a(x-\frac{\lambda}{2}+r)^2}{2^l}} = \sum_{x=0}^{2^l-1} e^{\frac{2\pi \mathrm{i} a(x-\frac{\lambda}{2})^2}{2^l}} \quad \text{for every} \quad r \in \mathbb{Z}\,.$$

So it suffices to consider the two types $g(0/2; 2^l; a)$ respectively $g(1/2; 2^l; a)$ depending on whether λ is even or λ is odd. As in Lemma 1.5(a) we have

$$g(\lambda/2; r\cdot 2^l; ra) = r \cdot g(\lambda/2; 2^l; a) \quad \text{for all} \quad r \in \mathbb{N}\,.$$

Now let λ be even. From [BEW], Proposition 1.5.3 we receive

$$e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}} \cdot g(0/2; 2^l; a) = e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}}\, 2^{\mathrm{ord}_2(a)} \begin{cases} \left(\frac{2^{l-\mathrm{ord}_2(a)}}{a\cdot 2^{-\mathrm{ord}_2(a)}}\right) (1+\mathrm{i}^{\,a\,2^{-\mathrm{ord}_2(a)}})\, 2^{\frac{l-\mathrm{ord}_2(a)}{2}} & \text{if } l-\mathrm{ord}_2(a) \geq 2 \\ 0 & \text{else} \end{cases}$$

If λ is odd we use Ex. 23 of [BEW], chap. 1 and obtain

$$e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}} \cdot g(1/2; 2^l; a) = e^{-\frac{2\pi \mathrm{i} a\lambda^2}{2^{l+2}}}\, 2^{\mathrm{ord}_2(a)} \begin{cases} 0 & \text{if } l-\mathrm{ord}_2(a) \geq 2 \\ \left(\frac{2^{l-\mathrm{ord}_2(a)}}{a\cdot 2^{-\mathrm{ord}_2(a)}}\right) (1+\mathrm{i}^{\,a\,2^{-\mathrm{ord}_2(a)}})\, 2^{\frac{l-\mathrm{ord}_2(a)}{2}} & \text{if } l-\mathrm{ord}_2(a) = 1 \end{cases}$$

□

Now we consider the representation number with respect to

$$f_2(x) = g_2(x) - \Delta = (x_1^2 - \lambda_1 x_1) + \cdots + (x_n^2 - \lambda_n x_1) - \Delta$$

and again make use of the fact

$$2^l\, \mathrm{N}_{2^l}(f_2, \Delta) = \sum_{\mu\in\mathbb{Z}/2^l\mathbb{Z}} \sum_{x\in\mathbb{Z}^n/2^l\mathbb{Z}^n} e^{\frac{2\pi \mathrm{i}}{2^l}\mu f_2(x)} = \sum_{\mu=1}^{2^l} G_{2^l}(\mu g_2) e^{-\frac{2\pi \mathrm{i}}{2^l}\mu\Delta} \tag{4}$$

where for $\mu \neq 2^l$ we have

$$\begin{aligned} G_{2^l}(\mu g_2) &= \prod_{k=1}^{n} \sum_{x=0}^{2^l-1} e^{\frac{2\pi \mathrm{i}\mu(x^2-\lambda_k x)}{2^l}} \\ &= \frac{(1+\mathrm{i}^{\,\mu_{\bar{2}}})^n}{e^{\frac{2\pi \mathrm{i}\mu(\lambda,\lambda)}{2^{l+2}}}} \left(\frac{2^{n(l-\mathrm{ord}_2(\mu))}}{\mu_{\bar{2}}}\right) 2^{\frac{n}{2}(l+\mathrm{ord}_2(\mu))} \cdot \begin{cases} \mathrm{sign}(l-\mathrm{ord}_2(\mu)-1) & \text{if } \lambda_1 \equiv \ldots \equiv \lambda_n \equiv 0 \bmod 2 \\ 1-\mathrm{sign}(l-\mathrm{ord}_2(\mu)-1) & \text{if } \lambda_1 \equiv \ldots \equiv \lambda_n \equiv 1 \bmod 2 \\ 0 & \text{else} \end{cases} \end{aligned} \tag{5}$$

and $G_{2^l}(2^l\, g_2) = 2^{nl}$ else. Here we have written $(\cdot,\cdot)$ for the quadratic form which is induced by $\mathrm{diag}(2,\ldots,2)$

Theorem 3.3 *Let n be an even integer and let $l \in \mathbb{N}$. Let $\lambda \in \mathbb{Z}^n$ and write*

$$(\lambda, \lambda) = \lambda_1^2 + \cdots + \lambda_n^2 .$$

(a) If not all λ_k are equivalent $\bmod 2$ *we have*

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = 1 .$$

(b) If all $\lambda_k \equiv 1 \bmod 2$ we have

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = 1 + (-1)^{\Delta} .$$

(c) Put $\kappa := \frac{(\lambda,\lambda)}{4} + \Delta \in \mathbb{Z}$. Assume that all $\lambda_k \equiv 0 \bmod 2$. If 2 and κ are coprime we have

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = \begin{cases} 1 - (-1)^{\frac{n}{4} + \frac{\kappa}{2}} 2^{1-\frac{n}{2}} & \text{if } l \geq 2 \text{ and } n \equiv 2 \bmod 4 \\ 1 & \text{if } l = 1 \quad \text{or} \quad l \geq 2 \text{ and } n \equiv 0 \bmod 4 \end{cases}$$

If $l \leq \mathrm{ord}_2(\kappa)$ we have

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = \begin{cases} 1 - (-1)^{\frac{n}{4}} \frac{1 - 2^{(l-1)(1-\frac{n}{2})}}{1 - 2^{\frac{n}{2}-1}} & \text{if } n \equiv 0 \bmod 4 \\ 1 & \text{if } n \equiv 2 \bmod 4 \end{cases}$$

If $l = \mathrm{ord}_2(\kappa) + 1$ we have

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = \begin{cases} 1 & \text{if } n \equiv 2 \bmod 4 \\ 1 + (-1)^{\frac{n}{4}} 2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})} \left\{ \frac{2^{\mathrm{ord}_2(\kappa)(\frac{n}{2}-1)} - 1}{2^{\frac{n}{2}-1} - 1} - 2 \right\} & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

and if $l \geq \mathrm{ord}_2(\kappa) + 2$

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = \begin{cases} 1 - (-1)^{\frac{n}{4} + \frac{\kappa_{\bar{2}}}{2}} 2^{(\mathrm{ord}_2(\kappa)+1)(1-\frac{n}{2})} & \text{if } n \equiv 2 \bmod 4 \\ 1 - (-1)^{\frac{n}{4}} \left(\frac{1 - 2^{(\mathrm{ord}_2(\kappa)-1)(1-\frac{n}{2})}}{1 - 2^{\frac{n}{2}-1}} + 2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})} \right) & \text{if } n \equiv 0 \bmod 4 \end{cases} .$$

Proof
We have to consider three cases according to (5). If not all λ_k are equivalent $\bmod 2$ we simply have

$$2^{l(1-n)}\, \mathrm{N}_{2^l}(f_2, \Delta) = 1 .$$

For the two remaining cases we perform our usual decomposition of the summands as

$$2^l\, \mathrm{N}_{2^l}(f_2, \Delta) = 2^{nl} + \sum_{\mu=1}^{2^l - 1} G_{2^l}(\mu g_2) e^{-\frac{2\pi \mathrm{i}}{2^l} \mu \Delta} = 2^{nl} + S_{\mathcal{T}} + S_{\mathcal{T}^{\mathrm{C}}}$$

where

$$\mathcal{T} = \left\{ d \in [1, 2^l - 1] \cap \mathbb{Z} \, ; \, 2 \mid d \right\} \subseteq [1, 2^l - 1] \cap \mathbb{Z} .$$

Now let us consider the case where all $\lambda_k \equiv 1 \bmod 2$. If we put

$$\mathcal{T}_\mu = \{ t \in \mathcal{T} \, ; \, \mathrm{ord}_2(t) = \mu \} = \left\{ \sum_{j=\mu}^{l-1} \alpha_j 2^j \, ; \, \alpha_j \in \{0, \ldots, 2-1\} \, , \, \alpha_\mu \neq 0 \right\}$$

equation (5) tells us that only the elements of $\mathcal{T}_{l-1}$ can give a contribition to $S_{\mathcal{T}}$. Hence

$$\begin{aligned} S_{\mathcal{T}} = \sum_{\mu \in \mathcal{T}} G_{2^l}(\mu R) e^{-\frac{2\pi \mathrm{i}}{2^l} \mu \Delta} &= 2^{\frac{n}{2}(2l-1)} \sum_{t \in \mathcal{T}_{l-1}} \frac{(1 + \mathrm{i}^{t_{\bar{2}}})^n}{e^{\frac{2\pi \mathrm{i} t (\lambda,\lambda)}{2^{l+2}}}} e^{-\frac{2\pi \mathrm{i}}{2^l} t \Delta} \\ &= 2^{\frac{n}{2}(2l-1)} \sum_{\substack{t=1 \\ 2 \nmid t}}^{2-1} \frac{(1 + \mathrm{i}^{t_{\bar{2}}})^n}{e^{\frac{2\pi \mathrm{i} t (\lambda,\lambda)}{2^3}}} e^{-\frac{2\pi \mathrm{i}}{2} t \Delta} \\ &= 2^{\frac{n}{2}(2l-1)} \frac{(1 + \mathrm{i})^n}{e^{\frac{2\pi \mathrm{i} (\lambda,\lambda)}{2^3}}} e^{-\frac{2\pi \mathrm{i}}{2} \Delta} \end{aligned}$$

if $l>1$ and $S_{\mathcal{T}}=0$ if $l=1$. The complementary sum reads

$$S_{\mathcal{T}^{\mathrm{C}}}=\begin{cases}2^{\frac{nl}{2}}\sum\limits_{\substack{t=1\\2\nmid t}}^{2^l-1}\dfrac{(1+\mathrm{i}^{t_2})^n}{e^{\frac{2\pi i t(\lambda,\lambda)}{2^{l+2}}}}\,e^{-\frac{2\pi\mathrm{i}}{2^l}t\Delta} & \text{if } l=1\\ 0 & \text{if } l>1\end{cases}$$

Since $1+i=\sqrt{2}\,e^{2\pi\mathrm{i}/8}$ and $\frac{n-(\lambda,\lambda)}{4}\equiv 0 \bmod 2$ by assumption we finally receive

$$2^l\,\mathrm{N}_{2^l}(f_2,\Delta)=2^{nl}+2^{\frac{n}{2}(2l-1)}\frac{(1+\mathrm{i})^n}{e^{\frac{2\pi\mathrm{i}(\lambda,\lambda)}{2^3}}}\,e^{-\frac{2\pi\mathrm{i}}{2}\Delta}=2^{nl}+2^{nl}(-1)^{\Delta+\frac{n-(\lambda,\lambda)}{4}}=2^{nl}+2^{nl}(-1)^{\Delta}\,.$$

It remains to consider the case where all $\lambda_k\equiv 0 \bmod 2$. We have

$$S_{\mathcal{T}}=\begin{cases}\sum\limits_{\mu=1}^{l-2}2^{\frac{n}{2}(l+\mu)}\sum\limits_{\substack{t=1\\2\nmid t}}^{2^{l-\mu}-1}\dfrac{(1+\mathrm{i}^{\,t})^n}{e^{\frac{2\pi\mathrm{i}t(\lambda,\lambda)}{2^{l-\mu+2}}}}\,e^{-\frac{2\pi\mathrm{i}}{2^{l-\mu}}t\Delta} & \text{if } l>2\\ 0 & \text{if } l\le 2\end{cases}$$

and

$$S_{\mathcal{T}^{\mathrm{C}}}=\begin{cases}2^{\frac{nl}{2}}\sum\limits_{\substack{t=1\\2\nmid t}}^{2^l-1}\dfrac{(1+\mathrm{i}^{\,t})^n}{e^{\frac{2\pi\mathrm{i}t(\lambda,\lambda)}{2^{l+2}}}}\,e^{-\frac{2\pi\mathrm{i}}{2^l}t\Delta} & \text{if } l>1\\ 0 & \text{else}\end{cases}$$

Now if $l>2$ we have

$$S_{\mathcal{T}}=\sum_{\mu=1}^{l-2}2^{\frac{n}{2}(l+\mu)}\left[(1+i)^n\,e^{\frac{-2\pi\mathrm{i}(\lambda,\lambda)}{2^{l-\mu+2}}}\cdot e^{\frac{-2\pi\mathrm{i}\Delta}{2^{l-\mu}}}+(1-i)^n\,e^{\frac{-6\pi\mathrm{i}(\lambda,\lambda)}{2^{l-\mu+2}}}\cdot e^{\frac{-6\pi\mathrm{i}\Delta}{2^{l-\mu}}}\right]\sum_{t=0}^{2^{l-\mu-2}-1}e^{\frac{-2\pi\mathrm{i}t(\lambda,\lambda)}{2^{l-\mu}}}\,e^{\frac{-2\pi\mathrm{i}t\Delta}{2^{l-\mu-2}}}$$

and for $l>1$

$$S_{\mathcal{T}^{\mathrm{C}}}=2^{\frac{nl}{2}}\left[(1+i)^n\,e^{\frac{-2\pi\mathrm{i}(\lambda,\lambda)}{2^{l+2}}}\cdot e^{\frac{-2\pi\mathrm{i}\Delta}{2^l}}+(1-i)^n\,e^{\frac{-6\pi\mathrm{i}(\lambda,\lambda)}{2^{l+2}}}\cdot e^{\frac{-6\pi\mathrm{i}\Delta}{2^l}}\right]\sum_{t=0}^{2^{l-2}-1}e^{\frac{-2\pi\mathrm{i}t(\lambda,\lambda)}{2^l}}\,e^{\frac{-2\pi\mathrm{i}t\Delta}{2^{l-2}}}$$

We now consider several cases depending on the factorization of $\frac{(\lambda,\lambda)}{4}+\Delta$ as a power of 2.

(i) The first case we consider is when 2 and $\kappa:=\frac{(\lambda,\lambda)}{4}+\Delta$ are coprime. By the formula of the geometric sum we immediately see that for $l>2$

$$S_{\mathcal{T}}=2^{n(l-1)}\left[(1+i)^n\,e^{\frac{-2\pi\mathrm{i}(\lambda,\lambda)}{2^4}}\cdot e^{\frac{-2\pi\mathrm{i}\Delta}{2^2}}+(1-i)^n\,e^{\frac{-6\pi\mathrm{i}(\lambda,\lambda)}{2^4}}\cdot e^{\frac{-6\pi\mathrm{i}\Delta}{2^2}}\right]$$

Hence we obtain

$$\begin{aligned}2^l\,\mathrm{N}_{2^l}(f_2,\Delta)&=\begin{cases}2^{nl}+2^{n(l-1)}\left[(1+i)^n\,e^{\frac{-2\pi\mathrm{i}(\lambda,\lambda)}{2^4}}\cdot e^{\frac{-2\pi\mathrm{i}\Delta}{2^2}}+(1-i)^n\,e^{\frac{-6\pi\mathrm{i}(\lambda,\lambda)}{2^4}}\cdot e^{\frac{-6\pi\mathrm{i}\Delta}{2^2}}\right] & \text{if } l\ge 2\\ 2^{nl} & \text{if } l=1\end{cases}\\ &=\begin{cases}2^{nl}+2^{n(l-1)}\left[(2i)^{\frac{n}{2}}\,(-i)^{\kappa}+(-2i)^{\frac{n}{2}}\,(-i)^{3\kappa}\right] & \text{if } l\ge 2\\ 2^{nl} & \text{if } l=1\end{cases}\\ &=\begin{cases}2^{nl}+2^{n(l-1)}(2\mathrm{i})^{\frac{n}{2}}\,\mathrm{i}^{\kappa}\left[-1+(-1)^{\frac{n}{2}}\right] & \text{if } l\ge 2\\ 2^{nl} & \text{if } l=1\end{cases}\end{aligned}$$

(ii) Now assume that $l\le \mathrm{ord}_2(\kappa)$. Then we have for $l\ge 2$

$$S_{\mathcal{T}}=\sum_{\mu=1}^{l-2}2^{\frac{n}{2}(l+\mu)}\,2^{l-\mu-2}\,\left[(1+\mathrm{i})^n+(1-\mathrm{i})^n\right]=2^{l(\frac{n}{2}+1)-2}\left[(2\mathrm{i})^{\frac{n}{2}}+(-2\mathrm{i})^{\frac{n}{2}}\right]\left\{\frac{2^{(l-1)(\frac{n}{2}-1)}-1}{2^{\frac{n}{2}-1}-1}-1\right\}$$

and

$$S_{\mathcal{T}^{\mathrm{C}}} = 2^{l(\frac{n}{2}+1)-2}[(2\mathrm{i})^{\frac{n}{2}} + (-2\mathrm{i})^{\frac{n}{2}}]$$

if $l \geq 2$. Hence we have for $l \geq 1$

$$2^l\,\mathrm{N}_{2^l}(f_2,\Delta) = 2^{nl} + 2^{l(\frac{n}{2}+1)-2}\,(2\mathrm{i})^{\frac{n}{2}}[1+(-1)^{\frac{n}{2}}]\,\frac{2^{(l-1)(\frac{n}{2}-1)}-1}{2^{\frac{n}{2}-1}-1}$$

(iii) We finally consider the case where $\mathrm{ord}_2(\kappa) > 0$ and $l > \mathrm{ord}_2(\kappa)$. We first note that

$$\sum_{t=0}^{2^{l-\mu-2}-1} e^{\frac{-2\pi \mathrm{i} t\kappa}{2^{l-\mu-2}}} = \begin{cases} 0 & \text{if } \mathrm{ord}_2(\kappa) < l-\mu-2 \\ 2^{l-\mu-2} & \text{if } \mathrm{ord}_2(\kappa) \geq l-\mu-2 \end{cases}$$

We first consider the case where $l > 2$.

(a) $l - \mathrm{ord}_2(\kappa) = 1$. Then we have

$$S_{\mathcal{T}} = \sum_{\mu=1}^{l-2} 2^{\frac{n}{2}(l+\mu)}\,2^{l-\mu-2}\,[(1+\mathrm{i})^n + (1-\mathrm{i})^n] = 2^{l(\frac{n}{2}+1)-2}[(2\mathrm{i})^{\frac{n}{2}} + (-2\mathrm{i})^{\frac{n}{2}}]\left\{\frac{2^{(l-1)(\frac{n}{2}-1)}-1}{2^{\frac{n}{2}-1}-1} - 1\right\}$$

and

$$S_{\mathcal{T}^{\mathrm{C}}} = 2^{l(\frac{n}{2}+1)-2}[(1+\mathrm{i})^n(-1)^{\kappa_{\bar{2}}} + (1-\mathrm{i})^n(-1)^{3\kappa_{\bar{2}}}] = -\,2^{l(\frac{n}{2}+1)-2}[(1+\mathrm{i})^n + (1-\mathrm{i})^n]\,.$$

This yields

$$2^l\,\mathrm{N}_{2^l}(f_2,\Delta) = 2^{nl} + 2^{l(\frac{n}{2}+1)-2}[(2\mathrm{i})^{\frac{n}{2}} + (-2\mathrm{i})^{\frac{n}{2}}]\left\{\frac{2^{(l-1)(\frac{n}{2}-1)}-1}{2^{\frac{n}{2}-1}-1} - 2\right\}$$

and

$$\begin{aligned} 2^{l(1-n)}\,\mathrm{N}_{2^l}(f_2,\Delta) &= \begin{cases} 1 & \text{if } n \equiv 2 \bmod 4 \\ 1 + 2^{l(1-\frac{n}{2})-1}2^{\frac{n}{2}}(-1)^{\frac{n}{4}}\left\{\frac{2^{(l-1)(\frac{n}{2}-1)}-1}{2^{\frac{n}{2}-1}-1} - 2\right\} & \text{if } n \equiv 0 \bmod 4 \end{cases} \\ &= \begin{cases} 1 & \text{if } n \equiv 2 \bmod 4 \\ 1 + (-1)^{\frac{n}{4}}2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})}\left\{\frac{2^{\mathrm{ord}_2(\kappa)(\frac{n}{2}-1)}-1}{2^{\frac{n}{2}-1}-1} - 2\right\} & \text{if } n \equiv 0 \bmod 4 \end{cases} \end{aligned}$$

(b) $l - \mathrm{ord}_2(\kappa) = 2$. We obtain

$$\begin{aligned} S_{\mathcal{T}} = {} & 2^{\frac{n}{2}(l+1)}2^{l-3}[(1+i)^n(-1)^{\kappa_{\bar{2}}} + (1-i)^n(-1)^{3\kappa_{\bar{2}}}] + \sum_{\mu=2}^{l-2} 2^{\frac{n}{2}(l+\mu)}\,2^{l-\mu-2}\,[(1+\mathrm{i})^n + (1-\mathrm{i})^n] \\ = {} & -\,2^{l(\frac{n}{2}+1)-3}2^{\frac{n}{2}}2^{\frac{n}{2}}\mathrm{i}^{\frac{n}{2}}[1+(-1)^{\frac{n}{2}}] + 2^{l(\frac{n}{2}+1)-2}[(2\mathrm{i})^{\frac{n}{2}} + (-2\mathrm{i})^{\frac{n}{2}}]\frac{2^{(l-1)(\frac{n}{2}-1)} - 2^{2(\frac{n}{2}-1)}}{2^{\frac{n}{2}-1}-1} \\ = {} & 2^{l(\frac{n}{2}+1)-2}2^{\frac{n}{2}}\mathrm{i}^{\frac{n}{2}}[1+(-1)^{\frac{n}{2}}]\left(\frac{2^{(l-1)(\frac{n}{2}-1)} - 2^{2(\frac{n}{2}-1)}}{2^{\frac{n}{2}-1}-1} - 2^{\frac{n}{2}-1}\right) \end{aligned}$$

and

$$S_{\mathcal{T}^{\mathrm{C}}} = 2^{l(\frac{n}{2}+1)-2}[(1+\mathrm{i})^n(-i)^{\kappa_{\bar{2}}} + (1-\mathrm{i})^n(-i)^{3\kappa_{\bar{2}}}] = 2^{\frac{n}{2}}\mathrm{i}^{\frac{n}{2}}\,\mathrm{i}^{\kappa_{\bar{2}}}\,2^{l(\frac{n}{2}+1)-2}[-1+(-1)^{\frac{n}{2}}]\,.$$

Hence

$$2^l\,\mathrm{N}_{2^l}(f_2,\Delta) = \begin{cases} 2^{nl} + 2^{l(\frac{n}{2}+1)-1}2^{\frac{n}{2}}(-1)^{\frac{n}{4}}\left(\frac{2^{(l-1)(\frac{n}{2}-1)}-2^{2(\frac{n}{2}-1)}}{2^{\frac{n}{2}-1}-1} - 2^{\frac{n}{2}-1}\right) & \text{if } n \equiv 0 \bmod 4 \\ 2^{nl} - 2^{\frac{n}{2}}(-1)^{\frac{n}{4}+\frac{\kappa_{\bar{2}}}{2}}\,2^{l(\frac{n}{2}+1)-1} & \text{if } n \equiv 2 \bmod 4 \end{cases}$$

and therefore

$$\begin{aligned} 2^{l(1-n)}\,\mathrm{N}_{2^l}(f_2,\Delta) &= \begin{cases} 1 + (-1)^{\frac{n}{4}}2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})}2^{1-\frac{n}{2}}\left(\frac{2^{(\mathrm{ord}_2(\kappa)+1)(\frac{n}{2}-1)}-2^{2(\frac{n}{2}-1)}}{2^{\frac{n}{2}-1}-1} - 2^{\frac{n}{2}-1}\right) & \text{if } n \equiv 0 \bmod 4 \\ 1 - (-1)^{\frac{n}{4}+\frac{\kappa_{\bar{2}}}{2}}2^{1-\frac{n}{2}}\,2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})} & \text{if } n \equiv 2 \bmod 4 \end{cases} \\ &= \begin{cases} 1 + (-1)^{\frac{n}{4}}2^{(\mathrm{ord}_2(\kappa)+1)(1-\frac{n}{2})}\left(\frac{2^{(\mathrm{ord}_2(\kappa)+1)(\frac{n}{2}-1)}-2^{2(\frac{n}{2}-1)}}{2^{\frac{n}{2}-1}-1} - 2^{\frac{n}{2}-1}\right) & \text{if } n \equiv 0 \bmod 4 \\ 1 - (-1)^{\frac{n}{4}+\frac{\kappa_{\bar{2}}}{2}}\,2^{(\mathrm{ord}_2(\kappa)+1)(1-\frac{n}{2})} & \text{if } n \equiv 2 \bmod 4 \end{cases} \end{aligned}$$

(c) $l-\mathrm{ord}_2(\kappa)\geq 3$. We obtain

$$\begin{aligned}
S_{\mathcal{T}} = \;& 2^{\frac{n}{2}(2l-\mathrm{ord}_2(\kappa)-2)}2^{\mathrm{ord}_2(\kappa)}[(1+i)^n(-i)^{\kappa_{\bar{2}}}+(1-i)^n(-i)^{3\kappa_{\bar{2}}}]\\
+\;& 2^{\frac{n}{2}(2l-\mathrm{ord}_2(\kappa)-1)}2^{\mathrm{ord}_2(\kappa)-1}[(1+i)^n(-1)^{\kappa_{\bar{2}}}+(1-i)^n(-1)^{3\kappa_{\bar{2}}}]\\
+\;& \sum_{\mu=l-\mathrm{ord}_2(\kappa)}^{l-2} 2^{\frac{n}{2}(l+\mu)}\,2^{l-\mu-2}\,[(1+\mathrm{i})^n+(1-\mathrm{i})^n]\\
=\;& 2^{\frac{n}{2}(2l-\mathrm{ord}_2(\kappa)-2)}2^{\mathrm{ord}_2(\kappa)}2^{\frac{n}{2}}\,\mathrm{i}^{\frac{n}{2}}\,\mathrm{i}^{\kappa_{\bar{2}}}[-1+(-1)^{\frac{n}{2}}]\\
-\;& 2^{\frac{n}{2}(2l-\mathrm{ord}_2(\kappa)-1)}2^{\mathrm{ord}_2(\kappa)-1}2^{\frac{n}{2}}\,\mathrm{i}^{\frac{n}{2}}[1+(-1)^{\frac{n}{2}}]\\
+\;& 2^{l(\frac{n}{2}+1)-2}2^{\frac{n}{2}}\,\mathrm{i}^{\frac{n}{2}}[1+(-1)^{\frac{n}{2}}]\frac{2^{(l-1)(\frac{n}{2}-1)}-2^{(l-\mathrm{ord}_2(\kappa))(\frac{n}{2}-1)}}{2^{\frac{n}{2}-1}-1}\\
=\;& \begin{cases}(-1)^{\frac{n}{4}+\frac{\kappa_{\bar{2}}}{2}+1}2^{\frac{n}{2}(2l-\mathrm{ord}_2(\kappa)-2)}2^{\mathrm{ord}_2(\kappa)}2^{\frac{n}{2}+1} & \text{if } n\equiv 2 \bmod 4\\ (-1)^{\frac{n}{4}}\left(2^{nl-1}2^{\frac{n}{2}}2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})}\frac{1-2^{(\mathrm{ord}_2(\kappa)-1)(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}-2^{nl-\frac{n\,\mathrm{ord}_2(\kappa)}{2}}2^{\mathrm{ord}_2(\kappa)}\right) & \text{if } n\equiv 0 \bmod 4\end{cases}
\end{aligned}$$

Hence

$$2^l\,\mathrm{N}_{2^l}(f_2,\Delta)=\begin{cases}2^{nl}+(-1)^{\frac{n}{4}+\frac{\kappa_{\bar{2}}}{2}+1}2^{\frac{n}{2}(2l-\mathrm{ord}_2(\kappa)-2)}2^{\mathrm{ord}_2(\kappa)}2^{\frac{n}{2}+1} & \text{if } n\equiv 2 \bmod 4\\ 2^{nl}+(-1)^{\frac{n}{4}}\left(2^{nl-1}2^{\frac{n}{2}}2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})}\frac{1-2^{(\mathrm{ord}_2(\kappa)-1)(\frac{n}{2}-1)}}{1-2^{\frac{n}{2}-1}}-2^{nl-\frac{n\,\mathrm{ord}_2(\kappa)}{2}}2^{\mathrm{ord}_2(\kappa)}\right) & \text{if } n\equiv 0 \bmod 4\end{cases}$$

and

$$2^{l(1-n)}\,\mathrm{N}_{2^l}(f_2,\Delta)=\begin{cases}1-(-1)^{\frac{n}{4}+\frac{\kappa_{\bar{2}}}{2}}2^{(\mathrm{ord}_2(\kappa)+1)(1-\frac{n}{2})} & \text{if } n\equiv 2 \bmod 4\\ 1-(-1)^{\frac{n}{4}}\left(\frac{1-2^{(\mathrm{ord}_2(\kappa)-1)(1-\frac{n}{2})}}{1-2^{\frac{n}{2}-1}}+2^{\mathrm{ord}_2(\kappa)(1-\frac{n}{2})}\right) & \text{if } n\equiv 0 \bmod 4\end{cases}$$

If $l=2$ we necessarily have $\mathrm{ord}_2(\kappa)=1$ and thus

$$2^l\,\mathrm{N}_{2^l}(f_2,\Delta)=2^{nl}-2^{\frac{3n}{2}}\,\mathrm{i}^{\frac{n}{2}}(1+(-1)^{\frac{n}{2}})\,. \qquad \square$$

References

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